Ch 3.1: 2nd Order Linear Homogeneous Equations-Constant Coefficients

- A **second order ordinary differential equation** has the general form
$$y'' = f(t, y, y')$$
where f is some given function.

- This equation is said to be **linear** if f is linear in y and y':
$$y'' = g(t) - p(t)y' - q(t)y$$
Otherwise the equation is said to be **nonlinear**.

- In general, a second order linear equation often appears as
$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

- If $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Homogeneous Equations, Initial Values

- In Sections 3.5 and 3.6, we will see that once a solution to a homogeneous equation is found, then it is possible to solve the corresponding nonhomogeneous equation, or at least express the solution in terms of an integral.

- The focus of this chapter is thus on homogeneous equations; and in particular, those with constant coefficients:

$$ay'' + by' + cy = 0$$

  We will examine the variable coefficient case in Chapter 5.

- Initial conditions typically take the form: $y(t_0) = y_0, \; y'(t_0) = y_0'$

- Thus, solution passes through (t_0, y_0), and slope of solution at (t_0, y_0) is equal to y_0'.

How do we find a general solution?

(Example) Find a solution of the ODE: $$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0$$

- (Hint) Consider the first order DE: $$\frac{dy}{dt} - 2y = 0$$

- Since the two ODEs have constant coefficients, the same kind of functions may be solutions of the ODEs (?)

- We will develop rules and principles to get a general solution of a given ODE in Sec 3.2.

Example 1: Infinitely Many Solutions (1 of 3)

- Consider the second order linear differential equation $y'' - y = 0$
- Two solutions of this equation are $y_1(t) = e^t, \quad y_2(t) = e^{-t}$
- Other solutions include $y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$

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- Other solutions include $y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$
- Based on these observations, we see that there are infinitely many solutions of the form $$y(t) = c_1 e^t + c_2 e^{-t}$$
- (Question) How do we find all the solutions of the DE?
- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form (It is called super-position)

Example 1: Initial Conditions (2 of 3)

- Now consider the following initial value problem for our equation:

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1$$

- We have found a general solution of the form: $y(t) = c_1 e^t + c_2 e^{-t}$

- Using the initial equations, can we get the coefficients ?

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- We have found a general solution of the form: $y(t) = c_1 e^t + c_2 e^{-t}$

- Using the initial equations, $\left.\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= c_1 - c_2 = -1 \end{aligned}\right\} \Rightarrow c_1 = 1/2,\ c_2 = 3/2$

- Thus $y(t) = 1/2\ e^t + 3/2\ e^{-t}$

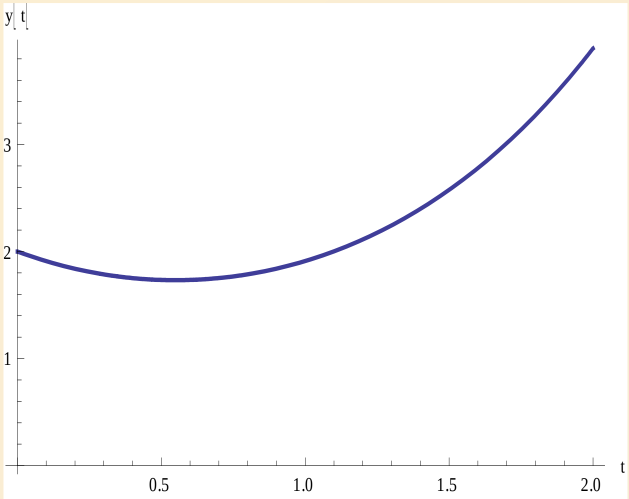
Example 1: Solution Graphs (3 of 3)

- Our initial value problem and solution are

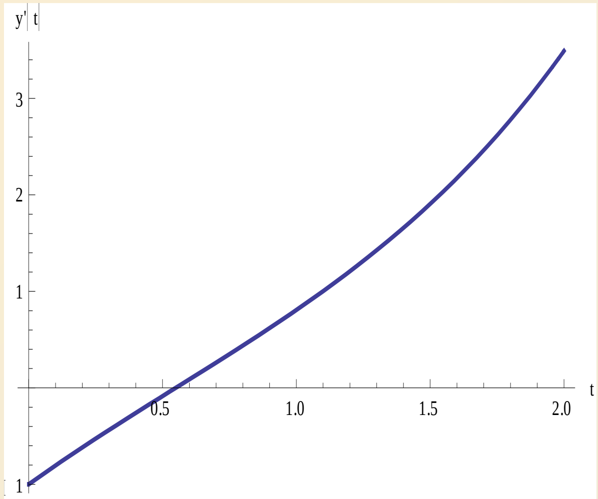
$$y'' - y = 0, \quad y(0) = 2, \; y'(0) = -1 \;\Rightarrow\; y(t) = 1/2\, e^{t} + 3/2\, e^{-t}$$

- Graphs of both y(t) and y′(t) are given below. Observe that both initial conditions are satisfied.

$$y(t) = 1/2\, e^{t} + 3/2\, e^{-t}$$



$$y'(t) = 1/2\, e^{t} - 3/2\, e^{-t}$$



Characteristic Equation

- To solve the 2nd order equation with constant coefficients, $ay'' + by' + cy = 0,$ we begin by assuming a solution of the form $\boldsymbol{y = e^{rt}}$**.**

- Substituting this into the differential equation, we obtain

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

- Simplifying, $$e^{rt}(ar^2 + br + c) = 0$$

  and hence $$ar^2 + br + c = 0$$

- This last equation is called the **characteristic equation** of the differential equation.

