Ch 3.2: Fundamental Solutions of Linear Homogeneous Equations

- Let $p, q$ be continuous functions on an interval $I = (\alpha, \beta)$, which could be infinite. For any function $y$ that is twice differentiable on $I$, define the differential operator $L$ by

$$L[y] = y'' + p \, y' + q \, y$$

- Note that $L[y]$ is a function on $I$, with output value

$$L[y](t) = y''(t) + p(t) \, y'(t) + q(t) \, y(t)$$

- For example, $p(t) = t^2$, $q(t) = e^{2t}$, $y(t) = \sin(t)$, $I = (0, 2\pi)$

$$L[y](t) = -\sin(t) + t^2 \cos(t) + 2e^{2t} \sin(t)$$

(Question) How do we find a general solution of ODE?
In this section we will discuss the second order linear homogeneous equation $L[y](t) = 0$, along with initial conditions as indicated below:

\[
L[y] = y'' + p(t) y' + q(t) y = 0
\]

\[
y(t_0) = y_0, \quad y'(t_0) = y_1
\]

We would like to know if there are solutions to this initial value problem, and if so, are they unique.

Also, we would like to know what can be said about the form and structure of solutions that might be helpful in finding solutions to particular problems.

These questions are addressed in the theorems of this section.
Theorem 3.2.1 (Existence and Uniqueness)

• Consider the initial value problem

\[ y'' + p(t)\ y' + q(t)\ y = g(t) \]
\[ y(t_0) = y_0, \ y'(t_0) = y'_0 \]

• where \( p, q, \) and \( g \) are continuous on an open interval \( I \) that contains \( t_0 \). Then there exists a unique solution \( y = \phi(t) \) on \( I \).

• Note: While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression (or general formula) for the solution. This is a major difference between first and second order linear equations.
Example 1

\[ y'' + p(t) y' + q(t) y = g(t) \]
\[ y(t_0) = y_0, \quad y'(t_0) = y_1 \]

• Consider the second order linear initial value problem
  
  \[(t^2 - 3t) y'' + t y' - (t + 3) y = 0, \quad y(1) = 2, \quad y'(1) = 1\]

• Writing the differential equation in the form:
  
  \[ y'' + p(t) y' + q(t) y = g(t) \]

  \[ p(t) = 1/(t - 3), \quad q(t) = -(t + 3)/(t(t - 3)) \] and \( g(t) = 0 \)

• The only points of discontinuity for these coefficients are \( t = 0 \) and \( t = 3 \). So the longest open interval containing the initial point \( t = 1 \) in which all the coefficients are continuous is \( 0 < t < 3 \)

• Therefore, the longest interval in which Theorem 3.2.1 guarantees the existence of the solution is \( 0 < t < 3 \)
Example 2

- Consider the second order linear initial value problem

\[ y'' + p(t)y' + q(t)y = 0, \quad y(0) = 0, \quad y'(0) = 0 \]

where \( p, q \) are continuous on an open interval \( I \) containing \( t_0 \).

- In light of the initial conditions, note that \( y = 0 \) is a solution to this homogeneous initial value problem.

- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that \( y = 0 \) is the only solution of this problem.
Theorem 3.2.2 (Principle of Superposition)

- If $y_1$ and $y_2$ are solutions to the equation $L[y] = y'' + p(t) y' + q(t) y = 0$, then the linear combination $c_1 y_1 + y_2 c_2$ is also a solution, for all constants $c_1$ and $c_2$.

- To prove this theorem, substitute $c_1 y_1 + y_2 c_2$ in for $y$ in the equation above, and use the fact that $y_1$ and $y_2$ are solutions.

- Thus for any two solutions $y_1$ and $y_2$, we can construct an infinite family of solutions, each of the form $y = c_1 y_1 + c_2 y_2$.

- (Question) Can all solutions be written this way? Does any solution have a different form (expression) altogether?

