

Ch 2.7: Numerical Approximations: Euler's Method

- Recall that a first order initial value problem has the form

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

- If f and $\partial f / \partial y$ are continuous, then this IVP has a unique solution $y = \phi(t)$ in some interval about t_0 .
- When the differential equation is linear, separable or exact, we can find the solution by symbolic manipulations.
- However, the solutions for most differential equations of this form cannot be found by analytical means.
- Therefore it is important to be able to approach the problem in other ways.

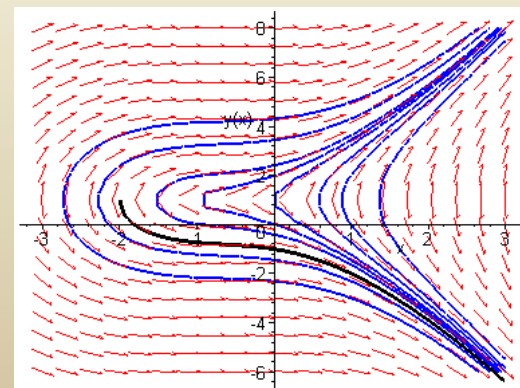
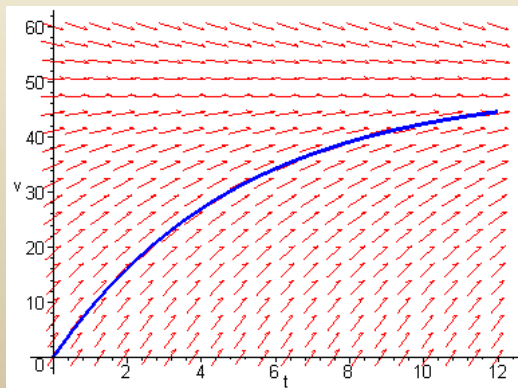
Direction Fields

- For the first order initial value problem (IVP)

$$y' = f(t, y), \quad y(t_0) = y_0,$$

we can sketch a **direction field** and visualize the behavior of solutions.

- *** This has the **advantage** of being a relatively simple process, even for complicated equations.
- *** However, **direction fields** do not lend themselves to quantitative computations or comparisons (with **experiment data**)



Numerical Methods

- For our first order initial value problem $y' = f(t, y), y(t_0) = y_0,$

an alternative is to compute approximate values of the solution $y = \phi(t)$ at a selected set of t -values.
- Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.
- There are many numerical methods that produce numerical approximations to solutions of differential equations, some of which are discussed in Chapter 8.
- In this section, we examine the tangent line method, which is also called Euler's Method.

Tangent line method: approximate the unknown solution $y(t)$ by tangent lines

Euler's Method: Tangent Line Approximation

- For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$,

we begin by approximating solution $y = \phi(t)$ at initial point t_0 .

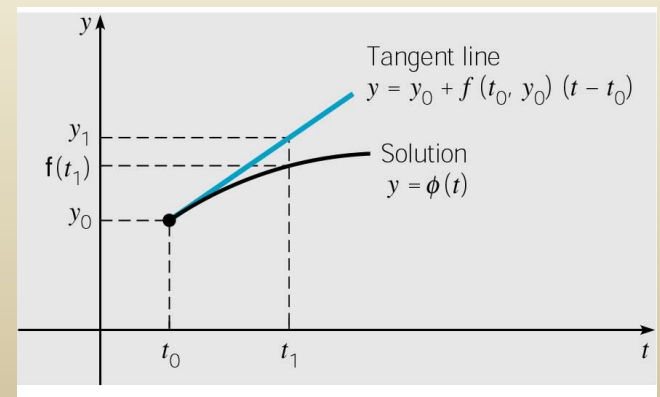
- The solution passes through initial point (t_0, y_0) with slope $f(t_0, y_0)$. The line tangent to the solution at this initial point is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

- The tangent line is a good approximation to solution curve on an interval short enough.

