Boyce/DiPrima 10th ed, Ch1.3: Classification of Differential Equations

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- The main purpose of this course is to discuss properties of solutions of differential equations, and to present methods of finding solutions or approximating them.

- To provide a framework for this discussion, in this section we give several ways of classifying differential equations.

Ordinary Differential Equations

- When the unknown function depends on a single independent variable, only ordinary derivatives appear in the equation.

- In this case the equation is said to be an ordinary differential equations (ODE).

- The equations discussed in the preceding two sections are ordinary differential equations. For example,

$$\frac{dv}{dt} = 9.8 - 0.2v, \quad \frac{dp}{dt} = 0.5p - 450$$

Partial Differential Equations

- When the unknown function depends on several independent variables, partial derivatives appear in the equation.

- In this case the equation is said to be a partial differential equation (PDE).

- Examples:

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{(heat equation)}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave equation)}$$

Systems of Differential Equations

- Another classification of differential equations depends on the number of unknown functions that are involved.

- If there is a single unknown function to be found, then one equation is sufficient. If there are two or more unknown functions, then a system of equations is required.

- For example, predator-prey equations have the form

$$dx/dt = a\,x - \alpha\,xy$$
$$dy/dt = -cy + \gamma\,xy$$

  where $x(t)$ and $y(t)$ are the respective populations of prey and predator species. The constants a, c, α, γ depend on the particular species being studied.
- Systems of equations are discussed in Chapter 7.

Order of Differential Equations

- The **order** of a differential equation is the order of the highest derivative that appears in the equation.

- Examples:

$$y' + 3y = 0$$

$$y'' + 3y' - 2t = 0$$

$$\frac{d^4 y}{dt^4} - \frac{d^2 y}{dt^2} + 1 = e^{2t}$$

$$u_{xx} + u_{yy} = \sin t$$

- We will be studying differential equations for which the highest derivative can be isolated:

$$y^{(n)}(t) = f\left(t, y, y', y'', y''', \ldots, y^{(n-1)}\right)$$

Linear & Nonlinear Differential Equations

- An ordinary differential equation

$$F\left(t, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$$

  is **linear** if F is linear (?) in the variables

$$y, y', y'', y''', \ldots, y^{(n)}$$

- Thus the general linear ODE has the form

$$a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \cdots + a_n(t) y = g(t)$$

- Example: Determine whether the equations below are linear or nonlinear.

(1) $y' + 3y = 0$ (2) $y'' + 3e^y y' - 2t = 0$ (3) $y'' + 3y' - 2t^2 = 0$

(4) $\dfrac{d^4 y}{dt^4} - t\dfrac{d^2 y}{dt^2} + 1 = t^2$ (5) $u_{xx} + u u_{yy} = \sin t$ (6) $u_{xx} + \sin(u) u_{yy} = \cos t$

Solutions to Differential Equations

- A solution $\phi(t)$ to an ordinary differential equation

$$y^{(n)}(t) = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

- satisfies the equation:

$$\phi^{(n)}(t) = f\left(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)}\right)$$

- Example: Verify the following solutions of the ODE

$$y'' + y = 0; \;\; y_1(t) = \sin t, \;\; y_2(t) = -\cos t, \;\; y_3(t) = 2\sin t$$

Solutions to Differential Equations

- Three important questions in the study of differential equations:
  - Is there a solution? (Existence)
  - If there is a solution, is it unique? (Uniqueness)
  - If there is a solution, how do we find it?

    (Analytical Solution, Numerical Approximation, etc)