--- To answer this question, we use the Wronskian determinant.
The Wronskian Determinant  (1 of 3)

• Suppose \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) y' + q(t) y = 0
\]

• From Theorem 3.2.2 (Superposition), we know that \( y = c_1 y_1 + c_2 y_2 \) is a solution to this equation.

• Next, find coefficients such that \( y = c_1 y_1 + c_2 y_2 \) satisfies the initial conditions

\[
y(t_0) = y_0, \quad y'(t_0) = y'_0
\]

• To do so, we need to solve the following equations:

\[
c_1 y_1(t_0) + c_2 y_2(t_0) = y_0
\]
\[
c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0
\]
Matrix (Linear Algebra)

• What is a matrix?

• Operations of matrices: addition/subtraction, multiplication

• Determinant of a matrix

• Inverse matrix

• Linear system of equations
The Wronskian Determinant  (2 of 3)

- Solving the equations, we obtain
  \[ c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \]
  \[ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \]

- In terms of determinants:

\[
c_1 = \frac{y_0 y'_2(t_0) - y'_0 y_2(t_0)}{y_1(t_0) y'_2(t_0) - y'_1(t_0) y_2(t_0)}
\]

\[
c_2 = \frac{-y_0 y'_1(t_0) + y'_0 y_1(t_0)}{y_1(t_0) y'_2(t_0) - y'_1(t_0) y_2(t_0)}
\]

\[
c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}
\]
The Wronskian Determinant  (3 of 3)

• In order for these formulas to be valid, the determinant $W$ in the denominator cannot be zero:

$$c_1 = \frac{y_0 y_2'(t_0) - y_2(t_0) y_0'}{W}, \quad c_2 = \frac{y_1(t_0) y_2(t_0) - y_1'(t_0) y_2'(t_0)}{W}$$

$$W = \det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)$$

• $W$ is called the **Wronskian determinant**, or more simply, the **Wronskian** of the solutions $y_1$ and $y_2$. We will sometimes use the notation

$$W(y_1, y_2)(t_0) = \det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$$
(Example) Compute the Wronskian of the two functions:

(1) \( f(x) = e^x \) and \( g(x) = e^{-x} \).

(2) \( f(t) = \sin(t) \) and \( g(t) = \cos(t) \).
Theorem 3.2.3

• Suppose $y_1$ and $y_2$ are solutions to the linear equation

$$L[y] = y'' + p(t) y' + q(t) y = 0$$

with the initial conditions $y(t_0) = y_0$, $y'(t_0) = y'_0$

Then it is always possible to choose constants $c_1$, $c_2$ so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix} = y_1 y'_2 - y'_1 y_2$$

is not zero at the point $t_0$. 
Example 3

• In Example 2 of Section 3.1, we found that \( y_1(t) = e^{-2t} \) and \( y_2(t) = e^{-3t} \) were solutions to the differential equation \( y'' + 5y' + 6y = 0 \).

• The Wronskian of these two functions is

\[
W = \det \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix} = e^{-2t}(-3)e^{-3t} - e^{-3t}(-2)e^{-2t} = -e^{-5t}
\]

• Since \( W \) is nonzero for all values of \( t \), the functions \( y_1 \) and \( y_2 \) can be used to construct solutions of the differential equation with initial conditions at any value of \( t \).
Theorem 3.2.4 (Fundamental Solutions)

• Suppose \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) y' + q(t) y = 0.
\]

Then the family of solutions \( y = c_1 y_1 + c_2 y_2 \) with arbitrary coefficients \( c_1, c_2 \) includes every solution to the differential equation if and only if there is a point \( t_0 \) such that \( W(y_1, y_2)(t_0) \neq 0 \).