- Thus if t_1 is close enough to t_0 , we can approximate $\phi(t_1)$ by

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$



Euler's Formula

- For a point t_2 close to t_1 , we approximate $\phi(t_2)$ using the line passing through (t_1, y_1) with slope $f(t_1, y_1)$:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

- Thus we create a sequence y_k of approximations to $\phi(t_k)$:

$$y_1 = y_0 + f_0 \cdot (t_1 - t_0)$$

$$y_2 = y_1 + f_1 \cdot (t_2 - t_1)$$

$$\vdots$$

$$y_{k+1} = y_k + f_k \cdot (t_{k+1} - t_k)$$

where $f_k = f(t_k, y_k)$.

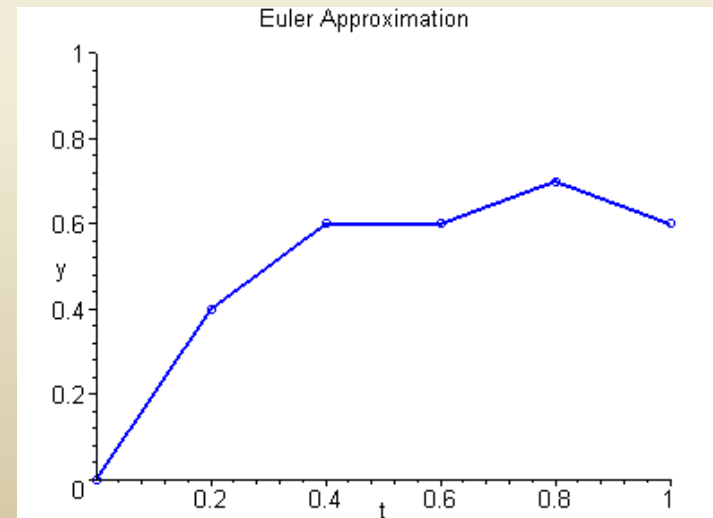
- For a uniform step size $h = t_k - t_{k-1}$, Euler's formula becomes

$$y_{k+1} = y_k + f_k h, \quad k = 0, 1, 2, \dots$$

Euler Approximation

- To graph an Euler approximation, we plot the points $(t_0, y_0), (t_1, y_1), \dots, (t_n, y_n)$, and then connect these points with line segments.

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n), \text{ where } f_n = f(t_n, y_n)$$



Example 1: Euler's Method (1 of 3)

- For the initial value problem $\frac{dy}{dt} = 3 - 2t - 0.5y$, $y(0) = 1$
- We can use Euler's method with $h = 0.2$ to approximate the solution at $t = 0.2, 0.4, 0.6, 0.8$, and 1.0 as shown below.

$$y_1 = y_0 + f_0 \cdot h = 1 + (3 - 0 - 0.5)(0.2) = 1 + 2.5(0.2) = 1.5$$

$$y_2 = y_1 + f_1 \cdot h = 1.5 + (3 - 2(0.2) - 0.5(1.5))(0.2) \approx 1.87$$

$$y_3 = y_2 + f_2 \cdot h = 1.87 + (3 - 2(0.4) - 0.5(1.87))(0.2) \approx 2.123$$

$$y_4 = y_3 + f_3 \cdot h = 2.123 + (3 - 2(0.6) - 0.5(2.123))(0.2) \approx 2.2707$$

$$y_5 = y_4 + f_4 \cdot h = 2.2707 + (3 - 2(0.8) - 0.5(2.2707))(0.2) \approx 2.32363$$

Example 1: Exact Solution (2 of 3)

- We can find the exact solution to our IVP, as in Chapter 2.1:

$$y' = 3 - 2t - 0.5y, \quad y(0) = 1$$

$$y' + 0.5y = 3 - 2t$$

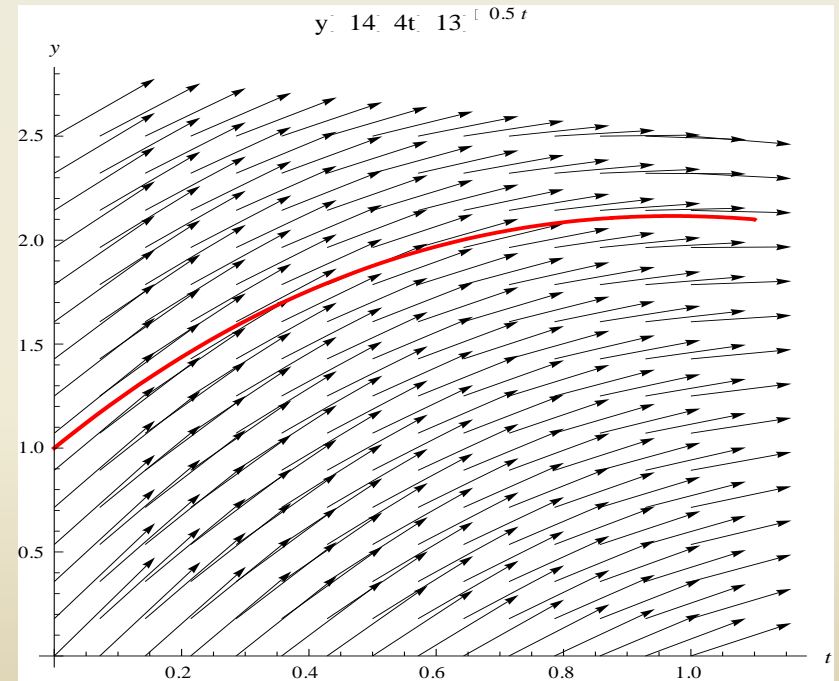
$$e^{0.5t}y' + 0.5e^{0.5t}y = 3e^{0.5t} - 2te^{0.5t}$$

$$e^{0.5t}y = 14e^{0.5t} - 4te^{0.5t} + k$$

$$y = 14 - 4t + ke^{-0.5t}$$

$$y(0) = 1 \Rightarrow k = -13$$

$$\Rightarrow y = 14 - 4t - 13e^{-0.5t}$$



Matlab files

- Files: [eul.m](#), [rk2.m](#), [rk4.m](#) (<http://math.rice.edu/~dfield>)
 - For using these files, Read the guideline in course main webpage: <http://www.math.purdue.edu/academic/courses/MA26600/>
- (Example) $y' = 2y - t + 1, \quad y(0) = 1$
(How to use `eul.m`)
$$y(t) = \frac{2t-1}{4} + \frac{5}{4}e^{2t}$$
 - (1) create M-file, `fcn1.m`
 - (2) The syntax for “`eul`”:
`> eul('fcn1',[t0, tf],y(0),h)`
 - `t0` = the initial value of `x` -- `tf` = the final value of `t`
 - `y(0)` = the initial value of `y` -- `h` = the step size.
 - (3) You can try `> eul('fcn1',[0,3],1,0.1)`
 - (4) To plot the numerical solution we can try
`> [t,y] = eul('fcn1',[0,3],1,0.1)`
`> plot(t,y)`
- You can also create your own M-files

Euler method (Tangent line method)

- (Example) By using the Euler method approximate the solution of the ODE with $h = 0.02$:

$$\frac{dy}{dt} = t^2 - y^2, \quad y(1) = 2.$$

- (Example) Estimate $\int_1^{1.8} \sin(x^2) dx$ by using “eul” with $h = 0.02$

(Hint) What if we set $y(t) = \int_1^t \sin(x^2) dx$? What is $\frac{dy}{dt} = ?$

- Remark: When we use these routines, things can go wrong !!
 - (1) Step size is too large or too small.
 - (2) Solution may not be defined at point in question

Error estimate

Example 1: Euler's Method (1 of 3)

- For the initial value problem $\frac{dy}{dt} = 3 - 2t - 0.5y, \quad y(0) = 1$

we can use Euler's method with various step sizes (h) to approximate the solution at $t = 1.0, 2.0, 3.0, 4.0$, and 5.0 and compare our results to the exact solution

$$y = 14 - 4t - 13e^{-0.5t}$$

at those values of t .

