Sec 2.3 Modeling with First Order Equations

- Mathematical models characterize physical systems, often using differential equations.

- **Model Construction**: Translating physical situation into mathematical terms. Clearly state physical principles believed to govern process. Differential equation is a mathematical model of process, typically an approximation.

- **Analysis of Model**: Solving equations or obtaining qualitative understanding of solution. May simplify model, as long as physical essentials are preserved.

- **Comparison with Experiment or Observation**: Verifies solution or suggests refinement of model.

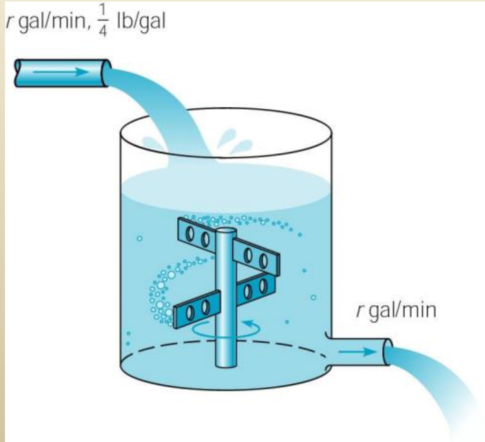
Example 1: Salt Solution (1 of 11)

(Ex 1) At time $t = 0$, a tank contains 5 lb of salt dissolved in 100 gal of (salt) water. Assume that water containing ¼ lb of salt/gal is entering tank at rate of r gal/min, and leaves at same rate.

(a) Set up IVP that describes this salt solution flow process.

(b) Find amount of salt $Q(t)$ in tank at any given time t.

(c) Find limiting amount Q_L of salt $Q(t)$ in tank after a very long time.



Example 1: Salt Solution   (2 of 11)

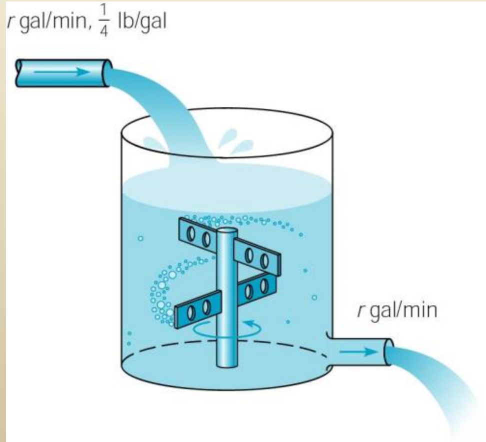
(Ex 1) At time $t = 0$, a tank contains 5 lb of salt dissolved in 100 gal of (salt) water. Assume that water containing ¼ lb of salt/gal is entering tank at rate of r gal/min, and leaves at same rate.

$Q(t)$ = the amount (lb) of the salt in the tank at time t.

dQ/dt = the rate of change of $Q(t)$

(a) Set up IVP that describes this salt solution flow process.

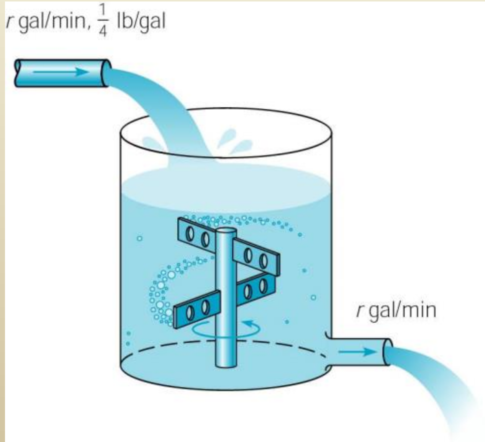
(b) Find amount of salt $Q(t)$ in tank at any given time t.



Example 1: Salt Solution

(Ex 1) At time $t = 0$, a tank contains 5 lb of salt dissolved in 100 gal of (salt) water. Assume that water containing ¼ lb of salt/gal is entering tank at rate of r gal/min, and leaves at same rate.

(c) Find limiting amount Q_L of salt $Q(t)$ in tank after a very long time.



Example 2: Salt Solution (4 of 11)

(Ex 2) At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of (salt) water. Assume that water containing ¼ lb of salt/gal is entering tank at rate of r gal/min, and leaves at same rate.

