3.4 Repeated Roots; Reduction of Order

- Recall our 2nd order linear homogeneous ODE $$ay'' + by' + cy = 0$$
  where a, b and c are constants.

- Assuming an exponential solution leads to characteristic equation:
$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 & r_2:
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, $r_1 = r_2 = -b/2a$, since method only gives one solution:
$$y_1(t) = ce^{-bt/2a}$$

(Example 1) Find a general solution of the ODE:

$$y'' - 2y' + y = 0$$

(Example 1) Find a general solution of the ODE:

$$y'' - 2y' + y = 0 \qquad r^2 - 2r + 1 = (r-1)^2 = 0$$

$$r = 1: \qquad y_1(t) = e^t \qquad y_2(t) = v(t)e^t \,?$$

$$v(t) = t\,?$$

Questions: Is it really a solution of the ODE?

Are the two solutions fundamental solutions?

How do we compute v(t)?

Second Solution: Multiplying Factor $v(t)$

- We know that $y_1(t)$ a solution $\Rightarrow y_2(t) = cy_1(t)$ a solution

- Since y_1 and y_2 are linearly dependent, we generalize this approach and multiply by a function v, and determine conditions for which y_2 is a solution:

- Then $y_1(t) = e^{-bt/2a}$ a solution $\Rightarrow$ try $y_2(t) = v(t)e^{-bt/2a}$

$$y_2(t) = v(t)e^{-bt/2a}$$

$$y_2'(t) = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y_2''(t) = v''(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

Finding Multiplying Factor $v(t)$

$$ay'' + by' + cy = 0$$

- Substituting derivatives into ODE, we seek a formula for *v*:

$$e^{-bt/2a}\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + cv(t)\right\} = 0$$

$$av''(t) - bv'(t) + \frac{b^2}{4a}v(t) + bv'(t) - \frac{b^2}{2a}v(t) + cv(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0 \quad \Leftrightarrow \quad av''(t) + \left(\frac{-b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0$$

$$av''(t) - \left(\frac{b^2 - 4ac}{4a}\right)v(t) = 0$$

$$v''(t) = 0 \quad \Rightarrow \quad v(t) = k_3 t + k_4$$

General Solution

- To find our general solution, we have:

$$
\begin{aligned}
y(t) &= k_1 e^{-bt/2a} + k_2 v(t) e^{-bt/2a} \\
&= k_1 e^{-bt/2a} + \left(k_3 t + k_4\right) e^{-bt/2a} \\
&= c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}
\end{aligned}
$$

- Thus the general solution for repeated roots is

$$
y(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}
$$

Wronskian

- The general solution is $y(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$

- Thus every solution is a linear combination of

$$y_1(t) = e^{-bt/2a}, \qquad y_2(t) = te^{-bt/2a}$$

- The Wronskian of the two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\dfrac{b}{2a} e^{-bt/2a} & \left(1 - \dfrac{bt}{2a}\right) e^{-bt/2a} \end{vmatrix}$$

$$= e^{-bt/a}\left(1 - \frac{bt}{2a}\right) + e^{-bt/a}\left(\frac{bt}{2a}\right)$$

$$= e^{-bt/a} \neq 0 \quad \text{for all } t$$

- Thus y_1 and y_2 form a fundamental solution set for equation.

(Example 2) Find a general solution of the ODE.

$$y'' + 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 3$$

Example 2 (1 of 2)

- Consider the initial value problem $y'' + 4y' + 4y = 0$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \Rightarrow r^2 + 4r + 4 = 0 \Leftrightarrow (r+2)^2 = 0 \Leftrightarrow r = -2$$

- So one solution is $y_1(t) = e^{-2t}$ and a second solution is found:

$$y_2(t) = v(t)e^{-2t}$$
$$y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$$
$$y_2''(t) = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}$$

- Substituting these into the differential equation and simplifying yields where c_1 and c_2 are arbitrary constants.

$$v''(t) = 0,\ v'(t) = k_1,\ v(t) = k_1 t + k_2$$

Example 2 (2 of 2)

- Letting $k_1 = 1$ and $k_2 = 0$, $v(t) = t$ and $y_2(t) = te^{-2t}$
- So the general solution is $y(t) = c_1 e^{-2t} + c_2 te^{-2t}$
- Note that both y_1 and y_2 tend to 0 as $t \to \infty$ regardless of the values of c_1 and c_2
- Using initial conditions