Example 1: Euler's Method (2 of 3)

- Comparison of exact solution with Euler's Method for $h = 0.1, 0.05, 0.25, 0.01$

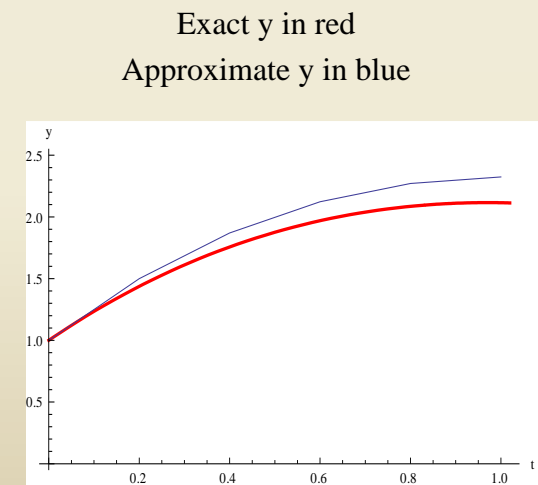
t	Exact y	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	2.1151	2.2164	2.1651	2.1399	2.1250
2.0	1.2176	1.3397	1.2780	1.2476	1.2295
3.0	-0.9007	-0.7903	-0.8459	-0.8734	-0.8898
4.0	-3.7594	-3.6707	-3.7152	-3.7373	-3.7506
5.0	-7.0671	-7.0337	-7.0337	-7.0504	-7.0604

Example 1: Error Analysis (3 of 3)

- From table below, we see that the errors start small, but get larger. This is most likely due to the fact that **the exact solution is not linear on $[0, 1]$** . Note:

$$\text{Percent Relative Error} = \frac{y_{\text{exact}} - y_{\text{approx}}}{y_{\text{exact}}} \times 100$$

t	Exact y	Approx y	Error	% Rel Error
0	1	1	0	0
0.2	1.43711	1.5	-0.06	-4.38
0.4	1.7565	1.87	-0.11	-6.46
0.6	1.96936	2.123	-0.15	-7.8
0.8	2.08584	2.2707	-0.18	-8.86
1	2.1151	2.32363	-0.2085	-9.8591083



Example 2: Euler's Method (1 of 3)

- For the initial value problem

$$\frac{dy}{dt} = 4 - t + 2y, \quad y(0) = 1$$

we can use Euler's method with $h = 0.1$ to approximate the solution at $t = 1, 2, 3$, and 4 , as shown below.

$$y_1 = y_0 + f_0 \cdot h = 1 + (4 - 0 + (2)(1))(0.1) = 1.6$$

$$y_2 = y_1 + f_1 \cdot h = 1.6 + (4 - 0.1 + (2)(1.6))(0.1) = 2.31$$

$$y_3 = y_2 + f_2 \cdot h = 2.31 + (4 - 0.2 + (2)(2.31))(0.1) \approx 3.15$$

$$y_4 = y_3 + f_3 \cdot h = 3.15 + (4 - 0.3 + (2)(3.15))(0.1) \approx 4.15$$

\vdots

- Exact solution (see Chapter 2.1): $y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$

Example 2: Error Analysis (2 of 3)

- The first ten Euler approxs are given in table below on left. A table of approximations for $t = 0, 1, 2, 3$ is given on right. See text for numerical results with $h = 0.05, 0.025, 0.01$.
- The errors are small initially, but quickly reach an unacceptable level. This suggests a nonlinear solution.

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
0.10	1.66	1.60	0.06	3.55
0.20	2.45	2.31	0.14	5.81
0.30	3.41	3.15	0.26	7.59
0.40	4.57	4.15	0.42	9.14
0.50	5.98	5.34	0.63	10.58
0.60	7.68	6.76	0.92	11.96
0.70	9.75	8.45	1.30	13.31
0.80	12.27	10.47	1.80	14.64
0.90	15.34	12.89	2.45	15.96
1.00	19.07	15.78	3.29	17.27

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution :

