Ch 3.2: Fundamental Solutions of Linear Homogeneous Equations

- Let p, q be continuous functions on an interval $I = (\alpha, \beta)$, which could be infinite. For any function y that is twice differentiable on I, define the differential operator L by

$$L[y] = y'' + p\,y' + q\,y$$

- Note that $L[y]$ is a function on I, with output value

$$L[y](t) = y''(t) + p(t)\,y'(t) + q(t)\,y(t)$$

- For example, $p(t) = t^2, \quad q(t) = e^{2t}, \qquad y(t) = \sin(t), \; I = (0, 2\pi)$

$$L[y](t) \quad = \quad -\sin(t) + t^2 \cos(t) + 2e^{2t} \sin(t)$$

(Question) How do we find a **general solution** of ODE?

Differential Operator Notation

- In this section we will discuss the second order linear homogeneous equation $L[y](t) = 0$, along with initial conditions as indicated below:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$
$$y(t_0) = y_0,\; y'(t_0) = y_1$$

- We would like to know if there are solutions to this initial value problem, and if so, are they unique.
- Also, we would like to know what can be said about the form and structure of solutions that might be helpful in finding solutions to particular problems.
- These questions are addressed in the theorems of this section.

Theorem 3.2.1 (Existence and Uniqueness)

- Consider the initial value problem
$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y_0'$$

- where *p*, *q*, and *g* are continuous on an open interval *I* that contains t_0. Then there exists a unique solution $y = \phi(t)$ on *I*.

- Note: While this theorem says that a solution to the initial value problem above exists, it is often **not possible** to write down a useful expression (or general formula) for the solution. This is a major difference between first and second order linear equations.

Example 1

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\; y'(t_0) = y_1$$

- Consider the second order linear initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0,\; y(1) = 2,\;\; y'(1) = 1$$

- Writing the differential equation in the form : $y'' + p(t)y' + q(t)y = g(t)$

$$p(t) = 1/(t-3),\; q(t) = -(t+3)/(t(t-3)) \text{ and } g(t) = 0$$

- The only points of discontinuity for these coefficients are $t = 0$ and $t = 3$. So the longest open interval containing the initial point $t = 1$ in which all the coefficients are continuous is $0 < t < 3$

- Therefore, the longest interval in which Theorem 3.2.1 guarantees the existence of the solution is $0 < t < 3$

Example 2

- Consider the second order linear initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(0) = 0, \quad y'(0) = 0$$

where p, q are continuous on an open interval I containing t_0.

- In light of the initial conditions, note that $y = 0$ is a solution to this homogeneous initial value problem.

- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that $y = 0$ is the only solution of this problem.

Theorem 3.2.2 (Principle of Superposition)

- If y_1 and y_2 are solutions to the equation $L[y] = y'' + p(t)\,y' + q(t)\,y = 0$

  then the linear combination $c_1y_1 + y_2c_2$ is also a solution, for all constants c_1 and c_2.

- To prove this theorem, substitute $c_1y_1 + y_2c_2$ in for y in the equation above, and use the fact that y_1 and y_2 are solutions.

- Thus for any two solutions y_1 and y_2, we can construct an infinite family of solutions, each of the form $y = c_1y_1 + c_2y_2$.

- (Question) Can all solutions be written this way?

  Does any solution have a different form (expression) altogether?

--- To answer this question, we use the Wronskian determinant.

The Wronskian Determinant (1 of 3)

- Suppose **y_1 and y_2** are solutions to the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

- From Theorem 3.2.2 (Superposition), we know that $y = c_1 y_1 + c_2 y_2$ is a solution to this equation.

- Next, find coefficients such that $y = c_1 y_1 + c_2 y_2$ satisfies the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

- To do so, we need to solve the following equations:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

Matrix (Linear Algebra)

- What is a matrix?
- Operations of matrices: addition/subtraction, multiplication
- Determinant of a matrix
- Inverse matrix
- Linear system of equations

The Wronskian Determinant (2 of 3)

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

- Solving the equations, we obtain

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$
$$c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

- In terms of determinants:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

The Wronskian Determinant (3 of 3)

- In order for these formulas to be valid, the determinant W in the denominator cannot be zero:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

$$W = \det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

- W is called the **Wronskian determinant**, or more simply, the **Wronskian** of the solutions y_1 and y_2. We will sometimes use the notation

$$W(y_1, y_2)(t_0) = \det\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$$

(Example) Compute the Wronskian of the two functions:

(1) $f(x) = e^{x}$ and $g(x) = e^{-x}$.

(2) $f(t) = \sin(t)$ and $g(t) = \cos(t)$.

Theorem 3.2.3

- Suppose y_1 and y_2 are solutions to the linear equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

with the initial conditions $y(t_0) = y_0, \qquad y'(t_0) = y'_0$

Then it is always possible to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

is not zero at the point t_0

Example 3

- In Example 2 of Section 3.1, we found that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$

  were solutions to the differential equation $y'' + 5y' + 6y = 0$

- The Wronskian of these two functions is

$$W = \det\begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix} = e^{-2t}(-3)e^{-3t} - e^{-3t}(-2)e^{-2t} = -e^{-5t}$$

- Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the differential equation with initial conditions at any value of t

Theorem 3.2.4 (Fundamental Solutions)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0.$$

Then the family of solutions $y = c_1 y_1 + c_2 y_2$ with arbitrary coefficients c_1, c_2 includes **every solution** to the differential equation if an only if there is a point t_0 such that $W(y_1, y_2)(t_0) \neq 0$.