- We then solve for *r* by factoring or using quadratic formula.

General Solution

- Using the quadratic formula on the characteristic equation

$$ar^2 + br + c = 0,$$

we obtain two solutions, r_1 and r_2.

- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

- In this case, the general solution has the form

Examples

Find solutions of the ODEs

(1) $$\frac{d^2y}{dt^2} - 4y = 0$$

(2) $$y'' - y' = 0$$

(3) $$y'' + 5y' + 6\mathrm{y} = 0$$

Example 2 (case 1: $r_1 \neq r_2$)

- Consider the linear differential equation $y'' + 5y' + 6y = 0$
- Assuming an exponential solution leads to the characteristic equation:
- Factoring the characteristic equation yields two solutions: r_1 = -2 and r_2 = -3
- Therefore, the general solution to this differential equation has the form

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 6 = 0 \;\Leftrightarrow\; (r+2)(r+3) = 0$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Initial Conditions

- For the initial value problem

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \;\; y'(t_0) = y_0',$$

we use the general solution $\quad y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

together with the initial conditions to find c_1 and c_2. That is,

$$\left. \begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y_0' \end{aligned} \right\} \Rightarrow c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 3 (IVP)

- Consider the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3$$

- From the preceding example, we know the general solution has the form:
- With derivative: $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$
- Using the initial conditions:

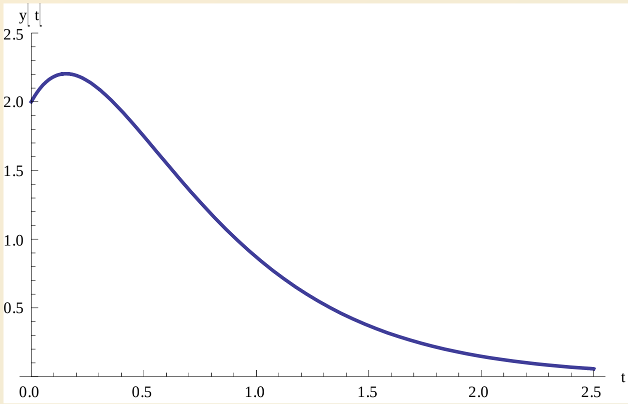
$$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

- Thus

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -2c_1 - 3c_2 &= 3 \end{aligned}\right\} \Rightarrow c_1 = 9, \; c_2 = -7$$

$$y(t) = 9e^{-2t} - 7e^{-3t}$$



(Ex) Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \ \ y'(0) = 1/2$$

Example 4: Initial Value Problem

- Consider the initial value problem $4y'' - 8y' + 3y = 0, \quad y(0) = 2, \; y'(0) = 1/2$

- Then $y(t) = e^{rt} \;\Rightarrow\; 4r^2 - 8r + 3 = 0 \;\Leftrightarrow\; (2r-3)(2r-1) = 0$

- Factoring yields two solutions, $r_1 = 3/2$ and $r_2 = ½$

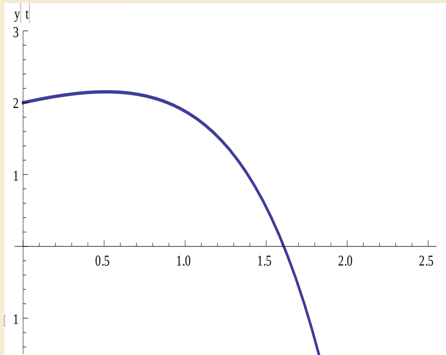
- The general solution has the form $y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ 3/2\,c_1 + 1/2\,c_2 &= 1/2 \end{aligned}\right\} \Rightarrow c_1 = -1/2\,,\; c_2 = 5/2$$

- Thus $y(t) = -1/2\; e^{3t/2} + 5/2\; e^{t/2}$

$y(t) = -1/2\; e^{3t/2} + 5/2\; e^{t/2}$



Example 5: Find Maximum Value

- For the initial value problem in Example 3,

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2,\ \ y'(0) = 3,$$

to find the maximum value attained by the solution, we set $y'(t) = 0$ and solve for t:

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

$$y'(t) = -18e^{-2t} + 21e^{-3t} \overset{set}{=} 0$$

$$6e^{-2t} = 7e^{-3t}$$

$$e^{t} = 7/6$$

$$t = \ln(7/6)$$

$$t \approx 0.1542$$

$$y \approx 2.204$$

$$y(t) = 9e^{-2t} - 7e^{-3t}$$