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

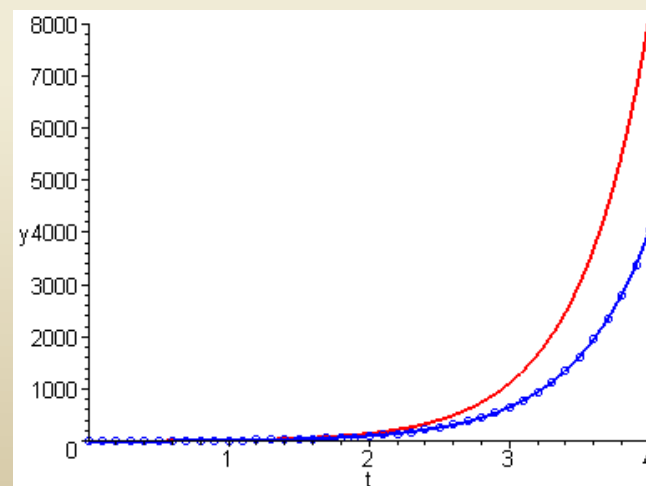
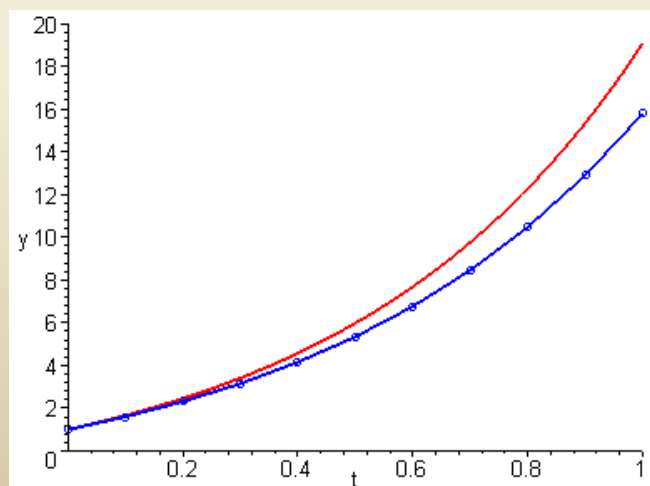
Example 2: Error Analysis & Graphs (3 of 3)

- Given below are graphs showing the exact solution (red) plotted together with the Euler approximation (blue).

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution :

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$



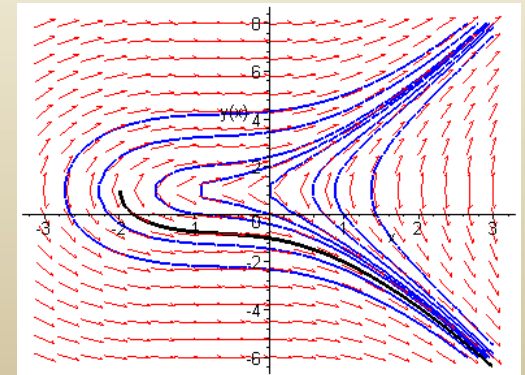
General Error Analysis Discussion (1 of 4)

- Recall that if f and $\partial f / \partial y$ are continuous, then our first order initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

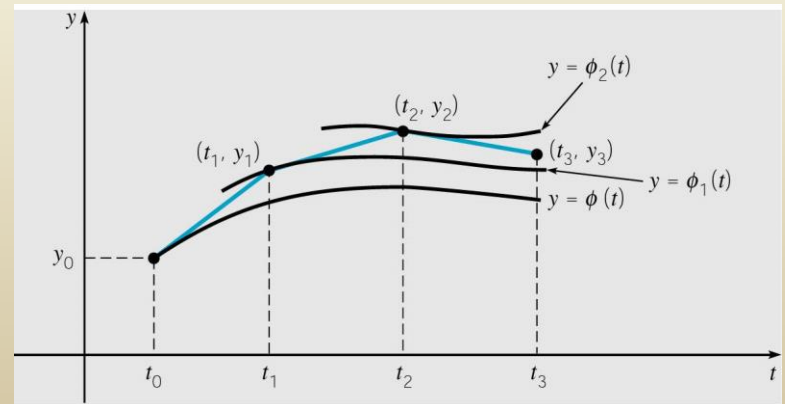
has a solution $y = \phi(t)$ in some interval about t_0 .

- In fact, the equation has infinitely many solutions, each one indexed by a constant c determined by the initial condition.
- Thus ϕ is the member of an infinite family of solutions that satisfies $\phi(t_0) = y_0$.