- The expression $y = c_1 y_1 + c_2 y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

(Example 4)

Find the Wronskian of the two functions:

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \quad r_1 \neq r_2$$

Case 1: two different real solutions $r^2 + pr + q = 0$

- Consider the general second order linear equation below, with the two solutions indicated:
$$y'' + p\,y' + q\,y = 0$$
- Suppose the functions below are solutions to this equation:
$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \quad r_1 \neq r_2$$
- The Wronskian of y_1 and y_2 is
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1 + r_2)t} \neq 0 \text{ for all } t.$$
- Thus y_1 and y_2 form a fundamental set of solutions to the equation, and can be used to construct all of its solutions.
- The general solution is
$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example 5: Solutions (1 of 2)

- Consider the following differential equation (Euler equation):

$$2t^2 y'' + 3t\, y' - y = 0, \;\; t > 0$$

- Show that the functions below are fundamental solutions:

$$y_1 = t^{1/2}, \qquad y_2 = t^{-1}$$

- To show this, first substitute y_1 into the equation:

$$2t^2\left(\frac{-t^{-3/2}}{4}\right) + 3t\left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0$$

- Thus y_1 is a indeed a solution of the differential equation.

- Similarly, y_2 is also a solution: $2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4-3-1)t^{-1} = 0$

(Question) The Wronskian of y_1 and y_2 is zero ?

Example 5: Fundamental Solutions (2 of 2)

- Recall that $y_1 = t^{1/2}, y_2 = t^{-1}$
- To show that y_1 and y_2 form a fundamental set of solutions, we evaluate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

- Since $W \neq 0$ for $t > 0$, y_1 and y_2 form a fundamental set of solutions for the differential equation

$$2t^2 y'' + 3t\,y' - y = 0, \ \ t > 0$$

Theorem 3.2.5: Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients p and q are continuous on some open interval I:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

- Let t_0 be a point in I, and y_1 and y_2 solutions of the equation with y_1 and satisfying initial conditions $y_1(t_0) = 1,\; y_1'(t_0) = 0$
  and satisfying initial conditions $y_2(t_0) = 0,\; y_2'(t_0) = 1$

- $W(y_1, y_2)(t_0) \neq 0$.

- Then, y_1, y_2 form a fundamental set of solutions to the given differential equation.

(Example 6) Find the solution of the IVP

(1) $y'' - y = 0, \quad y(0) = 0, \ y'(0) = 2$

(2) $y'' - y = 0, \quad y(0) = 0, \ y'(0) = 1$

Example 6: Apply Theorem 3.2.5 (1 of 3)

- Find the fundamental set specified by Theorem 3.2.5 for the differential equation and initial point $y'' - y = 0, \quad t_0 = 0$

- In Section 3.1, we found two solutions of this equation: $y_1 = e^t, y_2 = e^{-t}$

  The Wronskian of these solutions is $W(y_1, y_2)(t_0) = -2 \neq 0$ so they form a fundamental set of solutions.

- But these two solutions do not satisfy the initial conditions stated in Theorem 3.2.5, and thus they do not form the fundamental set of solutions mentioned in that theorem.

- Let y_3 and y_4 be the fundamental solutions of Theorem 3.2.5.

$$y_3(0) = 1, \; y_3'(0) = 0; \quad y_4(0) = 0, \; y_4'(0) = 1$$

Example 6: General Solution (2 of 3)

- Since y_1 and y_2 form a fundamental set of solutions,

$$y_3 = c_1 e^t + c_2 e^{-t}, \quad y_3(0) = 1, y_3'(0) = 0$$
$$y_4 = d_1 e^t + d_2 e^{-t}, \quad y_4(0) = 0, y_4'(0) = 1$$

- Solving each equation, we obtain

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t), \quad y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$$

- The Wronskian of y_3 and y_4 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0$$

- Thus $y_3, \ y_4$ form the fundamental set of solutions indicated in Theorem 3.2.5, with general solution in this case

$$y(t) = k_1 \cosh(t) + k_2 \sinh(t)$$

Example 6: Many Fundamental Solution Sets (3 of 3)

- Thus $$S_1 = \left\{ e^t, e^{-t} \right\}, \quad S_2 = \left\{ \cosh t, \sinh t \right\}$$

  both form fundamental solution sets to the differential equation and initial point $$y'' - y = 0, \quad t_0 = 0$$

- In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.

Theorem 3.2.6

Consider again the equation (2):

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

where p and q are continuous real-valued functions.

If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part u and its imaginary part v are also solutions of this equation.

Section 3.3: Case 2

Theorem 3.2.7 (Abel's Theorem)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0$$

where p and q are continuous on some open interval I. Then the $W(y_1,y_2)(t)$ is given by

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}$$

where c is a constant that depends on y_1 and y_2 but not on t.

- Note that $W(y_1,y_2)(t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Example 7 Apply Abel's Theorem

- Recall the following differential equation and its solutions:
  $2t^2 y'' + 3t\, y' - y = 0,\ \ t > 0$ with solutions $y_1 = t^{1/2}, y_2 = t^{-1}$
- We computed the Wronskian for these solutions to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{3}{2} t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

- Writing the differential equation in the standard form

$$y'' + 3/(2t)\, y' - 1/(2t^2)\, y = 0,\ \ t > 0$$

- So $p(t) = 3/(2t)$ and the Wronskian given by Thm.3.2.6 is

$$W(y_1, y_2)(t) = ce^{-\int 3/(2t)dt} = ce^{(-3/2\ \ln t\ )} = ct^{-3/2}$$

- This is the Wronskian for any pair of fundamental solutions. For the solutions given above, we must let c = -3/2

Summary

- To find a general solution of the differential equation

$$y'' + p(t)\, y' + q(t)\, y = 0, \;\; \alpha < t < \beta$$

we first find two solutions y_1 and y_2.

- Then make sure there is a point t_0 in the interval such that $W(y_1, y_2)(t_0) \neq 0$.

- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1 y_1 + c_2 y_2$.

- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.