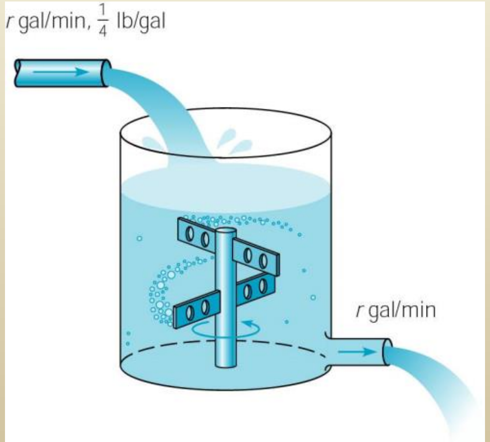
(a) Set up IVP that describes this salt solution flow process.

(b) Find amount of salt $Q(t)$ in tank at any given time t.

(c) Find limiting amount Q_L of salt $Q(t)$ in tank after a very long time.

(d) If $r = 3$ & $Q_0 = 2Q_L$, find time T after which salt is within 2% of Q_L.

(e) In (d) find flow rate r required if T is not to exceed 45 min.



Example 2: (a) Initial Value Problem

- At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume water containing ¼ lb of salt/gal enters tank at rate of r gal/min, and leaves at same rate.

- Assume salt is neither created or destroyed in tank, and distribution of salt in tank is uniform (stirred). Then $dQ/dt = \text{rate in} - \text{rate out}$

- Rate in: (1/4 lb salt/gal)(r gal/min) = (r/4) lb/min

- Rate out: If there is $Q(t)$ lbs salt in tank at time t, then concentration of salt is $Q(t)$ lb/100 gal, and it flows out at rate of $[Q(t)r/100]$ lb/min.

- Thus our IVP is $$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \quad Q(0) = Q_0$$

Example 2: (b) Find Solution $Q(t)$ (6 of 11)

- To find amount of salt $Q(t)$ in tank at any given time t, we need to solve the initial value problem

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}, \quad Q(0) = Q_0$$

- To solve, we use the method of integrating factors:

$$\mu(t) = e^{at} = e^{rt/100}$$

$$Q(t) = e^{-rt/100}\left[\int \frac{re^{rt/100}}{4}\,dt\right] = e^{-rt/100}\left[25e^{rt/100} + C\right] = 25 + Ce^{-rt/100}$$

$$Q(t) = 25 + \left[Q_0 - 25\right] e^{-rt/100}$$

or

$$Q(t) = 25\left(1 - e^{-rt/100}\right) + Q_0\, e^{-rt/100}$$

Example 2: Salt Solution (7 of 11)

(Ex 2) At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of (salt) water. Assume that water containing ¼ lb of salt/gal is entering tank at rate of r gal/min, and leaves at same rate.

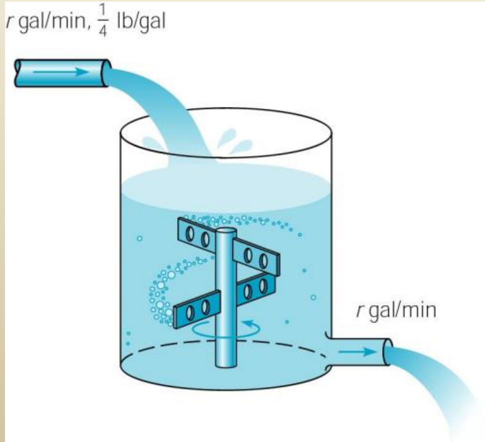
(a) Set up IVP that describes this salt solution flow process.

(b) Find amount of salt $Q(t)$ in tank at any given time t.

(c) Find limiting amount Q_L of salt $Q(t)$ in tank after a very long time.

(d) If $r = 3$ & $Q_0 = 2Q_L$, find time T after which salt is within 2% of Q_L.

(e) In (d) find flow rate r required if T is not to exceed 45 min $(T \leq 45)$.



