Ch 2.4: Differences Between Linear and Nonlinear Equations

- Recall that a first order ODE has the form $y' = f(t, y)$, and is linear if f is linear in y, and nonlinear if f is nonlinear in y.
- Examples: $y' = t\,y - e^{t}$, $y' = t\,y^2$.
- In this section, we will see that first order linear and nonlinear equations differ in a number of ways, including:
  - **The theory describing existence and uniqueness of solutions**, and corresponding domains, are different.
  - Well-posedness, the behavior of the solution.
  - Solutions to linear equations can be expressed in terms of a general solution, which is not usually the case for nonlinear equations: EASY !!
  - Linear equations have explicitly defined solutions while nonlinear equations typically do not, and nonlinear equations may or may not have implicitly defined solutions.
- For both types of equations, numerical and graphical construction of solutions are important.

(Example) Find the solution of the Initial Value Problem (IVP). How many solutions does IVP have?

$$y' = y^{1/3}, \;\; y(0) = 0 \qquad \left(t \geq 0\right)$$

(Question)

when does IVP have the unique (only one) solution?

Theorem 2.4.1

- Consider the linear first order initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

- If the functions p and g are continuous on an open interval (α, β) containing the point $t = t_0$, then there exists a unique solution $y = \phi(t)$ that satisfies the IVP for each t in (α, β).

- **Proof outline:** Use Sec 2.1 discussion and results:

$$y = \frac{\int_{t_0}^{t} \mu(s)g(s)ds + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

Theorem 2.4.2

- Consider the nonlinear first order initial value problem:

$$y' = f(t, y), \;\; y(0) = y_0$$

- Suppose f and $\partial f/\partial y$ are continuous on some open rectangle $(t, y) \in (\alpha, \beta) \text{ x } (\gamma, \delta)$ containing the point (t_0, y_0).
  Then in some interval $(t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$ there exists a unique solution $y = \phi(t)$ that satisfies the IVP.

- **Proof discussion:** Since there is no general formula for the solution of arbitrary nonlinear first order IVPs, this proof is difficult, and is beyond the scope of this course (: Picard Iteration)

- It turns out that conditions stated in Theorem 2.4.2 are sufficient but not necessary to guarantee existence of a solution, and continuity of f ensures existence but not uniqueness of ϕ.

Example 1: Linear IVP

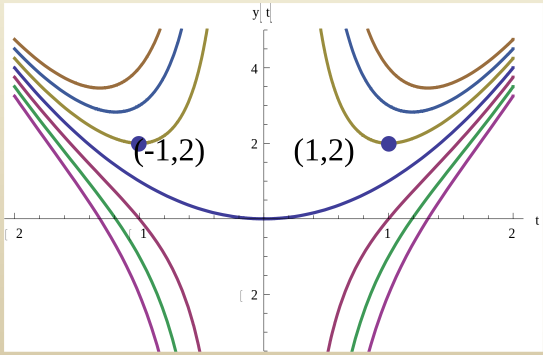
- Recall the initial value problem from Chapter 2.1 slides:

$$ty' + 2y = 4t^2, \quad y(1) = 2 \quad \Rightarrow \quad y = t^2 + \frac{1}{t^2}$$

- The solution to this initial value problem is defined for
  $t > 0$, the interval on which $p(t) = 2/t$ is continuous: t > 0   or   t < 0

- If the initial condition is $y(-1) = 2$, then the solution is given by same expression as above, but is defined on $t < 0$.

- In either case, Theorem 2.4.1 guarantees that solution is unique on corresponding interval.



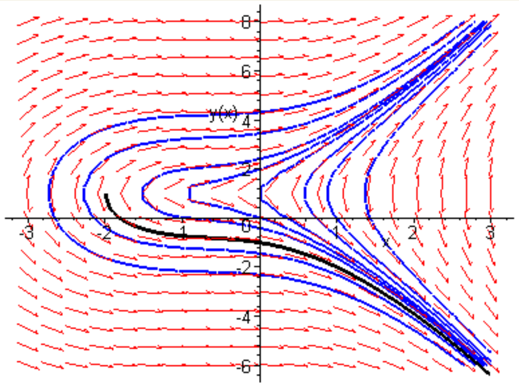
Example 2: Nonlinear IVP (1 of 2)

- Consider nonlinear initial value problem from Ch 2.2:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

