Ch 2.4: Differences Between Linear and Nonlinear Equations

• Recall that a first order ODE has the form \( y' = f(t, y) \), and is linear if \( f \) is linear in \( y \), and nonlinear if \( f \) is nonlinear in \( y \).

• Examples: \( y' = t y - e^t \), \( y' = ty^2 \).

• In this section, we will see that first order linear and nonlinear equations differ in a number of ways, including:
  – The theory describing existence and uniqueness of solutions, and corresponding domains, are different.
  – Well-posedness, the behavior of the solution.
  
    – Solutions to linear equations can be expressed in terms of a general solution, which is not usually the case for nonlinear equations: EASY !!
  – Linear equations have explicitly defined solutions while nonlinear equations typically do not, and nonlinear equations may or may not have implicitly defined solutions.

• For both types of equations, numerical and graphical construction of solutions are important.
(Example) Find the solution of the Initial Value Problem (IVP). How many solutions does IVP have?

\[ y' = y^{1/3}, \quad y(0) = 0 \quad (t \geq 0) \]

(Question) when does IVP have the unique (only one) solution?
Theorem 2.4.1

• Consider the linear first order initial value problem:

\[ y' + p(t)y = g(t), \quad y(t_0) = y_0 \]

• If the functions \( p \) and \( g \) are continuous on an open interval \((\alpha, \beta)\) containing the point \( t = t_0 \), then there exists a unique solution \( y = \phi(t) \) that satisfies the IVP for each \( t \) in \((\alpha, \beta)\).

• Proof outline: Use Sec 2.1 discussion and results:

\[
y = \frac{\int_{t_0}^{t} \mu(s)g(s)ds + y_0}{\mu(t)}, \quad \text{where} \quad \mu(t) = e^{\int_{t_0}^{t} p(s)ds}
\]
Theorem 2.4.2

• Consider the nonlinear first order initial value problem:

\[ y' = f(t, y), \quad y(0) = y_0 \]

• Suppose \( f \) and \( \frac{\partial f}{\partial y} \) are continuous on some open rectangle \( (t, y) \in (\alpha, \beta) \times (\gamma, \delta) \) containing the point \( (t_0, y_0) \). Then in some interval \( (t_0 - h, t_0 + h) \subseteq (\alpha, \beta) \) there exists a unique solution \( y = \phi(t) \) that satisfies the IVP.

• Proof discussion: Since there is no general formula for the solution of arbitrary nonlinear first order IVPs, this proof is difficult, and is beyond the scope of this course (: Picard Iteration)

• It turns out that conditions stated in Theorem 2.4.2 are sufficient but not necessary to guarantee existence of a solution, and continuity of \( f \) ensures existence but not uniqueness of \( \phi \).
Example 1: Linear IVP

- Recall the initial value problem from Chapter 2.1 slides:
  \[ ty' + 2y = 4t^2, \quad y(1) = 2 \quad \Rightarrow \quad y = t^2 + \frac{1}{t^2} \]

- The solution to this initial value problem is defined for \( t > 0 \), the interval on which \( p(t) = 2/t \) is continuous: \( t > 0 \) or \( t < 0 \)

- If the initial condition is \( y(-1) = 2 \), then the solution is given by same expression as above, but is defined on \( t < 0 \).

- In either case, Theorem 2.4.1 guarantees that solution is unique on corresponding interval.
Example 2: Nonlinear IVP  (1 of 2)

• Consider nonlinear initial value problem from Ch 2.2:

\[
\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1
\]

• The functions \( f \) and \( \frac{\partial f}{\partial y} \) are given by

\[
f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x, y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},
\]

and are continuous except on line \( y = 1 \).

• Thus we can draw an open rectangle about \((0, -1)\) on which \( f \) and \( \frac{\partial f}{\partial y} \) are continuous, as long as it doesn’t cover \( y = 1 \).

• How wide is rectangle? Recall solution defined for \( t > -2 \), with

\[
y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}
\]
Example 2: Change Initial Condition  (2 of 2)

• Our nonlinear initial value problem is

\[
\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1
\]

with

\[
f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x, y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},
\]

which are continuous except on line \(y = 1\).

• If we change initial condition to \(y(0) = 1\), then Theorem 2.4.2 is not satisfied. Solving this new IVP, we obtain

• Thus a solution exists but is not unique.

\[
y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}, \quad x > 0
\]
Example 3: Nonlinear IVP

- Consider nonlinear initial value problem
  \[ y' = y^{1/3}, \quad y(0) = 0 \quad (t \geq 0) \]

- The functions \( f \) and \( \frac{\partial f}{\partial y} \) are given by
  \[ f(t, y) = y^{1/3}, \quad \frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-2/3} \]

- Thus \( f \) continuous everywhere, but \( \frac{\partial f}{\partial y} \) doesn’t exist at \( y = 0 \), and hence Theorem 2.4.2 is not satisfied. Solutions exist but are not unique. Separating variables and solving, we obtain
  \[ y^{-1/3} \, dy = dt \quad \Rightarrow \quad \frac{3}{2} y^{2/3} = t + c \quad \Rightarrow \quad y = \pm \left( \frac{2}{3} t \right)^{3/2}, \quad t \geq 0 \]

- If initial condition is not on \( t \)-axis, then Theorem 2.4.2 does guarantee existence and uniqueness.
Example 4: Nonlinear IVP

- Consider nonlinear initial value problem
  \[ y' = y^2, \quad y(0) = 1 \]

- The functions \( f \) and \( \frac{\partial f}{\partial y} \) are given by
  \[ f(t, y) = y^2, \quad \frac{\partial f}{\partial y}(t, y) = 2y \]

- Thus \( f \) and \( \frac{\partial f}{\partial y} \) are continuous at \( t = 0 \), so Theorem 2.4.2 guarantees that solutions exist and are unique.