Example 2:
(c) Find Limiting Amount Q_L (8 of 11)

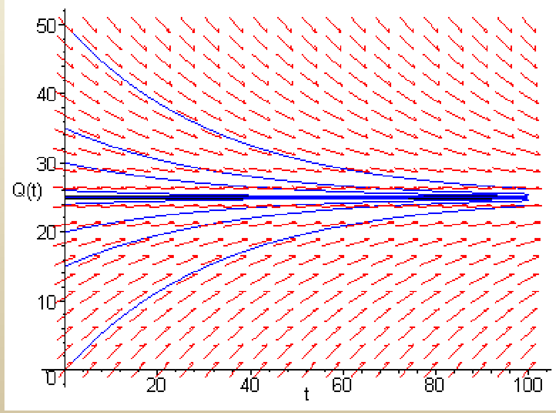
- Next, we find the limiting amount Q_L of salt $Q(t)$ in tank after a very long time:

$$Q_L = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(25 + [Q_0 - 25]e^{-rt/100}\right) = 25 \text{ lb}$$

- This result makes sense, since over time the incoming salt solution will replace original salt solution in tank. Since incoming solution contains 0.25 lb salt / gal, and tank is 100 gal, eventually tank will contain 25 lb salt.

- The graph shows integral curves for $r = 3$ and different values of Q_0.

$$Q(t) = 25\left(1 - e^{-rt/100}\right) + Q_0\, e^{-rt/100}$$



Example 2: (d) Find Time T

- Suppose $r = 3$ and $Q_0 = 2Q_L$. To find time T after which $Q(t)$ is within 2 % of Q_L, first note $Q_0 = 2Q_L = 50$ lb, hence
- Next, 2 % of 25 lb is 0.5 lb, and thus we solve

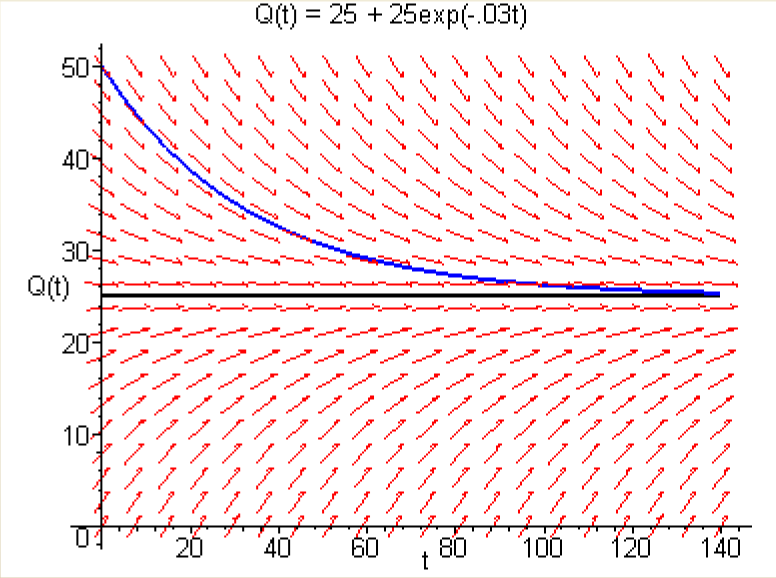
$$Q(t) = 25 + \left[Q_0 - 25\right]e^{-rt/100} = 25 + 25e^{-.03t}$$

$$25.5 = 25 + 25e^{-0.03T}$$

$$0.02 = e^{-0.03T}$$

$$\ln(0.02) = -0.03T$$

$$T = \frac{\ln(0.02)}{-0.03} \approx 130.4 \text{ min}$$



Example 2: (e) Find Flow Rate (10 of 11)

- To find flow rate r required if T is not to exceed 45 minutes, recall from part (d) that $Q_0 = 2Q_L = 50$ lb, with

$$Q(t) = 25 + 25e^{-rt/100}$$

and solution curves decrease from 50 to 25.5.

