# Ch 2.6 Exact Equations and Integrating Factors

- Need Bivariate Calculus

# EXAMPLE 1

(1) Find the first order partial derivatives $\psi_x$ and $\psi_y$ of

$$\psi(x, y) = x + xy^2.$$

(2) (Chain rule) If $y = y(x)$ then it may be possible to compute

$$\frac{d}{dx}\psi(x, y(x))$$

(3) What can we say about an implicit function $\psi(x, y) = c$ and the ODE:

$$1 + y^2 + 2xy\frac{dy}{dx} = 0$$

(Question) For given ODE how do we find the implicit solution $\psi(x, y) = c$ ?

# Example 2: Exact Equation

- Consider the equation: $$2x + y^2 + 2xy\frac{dy}{dx} = 0$$
- It is neither linear nor separable, and not homogeneous.

Can you find a function ϕ such that

$$\frac{\partial \phi}{\partial x} = 2x + y^2 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 2xy\,?$$

# Example 2: Exact Equation

- Consider the equation: $$2x + y^2 + 2xy\frac{dy}{dx} = 0$$
- It is neither linear nor separable, and not homogeneous. Can we find a function $\phi$ such that $$\frac{\partial \phi}{\partial x} = 2x + y^2 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 2xy \ ?$$

(Question) $\phi(x, y) = x^2 + xy^2$ ?

- Thinking of y as a function of x (: $y = y(x)$ ) and calling upon the chain rule, the differential equation and its solution become $$\frac{d\phi}{dx} = \frac{d}{dx}(x^2 + xy(x)^2) = 0 \quad \Rightarrow \quad \phi(x, y) = x^2 + xy^2 = c$$

# Ch 2.6 Exact Equations and Integrating Factors

- Consider a first order ODE of the form $$M(x,y)+N(x,y)\frac{dy}{dx}=0$$
- Suppose there is a function $\psi$ such that $$\psi_x(x,y)=M(x,y),\ \ \psi_y(x,y)=N(x,y)$$

  and such that $\psi(x,y) = c$ defines $y = \phi(x)$ implicitly. Then

$$M(x,y)+N(x,y)y'=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=\frac{d}{dx}\psi\left[x,\phi(x)\right]$$

  and hence the original ODE becomes $$\frac{d}{dx}\psi\left(x,\varphi(x)\right)=0.$$

- Thus $\psi(x,y) = c$ defines a solution implicitly.
- In this case, the ODE is said to be **exact**.

(Question) For any given ODE, how do we determine whether or not the DE is exact?

# Theorem 2.6.1

- Suppose an ODE can be written in the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \qquad (1)$$

where the functions $M$, $N$, $M_y$ and $N_x$ are all continuous in the rectangular region $R$: $(x, y) \in (\alpha, \beta) \times (\gamma, \delta)$.

Then Eq. (1) is an **exact** differential equation iff

$$M_y(x,y)=N_x(x,y), \quad \forall (x,y)\in R \qquad (2)$$

- That is, there exists a function $\psi$ satisfying the conditions

$$\psi_x(x,y)=M(x,y), \quad \psi_y(x,y)=N(x,y) \qquad (3)$$

if $M$ and $N$ satisfy Equation (2).

(Example 3) Solve the ODE.

$$2x - y + (2y - x)\frac{dy}{dx} = 0$$

(1) Is it exact?

(2) If it is exact solve the ODE

(Example 3) Consider a general solution of the ODE

$$(y\cos x + 2xe^{y}) + (\sin x + x^{2}e^{y} - 1)\frac{dy}{dx} = 0$$

(1) Is this ODE exact ?

(2) If it is exact, then find its an implicit solution by using the fact of Theorem (2.6.1).

# Example 3: Exact Equation (1 of 3)

- Consider the following differential equation:

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)\frac{dy}{dx} = 0$$

- Then $M(x,y) = y\cos x + 2xe^y,\ N(x,y) = \sin x + x^2 e^y - 1$

  and hence $\psi_{xy} = M_y(x,y) = \cos x + 2xe^y = N_x(x,y) = \psi_{yx} \quad \Rightarrow$ ODE is exact.