• The expression \( y = c_1 y_1 + c_2 y_2 \) is called the general solution of the differential equation above, and in this case \( y_1 \) and \( y_2 \) are said to form a fundamental set of solutions to the differential equation.
(Example 4)

Find the Wronskian of the two functions:

\[ y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}, \quad r_1 \neq r_2 \]
Case 1: two different real solutions \( r^2 + pr + q = 0 \)

- Consider the general second order linear equation below, with the two solutions indicated:
  \[ y'' + p y' + q y = 0 \]

- Suppose the functions below are solutions to this equation:
  \[ y_1 = e^{r_1 t}, \ y_2 = e^{r_2 t}, \ r_1 \neq r_2 \]

- The Wronskian of \( y_1 \) and \( y_2 \) is
  \[
  W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2) t} \neq 0 \text{ for all } t.
  \]

- Thus \( y_1 \) and \( y_2 \) form a fundamental set of solutions to the equation, and can be used to construct all of its solutions.

- The general solution is
  \[ y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]
Example 5: Solutions (1 of 2)

• Consider the following differential equation (Euler equation):

\[ 2t^2 y'' + 3t y' - y = 0, \quad t > 0 \]

• Show that the functions below are fundamental solutions:

\[ y_1 = t^{1/2}, \quad y_2 = t^{-1} \]

• To show this, first substitute \( y_1 \) into the equation:

\[
2t^2 \left(-\frac{t^{-3/2}}{4}\right) + 3t \left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0
\]

• Thus \( y_1 \) is a indeed a solution of the differential equation.

• Similarly, \( y_2 \) is also a solution:

\[
2t^2 \left(2t^{-3}\right) + 3t \left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0
\]

(Question) The Wronskian of \( y_1 \) and \( y_2 \) is zero?
Example 5: Fundamental Solutions (2 of 2)

- Recall that \( y_1 = t^{1/2}, y_2 = t^{-1} \)

- To show that \( y_1 \) and \( y_2 \) form a fundamental set of solutions, we evaluate the Wronskian of \( y_1 \) and \( y_2 \):

\[
W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2} t^{-3/2} = -\frac{3}{2} t^{-3/2} = -\frac{3}{2\sqrt{t^3}}
\]

- Since \( W \neq 0 \) for \( t > 0 \), \( y_1 \) and \( y_2 \) form a fundamental set of solutions for the differential equation

\[
2t^2 y'' + 3t y' - y = 0, \ t > 0
\]
Theorem 3.2.5: Existence of Fundamental Set of Solutions

• Consider the differential equation below, whose coefficients $p$ and $q$ are continuous on some open interval $I$:

$$L[y] = y'' + p(t) y' + q(t) y = 0$$

• Let $t_0$ be a point in $I$, and $y_1$ and $y_2$ solutions of the equation with $y_1$ and $y_2$ satisfying initial conditions $y_1(t_0) = 1$, $y_1'(t_0) = 0$ and satisfying initial conditions $y_2(t_0) = 0$, $y_2'(t_0) = 1$

• $W(y_1, y_2)(t_0) \neq 0$.

• Then, $y_1, y_2$ form a fundamental set of solutions to the given differential equation.
(Example 6) Find the solution of the IVP

(1) \( y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 2 \)

(2) \( y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1 \)
Example 6: Apply Theorem 3.2.5 (1 of 3)

• Find the fundamental set specified by Theorem 3.2.5 for the differential equation and initial point \( y'' - y = 0, \quad t_0 = 0 \)

• In Section 3.1, we found two solutions of this equation: \( y_1 = e^t, \quad y_2 = e^{-t} \)

The Wronskian of these solutions is \( W(y_1, y_2)(t_0) = -2 \neq 0 \) so they form a fundamental set of solutions.

• But these two solutions do not satisfy the initial conditions stated in Theorem 3.2.5, and thus they do not form the fundamental set of solutions mentioned in that theorem.