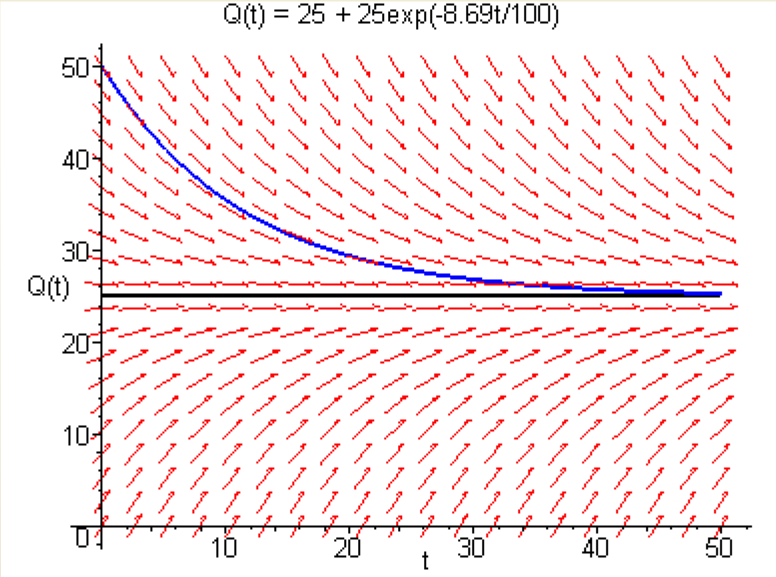
- Thus we solve

$$25.5 = 25 + 25e^{-\frac{45}{100}r}$$

$$0.02 = e^{-0.45r}$$

$$\ln(0.02) = -.45r$$

$$r = \frac{\ln(0.02)}{-0.45} \approx 8.69 \text{ gal/min}$$



Example 2: Discussion (11 of 11)

- As long as flow rates are accurate, and concentration of salt in tank is uniform, then differential equation is accurate description of flow process.

- Models of this kind are often used for pollution in lake, drug concentration in organ, etc. Flow rates may be harder to determine, or may be variable, and concentration may not be uniform. Also, rates of inflow and outflow may not be same, so variation in amount of liquid must be taken into account.

- Since situation is hypothetical, the model is valid.

Example 2: Compound Interest

- In general, if interest in an account is to be compounded *m* times a year, rather than continuously, the equation describing the amount in the account for any time *t*, measured in years, becomes:

$$S(t) = S_0(1 + r/m)^{mt}$$

- The relationship between these two results is clarified if we recall from calculus that

$$\lim_{m\to\infty} S_0(1 + r/m)^{mt} = S_0 e^{rt}$$

Growth of Capital at a Return Rate of *r* = 8% For Several Modes of Compounding: *S(t)/S(0)*			
t	*m = 4*	*m = 365*	*exp(rt)*
Years	Compounded Quarterly	Compounded Daily	Compounded Continuously
1	1.082432	1.083278	1.083287
2	1.171659	1.17349	1.173511
5	1.485947	1.491759	1.491825
10	2.20804	2.225346	2.225541
20	4.875439	4.952164	4.953032
30	10.76516	11.02028	11.02318
40	23.76991	24.52393	24.53253

A comparison of the accumulation of funds for quarterly, daily, and continuous compounding is shown for short-term and long-term periods.

- (Ex 2) If a sum of money is deposited in a bank that pays interest at an annual rate, r, compounded **continuously**,

(1) Find the ODE that the amount of money $S = S(t)$ at any time in the fund will satisfy:

(2) Solve the ODE with the initial amount $S(0) = S_0$

Example 2: Compound Interest (2 of 3)

- If a sum of money is deposited in a bank that pays interest at an annual rate, r, compounded **continuously**, the amount of money $S = S(t)$ at any time in the fund will satisfy the differential equation:

$$\frac{dS}{dt} = rS, \quad S(0) = S_0 \text{ where } S_0 \text{ represents the initial investment.}$$

- The solution to this differential equation, found by separating the variables and solving for S, becomes:

$$S(t) = S_0 e^{rt}, \text{ where } t \text{ is measured in years}$$

- Thus, with continuous compounding, the amount in the account grows exponentially over time.

Example 2: Deposits and Withdrawals (3 of 3)

- Returning now to the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to the accrual of interest, dividends, or capital gains. If we assume that the deposits or withdrawals take place at a constant rate *k*, this is described by the differential equation:

$$\frac{dS}{dt} = rS + k \text{ or in standard form } \frac{dS}{dt} - rS = k \text{ and } \mathrm{S}(0) = \mathrm{S}_0$$

  where k is positive for deposits and negative for withdrawals.

- We can solve this as a general linear equation to arrive at the solution:

$$S(t) = S_0 e^{rt} + (k/r)(e^{rt} - 1)$$

- To apply this equation, suppose that one opens an IRA at age 25 and makes annual investments of $2000 thereafter with r = 8 %.