General Error Analysis Discussion (2 of 4)

- The first step of Euler's method uses the tangent line to ϕ at the point (t_0, y_0) in order to estimate $\phi(t_1)$ with y_1 .
- The point (t_1, y_1) is typically **not on the graph of ϕ** , because y_1 is an approximation of $\phi(t_1)$.
- Thus the next iteration of Euler's method does not use a tangent line approximation to ϕ , but rather to a nearby solution ϕ_1 that passes through the point (t_1, y_1) .
- Thus Euler's method uses a succession of tangent lines to a sequence of different solutions $\phi, \phi_1, \phi_2, \dots$ of the differential equation.



Error Analysis Example:

Converging Family of Solutions (3 of 4)

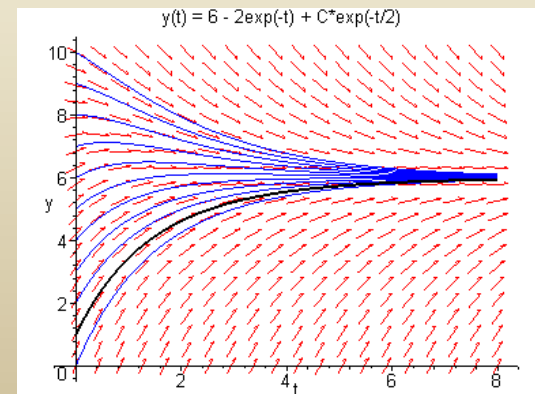
- Since Euler's method uses tangent lines to a sequence of different solutions, the accuracy after many steps depends on behavior of solutions passing through (t_n, y_n) , $n = 1, 2, 3, \dots$

- For example, consider the following initial value problem:

$$y' = 3 + e^{-t} - y/2, \quad y(0) = 1 \Rightarrow y = \phi(t) = 6 - 2e^{-t} - 3e^{-t/2}$$

- The direction field and graphs of a few solution curves are given below. Note that it doesn't matter which solutions we are approximating with tangent lines, as all solutions get closer to each other as t increases.

- Results of using Euler's method for this equation are given in text.

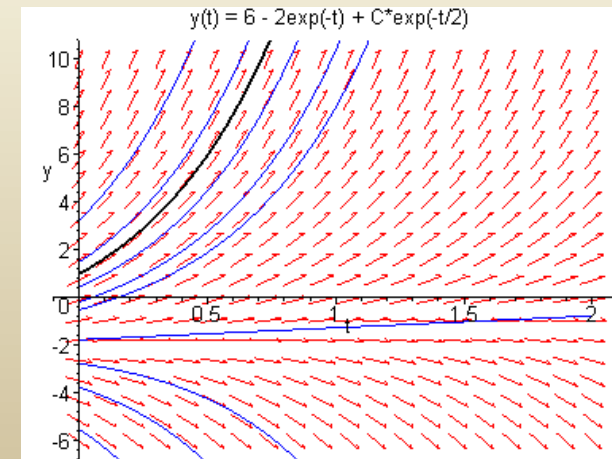


Error Analysis Example: Divergent Family of Solutions (4 of 4)

- Now consider the initial value problem for Example 2:

$$y' = 4 - t + 2y, \quad y(0) = 1 \Rightarrow y = -7/4 + t/2 + 11e^{2t}/4$$

- The direction field and graphs of solution curves are given below. Since the family of solutions is divergent, at each step of Euler's method we are following a different solution than the previous step, with each solution separating from the desired one more and more as t increases.



Error Bounds and Numerical Methods

- In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- In our examples, we compared approximations with exact solutions. However, numerical procedures are usually used when an exact solution is not available. What is needed are bounds for (or estimates of) errors, which do not require knowledge of exact solution. More discussion on these issues and other numerical methods is given in Chapter 8.
- Since numerical approximations ideally reflect behavior of solution, a member of a diverging family of solutions is harder to approximate than a member of a converging family.
- Also, direction fields are often a relatively easy first step in understanding behavior of solutions.