- The functions f and $\partial f/\partial y$ are given by

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$



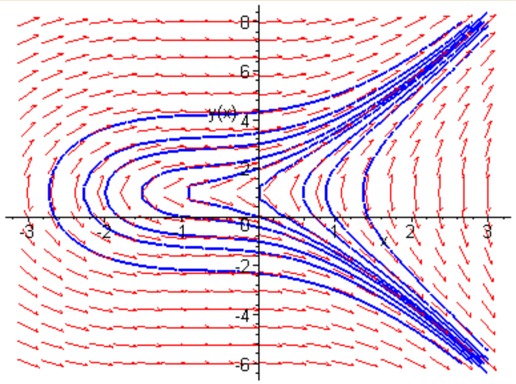
and are continuous except on line $y = 1$.

- Thus we can draw an open rectangle about (0, -1) on which f and $\partial f/\partial y$ are continuous, as long as it doesn't cover $y = 1$.

- How wide is rectangle? Recall solution defined for $t > -2$, with

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Example 2: Change Initial Condition (2 of 2)



- Our nonlinear initial value problem is

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

with

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x, y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

which are continuous except on line $y = 1$.

- If we change initial condition to $y(0) = 1$, then Theorem 2.4.2 is not satisfied. Solving this new IVP, we obtain

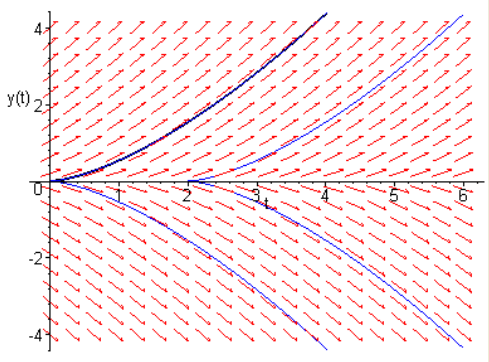
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}, \quad x > 0$$

- Thus a solution exists but is not unique.

Example 3: Nonlinear IVP

- Consider nonlinear initial value problem

$$y' = y^{1/3}, \;\; y(0) = 0 \qquad (t \geq 0)$$



- The functions f and $\partial f/\partial y$ are given by

$$f(t, y) = y^{1/3}, \;\; \frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-2/3}$$

- Thus f continuous everywhere, but $\partial f/\partial y$ doesn't exist at $y = 0$, and hence Theorem 2.4.2 is not satisfied. Solutions exist but are not unique. Separating variables and solving, we obtain

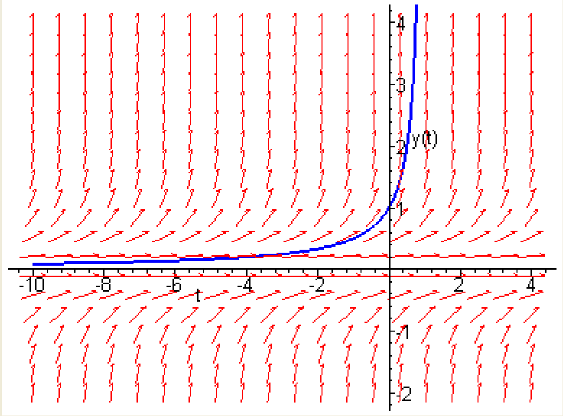
$$y^{-1/3} dy = dt \;\; \Rightarrow \;\; \frac{3}{2} y^{2/3} = t + c \;\Rightarrow\; y = \pm\left(\frac{2}{3} t\right)^{3/2}, \;\; t \geq 0$$

- If initial condition is not on t-axis, then Theorem 2.4.2 does guarantee existence and uniqueness.

Example 4: Nonlinear IVP

- Consider nonlinear initial value problem

$$y' = y^2, \;\; y(0) = 1$$



- The functions *f* and ∂*f*/∂*y* are given by

$$f(t,y) = y^2, \qquad \frac{\partial f}{\partial y}(t,y) = 2y$$

- Thus *f* and ∂*f*/∂*y* are continuous at $t = 0$, so Theorem 2.4.2 guarantees that solutions exist and are unique.

- Separating variables and solving, we obtain

$$y^{-2}dy = dt \;\Rightarrow\; -y^{-1} = t + c \;\Rightarrow\; y = \frac{-1}{t+c} \;\Rightarrow\; y = \frac{1}{1-t}$$

(Question) What is the interval of definition of the solution? $(1, \infty)$?

