3.3 Case 2: Complex Roots of Characteristic equation

- Recall our discussion of the equation $$ay'' + by' + cy = 0$$ where a, b and c are constants.

- Assuming an exponential soln leads to characteristic equation:
$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 & r_2:
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then complex roots: $r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$

- Thus $y_1(t) = e^{(\lambda + i\mu)t}, \; y_2(t) = e^{(\lambda - i\mu)t}$?

(Example 1) Find general (real-valued) solutions of the ODEs

(1) $$y'' + y = 0$$

(2) $$y'' + 4y' + 5y = 0$$

Euler's Formula; Complex Valued Solutions

- Substituting *it* into Taylor series for e^t, we obtain **Euler's formula**:

$$e^{it}=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}t^{2n-1}}{(2n-1)!}=\cos t+i\sin t$$

- Generalizing Euler's formula, we obtain $e^{i\mu t}=\cos(\mu t)+i\sin(\mu t)$

- Then $e^{(\lambda+i\mu)t}=e^{\lambda t}e^{i\mu t}=e^{\lambda t}\left[\cos\mu t+i\sin\mu t\right]=e^{\lambda t}\cos(\mu t)+ie^{\lambda t}\sin(\mu t)$

- Therefore

$$y_1(t)=e^{(\lambda+i\mu)t}=e^{\lambda t}\cos(\mu t)+ie^{\lambda t}\sin(\mu t)$$

$$y_2(t)=e^{(\lambda-i\mu)t}=e^{\lambda t}\cos(\mu t)-ie^{\lambda t}\sin(\mu t)$$

Real Valued Solutions

- Our two solutions thus far are complex-valued functions:

$$y_1(t) = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$$
$$y_2(t) = e^{\lambda t}\cos\mu t - ie^{\lambda t}\sin\mu t$$

- We would prefer to have real-valued solutions, since our differential equation has real coefficients.

- To achieve this, recall that linear combinations of solutions are themselves solutions:

$$y_1(t) + y_2(t) = 2e^{\lambda t}\cos(\mu t)$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t}\sin(\mu t)$$

- Ignoring constants, we obtain the two solutions

$$y_3(t) = e^{\lambda t}\cos(\mu t), \quad y_4(t) = e^{\lambda t}\sin(\mu t)$$

Real Valued Solutions: The Wronskian

- Thus we have the following real-valued functions:

$$y_3(t) = e^{\lambda t}\cos\mu t,\;\; y_4(t) = e^{\lambda t}\sin\mu t$$

- Checking the Wronskian, we obtain

$$W = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ e^{\lambda t}\left(\lambda\cos\mu t - \mu\sin\mu t\right) & e^{\lambda t}\left(\lambda\sin\mu t + \mu\cos\mu t\right) \end{vmatrix}$$

$$= \mu e^{2\lambda t} \neq 0$$

- Thus y_3 and y_4 form a fundamental solution set for our ODE, and (Case 2:) the general solution can be expressed as

$$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$$

(Example 2) Find a general solution of the ODE:

$$y'' + y' + 9.25y = 0$$

Example 2 (1 of 2)

- Consider the differential equation

$$y'' + y' + 9.25y = 0$$

- For an exponential solution, the characteristic equation is

$$y(t) = e^{rt} \Rightarrow r^2 + r + 9.25 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1 - 4*9.25}}{2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i$$

- Therefore, separating the real and imaginary components,

$$\lambda = -1/2,\ \mu = 3$$

and thus the general solution is

$$y(t) = c_1 e^{-t/2} \cos(3t) + c_2 e^{-t/2} \sin(3t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

Example 2 (2 of 2)

- Using the general solution just determined $y(t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$

- We can determine the particular solution that satisfies the initial conditions

$$y(0) = 2 \text{ and } y'(0) = 8$$

- So

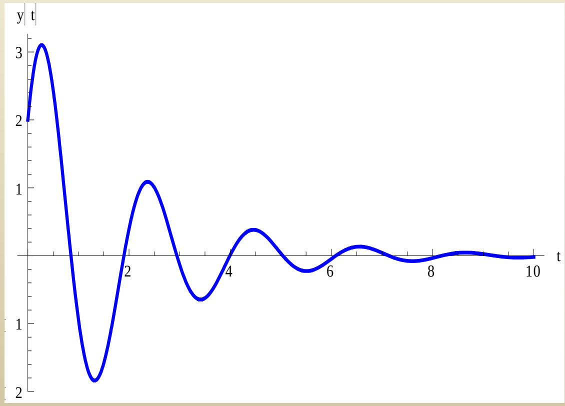
$$\left.\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= -1/2\, c_1 + 3c_2 = 8 \end{aligned}\right\} \Rightarrow c_1 = 2,\ c_2 = 3$$

- Thus the solution of this IVP is

$$y(t) = e^{-t/2}\left(2\cos(3t) + 3\sin(3t)\right)$$

- The solution is a decaying oscillation

$y(t) = e^{-t/2}\left(2\cos(3t) + 3\sin(3t)\right)$



(Example 3) Find the solution of the IVP

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1$$

Example 3

- Consider the initial value problem

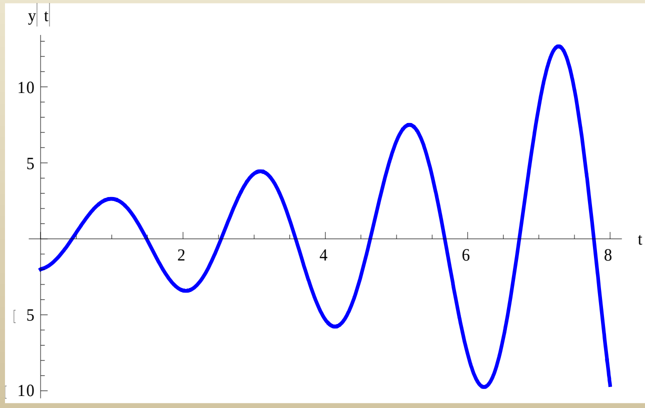
$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1$$

- Then $\quad y(t) = e^{rt} \Rightarrow 16r^2 - 8r + 145 = 0 \Leftrightarrow r = \dfrac{1}{4} \pm 3i$

- Thus the general solution is $\quad y(t) = c_1 e^{t/4}\cos(3t) + c_2 e^{t/4}\sin(3t)$

- And $\quad \left.\begin{array}{l} y(0) = c_1 = -2 \\ y'(0) = -1/4\,c_1 + 3c_2 = 1 \end{array}\right\} \Rightarrow c_1 = -2,\ c_2 = 1/2$

$y(t) = e^{t/4}\left(-2\cos(3t) + 1/2\ \sin(3t)\right)$



- The solution of the IVP is

$$y(t) = e^{t/4}\left(-2\cos(3t) + 1/2\ \sin(3t)\right)$$

- The solution is displays a growing oscillation.

Example 4

- Consider the equation $$y'' + 9y = 0$$
- Then $y(t) = e^{rt} \;\Rightarrow\; r^2 + 9 = 0 \;\Leftrightarrow\; r = \pm 3i$
- Therefore $\lambda = 0,\ \mu = 3$
- and thus the general solution is $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$
- Because $\lambda = 0$, there is no exponential factor in the solution, so the amplitude of each oscillation remains constant.

  The figure shows the graph of two typical solutions.

solid: $y = 2\cos(3t) + 2\sin(3t)$
dashed: $y = \cos(3t) + 1/2\,\sin(3t)$