- At age 65, $$S(40) = 0 * e^{0.08*40} + (2000/0.08)(e^{0.08*40} - 1) \approx \$588{,}313$$

Example 3: Pond Pollution (1 of 7)

- Consider a pond that initially contains 10 million gallons of fresh water. Water containing toxic waste flows into the pond at the rate of 5 million gal/year, and exits at same rate. The concentration $c(t)$ of toxic waste in the incoming water varies periodically with time: $c(t) = 2 + \sin 2t$ (g/gal)

  (a) Construct a mathematical model of this flow process and determine the amount $Q(t)$ of toxic waste in pond at time t.

  (b) Plot solution and describe in words the effect of the variation in the incoming concentration.

Example 3: (a) Initial Value Problem (2 of 7)

- Pond initially contains 10 million gallons of fresh water. Water containing toxic waste flows into pond at rate of 5 million gal/year, and exits pond at same rate. Concentration is $c(t) = 2 + \sin 2t$ g/gal of toxic waste in incoming water.
- Assume toxic waste is neither created or destroyed in pond, and distribution of toxic waste in pond is uniform (stirred).
- Then

$$dQ/dt = \text{rate in - rate out}$$

- Rate in: $(2 + \sin 2t \text{ g/gal})(5 \times 10^6 \text{ gal/year})$
- Rate out: If there is $Q(t)$ g of toxic waste in pond at time t, then concentration of salt is $Q(t)$ lb/10^7 gal, and it flows out at rate of $[Q(t) \text{ g}/10^7 \text{ gal}][5 \times 10^6 \text{ gal/year}]$

Example 3:
(a) Initial Value Problem, Scaling (3 of 7)

- Recall from previous slide that
  - Rate in: (2 + sin 2*t* g/gal)(5 x 10^6 gal/year)
  - Rate out: [$Q(t)$ g/10^7 gal][5 x 10^6 gal/year] = $Q(t)/2$ g/yr.
- Then initial value problem is

$$\frac{dQ}{dt} = (2 + \sin 2t)(5 \text{ x } 10^6) - \frac{Q(t)}{2}, \quad Q(0) = 0$$

- Change of variable (scaling): Let $q(t) = Q(t)/10^6$. Then

$$\frac{dq}{dt} + \frac{q(t)}{2} = 10 + 5\sin 2t, \quad q(0) = 0$$

Example 3:
(a) Solve Initial Value Problem (4 of 7)

- To solve the initial value problem

$$q' + q/2 = 10 + 5\sin 2t, \quad q(0) = 0$$

we use the method of integrating factors:

$$\mu(t) = e^{at} = e^{t/2}$$

$$q(t) = e^{-t/2}\int e^{t/2}\left(10 + 5\sin 2t\right)dt$$

- Using integration by parts (see next slide for details) and the initial condition, we obtain after simplifying,

$$q(t) = e^{-t/2}\left[20e^{t/2} - \frac{40}{17}e^{t/2}\cos 2t + \frac{10}{17}e^{t/2}\sin 2t + C\right]$$

$$q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t - \frac{300}{17}e^{-t/2}$$

Example 3: (a) Integration by Parts

$$\int e^{t/2}\sin 2t\,dt = \left[-\frac{1}{2}e^{t/2}\cos 2t + \frac{1}{4}\left(\int e^{t/2}\cos 2t\,dt\right)\right]$$

$$= \left[-\frac{1}{2}e^{t/2}\cos 2t + \frac{1}{4}\left(\frac{1}{2}e^{t/2}\sin 2t - \frac{1}{4}\int e^{t/2}\sin 2t\,dt\right)\right]$$

$$= \left[-\frac{1}{2}e^{t/2}\cos 2t + \frac{1}{8}e^{t/2}\sin 2t - \frac{1}{16}\int e^{t/2}\sin 2t\,dt\right]$$

$$\frac{17}{16}\int e^{t/2}\sin 2t\,dt = -\frac{1}{2}e^{t/2}\cos 2t + \frac{1}{8}e^{t/2}\sin 2t + C$$

$$\int e^{t/2}\sin 2t\,dt = -\frac{8}{17}e^{t/2}\cos 2t + \frac{2}{17}e^{t/2}\sin 2t + C$$

$$5\int e^{t/2}\sin 2t\,dt = -\frac{40}{17}e^{t/2}\cos 2t + \frac{10}{17}e^{t/2}\sin 2t + C$$