- From Theorem 2.6.1,

$$\psi_x(x,y) = M = y\cos x + 2xe^y,\ \ \psi_y(x,y) = N = \sin x + x^2 e^y - 1$$

- Thus

$$\psi(x,y) = \int \mathrm{M}(x,y)dx = \int \left(y\cos x + 2xe^y\right)dx = y\sin x + x^2 e^y + C(y)$$

# Example 3: Solution (2 of 3)

- We have $\psi_x(x,y) = M = y\cos x + 2xe^y, \ \psi_y(x,y) = N = \sin x + x^2e^y - 1$

and $\psi(x,y) = \int \mathrm{M}(x,y)dx = \int \left( y\cos x + 2xe^y \right) dx = y\sin x + x^2e^y + C(y)$

*** x and y are two independent variables

- It follows that $\psi_y(x,y) = \sin x + x^2e^y - 1 = \sin x + x^2e^y + C'(y)$

$$\Rightarrow \quad C'(y) = -1 \quad \Rightarrow \quad C(y) = -y + k$$

- Thus $\psi(x,y) = y\sin x + x^2e^y - y + k$

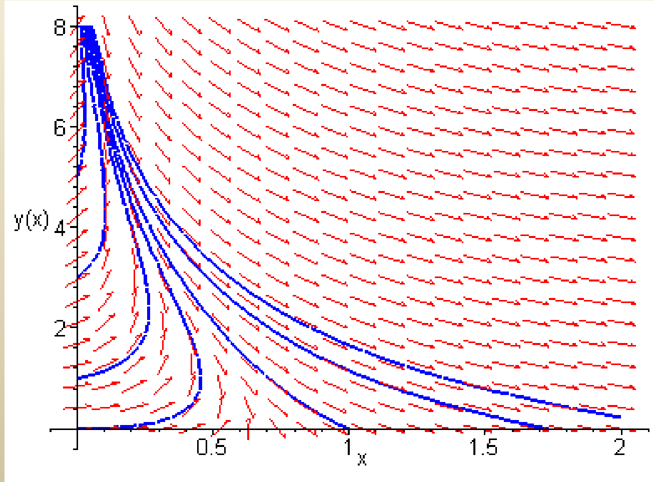
- By Theorem 2.6.1, the solution is given implicitly by $y\sin x + x^2e^y - y = c$

# Example 3:
# Direction Field and Solution Curves (3 of 3)

- Our differential equation and solutions are given by

$$(y\cos x + 2xe^{y}) + (\sin x + x^{2}e^{y} - 1)y' = 0,$$
$$y\sin x + x^{2}e^{y} - y = c$$

- A graph of the direction field for this differential equation, along with several solution curves, is given below.



(Example 4) Solve the ODE. Is it exact?

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

# Example 4: Non-Exact Equation (1 of 2)

- Consider the following differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Then $\quad M(x,y) = 3xy + y^2,\ N(x,y) = x^2 + xy$

and hence $\quad M_y(x,y) = 3x + 2y \neq 2x + y = N_x(x,y) \ \Rightarrow$ ODE is not exact

- To show that our differential equation cannot be solved by this method, let us seek a function $\psi$ such that

- Thus

$$\psi_x(x,y) = M = 3xy + y^2,\ \psi_y(x,y) = N = x^2 + xy$$

$$\psi(x,y) = \int \psi_x(x,y)dx = \int \left(3xy + y^2\right)dx = 3x^2y/2 + xy^2 + C(y)$$

# Example 4: Non-Exact Equation (2 of 2)

- We seek $\psi$ such that $\psi_x(x,y) = M = 3xy + y^2,\ \psi_y(x,y) = N = x^2 + xy$

and $\psi(x,y) = \int \psi_x(x,y)dx = \int \left(3xy + y^2\right)dx = 3x^2y/2 + xy^2 + C(y)$

- Then $\psi_y(x,y) = x^2 + xy = 3x^2/2 + 2xy + C'(y)$

$$\Rightarrow\ C'(y) \stackrel{?}{=} -xy - x^2/2$$

- Because $C'(y)$ depends on $x$ as well as $y$, there is no such function $\psi(x, y)$ such that

$$\frac{d\psi}{dx} = (3xy + y^2) + (x^2 + xy)y'$$

# Integrating Factors

- It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$:

$$M(x,y)+N(x,y)y'=0$$

$$\mu(x,y)M(x,y)+\mu(x,y)N(x,y)y'=0$$

- For this equation to be exact, we need

$$\left(\mu M\right)_y=\left(\mu N\right)_x \iff M\mu_y-N\mu_x+\left(M_y-N_x\right)\mu=0$$

- This partial differential equation may be difficult to solve. If $\mu$ is a function of $x$ alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx}=\frac{M_y-N_x}{N}\mu,$$

provided right side is a function of $x$ only. Similarly if $\mu$ is a function of $y$ alone. See text for more details.

# Example 4: Non-Exact Equation

- Consider the following non-exact differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Seeking an integrating factor, we solve the linear equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \iff \frac{d\mu}{dx} = \frac{\mu}{x} \implies \mu(x) = x$$

- Multiplying our differential equation by $\mu$, we obtain the exact equation

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0,$$

which has its solutions given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = c$$