• Let \( y_3 \) and \( y_4 \) be the fundamental solutions of Theorem 3.2.5.

\[
y_3(0) = 1, \quad y'_3(0) = 0; \quad y_4(0) = 0, \quad y'_4(0) = 1
\]
Example 6: General Solution (2 of 3)

• Since $y_1$ and $y_2$ form a fundamental set of solutions,
  
  \[ y_3 = c_1 e^t + c_2 e^{-t}, \quad y_3(0) = 1, y'_3(0) = 0 \]
  
  \[ y_4 = d_1 e^t + d_2 e^{-t}, \quad y_4(0) = 0, y'_4(0) = 1 \]

• Solving each equation, we obtain
  
  \[ y_3(t) = \frac{1}{2} e^t + \frac{1}{2} e^{-t} = \cosh(t), \quad y_4(t) = \frac{1}{2} e^t - \frac{1}{2} e^{-t} = \sinh(t) \]

• The Wronskian of $y_3$ and $y_4$ is
  
  \[ W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0 \]

• Thus $y_3, y_4$ form the fundamental set of solutions indicated in Theorem 3.2.5, with general solution in this case
  
  \[ y(t) = k_1 \cosh(t) + k_2 \sinh(t) \]
Example 6: Many Fundamental Solution Sets (3 of 3)

• Thus \( S_1 = \{ e^t, e^{-t} \} \), \( S_2 = \{ \cosh t, \sinh t \} \)

  both form fundamental solution sets to the differential equation and initial point
  \[ y'' - y = 0, \quad t_0 = 0 \]

• In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.
Consider again the equation (2):

\[ L[y] = y'' + p(t) y' + q(t) y = 0 \]

where \( p \) and \( q \) are continuous real-valued functions.

If \( y = u(t) + iv(t) \) is a complex-valued solution of Eq. (2), then its real part \( u \) and its imaginary part \( v \) are also solutions of this equation.

Section 3.3: Case 2
Theorem 3.2.7 (Abel’s Theorem)

• Suppose \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) y' + q(t) y = 0
\]

where \( p \) and \( q \) are continuous on some open interval \( I \). Then the \( W(y_1, y_2)(t) \) is given by

\[
W(y_1, y_2)(t) = ce^{-\int p(t) dt}
\]

where \( c \) is a constant that depends on \( y_1 \) and \( y_2 \) but not on \( t \).

• Note that \( W(y_1, y_2)(t) \) is either zero for all \( t \) in \( I \) (if \( c = 0 \)) or else is never zero in \( I \) (if \( c \neq 0 \)).
Example 7 Apply Abel’s Theorem

- Recall the following differential equation and its solutions:
  \[ 2t^2 y'' + 3t y' - y = 0, \quad t > 0 \]
  with solutions \[ y_1 = t^{1/2}, \quad y_2 = t^{-1} \]

- We computed the Wronskian for these solutions to be
  \[
  W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{3}{2} t^{-3/2} = -\frac{3}{2\sqrt{t^3}}
  \]

- Writing the differential equation in the standard form
  \[ y'' + 3/(2t) y' - 1/(2t^2) y = 0, \quad t > 0 \]

- So \( p(t) = 3/(2t) \) and the Wronskian given by Thm.3.2.6 is
  \[
  W(y_1, y_2)(t) = ce^{-\int 3/(2t) \, dt} = ce^{(-3/2 \ln t)} = ct^{-3/2}
  \]

- This is the Wronskian for any pair of fundamental solutions. For the solutions given above, we must let \( c = -3/2 \)
Summary

• To find a general solution of the differential equation

\[ y'' + p(t) y' + q(t) y = 0, \quad \alpha < t < \beta \]

we first find two solutions \( y_1 \) and \( y_2 \).

• Then make sure there is a point \( t_0 \) in the interval such that \( W(y_1, y_2)(t_0) \neq 0 \).

• It follows that \( y_1 \) and \( y_2 \) form a fundamental set of solutions to the equation, with general solution \( y = c_1 y_1 + c_2 y_2 \).

• If initial conditions are prescribed at a point \( t_0 \) in the interval where \( W \neq 0 \), then \( c_1 \) and \( c_2 \) can be chosen to satisfy those conditions.