Example 3: (b) Analysis of solution (6 of 7)

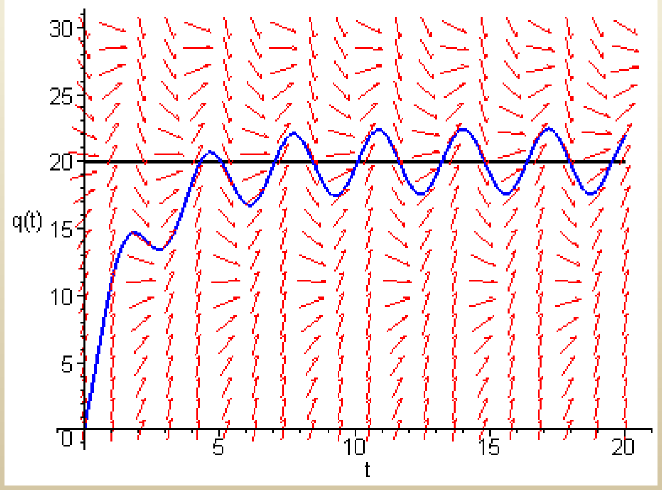
- Thus our initial value problem and solution is

$$\frac{dq}{dt} + \frac{1}{2} q = 10 + 5 \sin 2t, \quad q(0) = 0$$

$$q(t) = 20 - \frac{40}{17} \cos 2t + \frac{10}{17} \sin 2t - \frac{300}{17} e^{-t/2}$$

- A graph of solution along with direction field for differential equation is given below.

- Note that exponential term is important for small t, but decays away for large t. Also, $y = 20$ would be equilibrium solution if not for $\sin(2t)$ term.



Example 3:
(b) Analysis of Assumptions (7 of 7)

- Amount of water in pond controlled entirely by rates of flow, and none is lost by evaporation or seepage into ground, or gained by rainfall, etc.

- Amount of pollution in pond controlled entirely by rates of flow, and none is lost by evaporation, seepage into ground, diluted by rainfall, absorbed by fish, plants or other organisms, etc.

- Distribution of pollution throughout pond is uniform.

- (Ex 4) A body of mass *m* is projected away from the earth in a direction perpendicular to the earth's surface with initial velocity v_0 and no air resistance. Taking into account the variation of the earth's gravitational field with distance, the gravitational force acting on the mass is
- $$W(x) = -\frac{mgR^2}{(R+x)^2}$$: a function of x

where x is the distance above the earth's surface.

R is the radius of the earth and g is the acceleration due to gravity at the earth's surface.

(1) Set up an ODE for the velocity, and solve it.

(2) Find the maximum height (h) of the body.

(3) Find the limit of the initial velocity as $h \to \infty$.

Example 4: Escape Velocity (1 of 2)

- A body of mass *m* is projected away from the earth in a direction perpendicular to the earth's surface with initial velocity *v0* and no air resistance. Taking into account the variation of the earth's gravitational field with distance, the gravitational force acting on the mass is

  $$w(x) = -\frac{mgR^2}{(R+x)^2} \quad \text{where } x \text{ is the distance above the earth's surface}$$

  R is the radius of the earth and g is the acceleration due to gravity at the earth's surface. Using Newton's law F = ma,

  $$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \quad v(0) = v_0$$

- Since $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$ and cancelling the m's, the differential equation

becomes $$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}, \quad \text{since } x = 0 \text{ when } t = 0,\ v(0) = v_0$$

Example 4: Escape Velocity (2 of 2)

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2},\ v(0) = v_0$$

- We can solver the differential equation by separating the variables and integrating to arrive at:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + C = \frac{gR^2}{R+x} + \frac{\mathrm{v}_0{}^2}{2} - gR$$

- The maximum height (altitude) will be reached when the velocity is zero. Calling that maximum height ξ, we have

$$\xi = \frac{v_0{}^2 R}{2gR - v_0{}^2}$$

- We can now find the initial velocity required to lift a body to a height ξ :

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}}$$

and, taking the limit as $\xi \to \infty$, we get

the escape velocity, representing the initial velocity required to escape earth's gravitational force:

$$v_0 = \sqrt{2gR}$$

- Notice that this does not depend on the mass of the body.