- The solution $y(t)$ is defined on $(-\infty, 1)$. Note that the singularity at $t = 1$ is not obvious from original IVP statement.

Interval of Definition: Linear Equations

- By Theorem 2.4.1, the solution of a linear initial value problem

$$y' + p(t)y = g(t), \;\; y(0) = y_0$$

exists throughout any interval about $t = t_0$ on which p and g are continuous.

- Vertical asymptotes or other discontinuities of solution can only occur at points of discontinuity of p or g.

*** From the coefficients we can predict the solution’s shape and behavior.

- However, solution may be differentiable at points of discontinuity of p or g. See Chapter 2.1: Example 3 of text.

- Compare these comments with Example 1 and with previous linear equations in Chapter 1 and Chapter 2.

Interval of Definition: Nonlinear Equations

- In the nonlinear case, the interval (: domain of solution) on which a solution exists may be difficult to determine.

- The solution $y = \phi(t)$ exists as long as $(t, \phi(t))$ remains within rectangular region indicated in Theorem 2.4.2. This is what determines the value of h in that theorem. Since $\phi(t)$ is usually not known, it may be impossible to determine this region.

- In any case, the interval on which a solution exists may have no simple relationship to the function f in the differential equation $y' = f(t, y)$, in contrast with linear equations.

- Furthermore, any singularities in the solution may depend on the initial condition as well as the equation.

- Compare these comments to the preceding examples.

General Solutions

- For a first order linear equation, it is possible to obtain a solution (: general solution) containing one arbitrary constant, from which all solutions follow by specifying values for this constant.

- For nonlinear equations, such general solutions may not exist. That is, even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by specifying values for this constant.

- Consider Example 4: The function $y = 0$ is a solution of the differential equation, but it cannot be obtained by specifying a value for c in solution found using separation of variables:

$$\frac{dy}{dt} = y^2 \quad \Rightarrow \quad y = \frac{-1}{t + c}$$

Explicit Solutions: Linear Equations

- By Theorem 2.4.1, a solution of a linear initial value problem

$$y' + p(t)y = g(t), \quad y(0) = y_0$$

exists throughout any interval about $t = t_0$ on which p and g are **continuous**, and this solution is unique.

- The solution has an explicit representation,

$$y = \frac{\int_{t_0}^{t} \mu(t)g(t)dt + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s)ds},$$

and can be evaluated at any appropriate value of t, as long as the necessary integrals can be computed.

Explicit Solution Approximation

- For linear first order equations, an explicit representation for the solution can be found, as long as necessary integrals can be solved.

- If integrals can't be solved, then numerical methods are often used to approximate the integrals.

$$y = \frac{\int_{t_0}^{t} \mu(t) g(t) dt + C}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s) ds}$$

$$\int_{t_0}^{t} \mu(t) g(t) dt \approx \sum_{k=1}^{n} \mu(t_k) g(t_k) \Delta t_k$$

Implicit Solutions: Nonlinear Equations

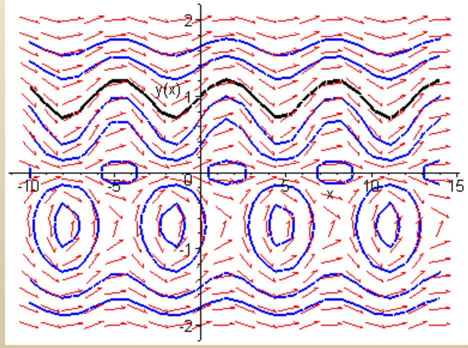
- For nonlinear equations, explicit representations of solutions may not exist.

- As we have seen, it may be possible to obtain an equation which implicitly defines the solution. If equation is simple enough, an explicit representation can sometimes be found.

- Otherwise, numerical calculations are necessary in order to determine values of *y* for given values of *t*. These values can then be plotted in a sketch of the integral curve.

- Recall the examples from earlier in the chapter and consider the following example

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1 \;\Rightarrow\; \ln y + y^3 = \sin x + 1$$



Direction Fields

- In addition to using numerical methods to sketch the integral curve, the nonlinear equation itself can provide enough information to sketch a direction field.

- The direction field can often show the qualitative form of solutions, and can help identify regions in the *ty*-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigations.

- Chapter 2.7 and Chapter 8 focus on numerical methods.