- Separating variables and solving, we obtain
  \[ y^{-2} \, dy = dt \quad \Rightarrow \quad -y^{-1} = t + c \quad \Rightarrow \quad y = \frac{-1}{t + c} \quad \Rightarrow \quad y = \frac{1}{1 - t} \]

(Question) What is the interval of definition of the solution? \((1, \infty)\)?

- The solution \( y(t) \) is defined on \((-\infty, 1)\). Note that the singularity at \( t = 1 \) is not obvious from original IVP statement.
Interval of Definition: Linear Equations

• By Theorem 2.4.1, the solution of a linear initial value problem

\[ y' + p(t)y = g(t), \quad y(0) = y_0 \]

exists throughout any interval about \( t = t_0 \) on which \( p \) and \( g \) are continuous.

• Vertical asymptotes or other discontinuities of solution can only occur at points of discontinuity of \( p \) or \( g \).

*** From the coefficients we can predict the solution’s shape and behavior.

• However, solution may be differentiable at points of discontinuity of \( p \) or \( g \).

See Chapter 2.1: Example 3 of text.

• Compare these comments with Example 1 and with previous linear equations in Chapter 1 and Chapter 2.
Interval of Definition: Nonlinear Equations

• In the nonlinear case, the interval (domain of solution) on which a solution exists may be difficult to determine.

• The solution $y = \phi(t)$ exists as long as $(t, \phi(t))$ remains within rectangular region indicated in Theorem 2.4.2. This is what determines the value of $h$ in that theorem. Since $\phi(t)$ is usually not known, it may be impossible to determine this region.

• In any case, the interval on which a solution exists may have no simple relationship to the function $f$ in the differential equation $y' = f(t, y)$, in contrast with linear equations.

• Furthermore, any singularities in the solution may depend on the initial condition as well as the equation.

• Compare these comments to the preceding examples.
General Solutions

- For a first order linear equation, it is possible to obtain a solution (general solution) containing one arbitrary constant, from which all solutions follow by specifying values for this constant.

- For nonlinear equations, such general solutions may not exist. That is, even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by specifying values for this constant.

- Consider Example 4: The function $y = 0$ is a solution of the differential equation, but it cannot be obtained by specifying a value for $c$ in solution found using separation of variables:

$$\frac{dy}{dt} = y^2 \quad \Rightarrow \quad y = \frac{-1}{t + c}$$
Explicit Solutions: Linear Equations

• By Theorem 2.4.1, a solution of a linear initial value problem

\[ y' + p(t)y = g(t), \quad y(0) = y_0 \]

exists throughout any interval about \( t = t_0 \) on which \( p \) and \( g \) are continuous, and this solution is unique.

• The solution has an explicit representation,

\[
y(t) = \frac{\int_{t_0}^{t} \mu(t)g(t)\,dt + y_0}{\mu(t)}, \quad \text{where} \quad \mu(t) = e^{\int_{t_0}^{t} p(s)\,ds},
\]

and can be evaluated at any appropriate value of \( t \), as long as the necessary integrals can be computed.
Explicit Solution Approximation

- For linear first order equations, an explicit representation for the solution can be found, as long as necessary integrals can be solved.

- If integrals can’t be solved, then numerical methods are often used to approximate the integrals.

\[
y = \frac{\int_{t_0}^{t} \mu(t) g(t) dt + C}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s) ds}
\]

\[
\int_{t_0}^{t} \mu(t) g(t) dt \approx \sum_{k=1}^{n} \mu(t_k) g(t_k) \Delta t_k
\]
Implicit Solutions: Nonlinear Equations

• For nonlinear equations, explicit representations of solutions may not exist.

• As we have seen, it may be possible to obtain an equation which implicitly defines the solution. If equation is simple enough, an explicit representation can sometimes be found.

• Otherwise, numerical calculations are necessary in order to determine values of $y$ for given values of $t$. These values can then be plotted in a sketch of the integral curve.

• Recall the examples from earlier in the chapter and consider the following example

$$y' = \frac{y \cos x}{1 + 3y^3}, \quad y(0) = 1 \quad \Rightarrow \quad \ln y + y^3 = \sin x + 1$$
Direction Fields

• In addition to using numerical methods to sketch the integral curve, the nonlinear equation itself can provide enough information to sketch a direction field.

• The direction field can often show the qualitative form of solutions, and can help identify regions in the $ty$-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigations.

• Chapter 2.7 and Chapter 8 focus on numerical methods.