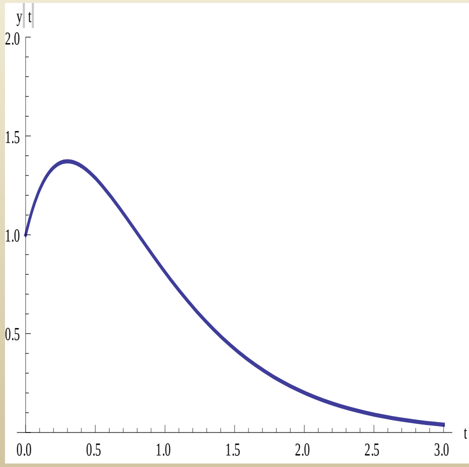
$$y(0) = 1 \text{ and } y'(0) = 3$$

$$\left.\begin{array}{rcrcl} c_1 & & & = & 1 \\ -2c_1 & + & c_2 & = & 3 \end{array}\right\} \Rightarrow c_1 = 1,\ c_2 = 5$$

- Therefore the solution to the IVP is

$$y(t) = e^{-2t} + 5te^{-2t}$$

$y(t) = (1 + 5t)e^{-2t}$



(Example 3) Find the solution of the IVP

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = 1/3$$

Example 3 (1 of 2)

- Consider the initial value problem

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = 1/3$$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \;\Rightarrow\; r^2 - r + 0.25 = 0 \;\Leftrightarrow\; (r - 1/2)^2 = 0 \;\Leftrightarrow r = 1/2$$

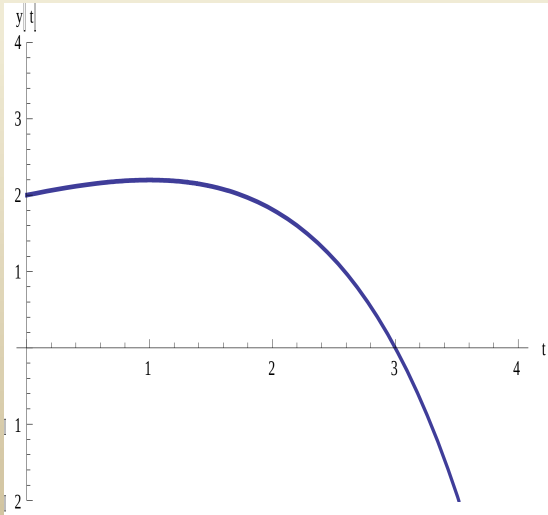
- Thus the general solution is $y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$

- Using the initial conditions:

$$\left.\begin{array}{rcl} c_1 & = & 2 \\ \frac{1}{2}c_1 + c_2 & = & \frac{1}{3} \end{array}\right\} \Rightarrow c_1 = 2,\; c_2 = -\frac{2}{3}$$

- Thus $y(t) = 2e^{t/2} - \frac{2}{3} t e^{t/2}$

$y(t) = e^{t/2}(2 - 2/3\, t)$



Example 3 (2 of 2)

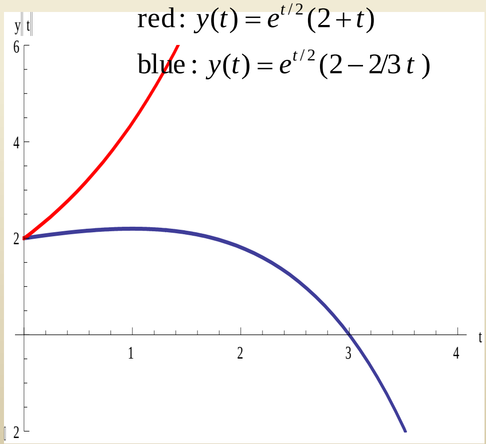
- Suppose that the initial slope in the previous problem was increased

$$y(0) = 2, \ \ y'(0) = 2$$

- The solution of this modified problem is

$$y(t) = 2e^{t/2} + te^{t/2}$$

- Notice that the coefficient of the second term is now positive. This makes a big difference in the graph, since the exponential function is raised to a positive power: $\lambda = 1/2 \ > 0$



Euler equations: $at^2y'' + bty' + cy = 0$

- Consider a second order DE with the following variable coefficients:
$$A(t)y'' + B(t)y' + C(t)y = 0$$

$$A(t) = at^2, \qquad B(t) = bt, \qquad C(t) = c$$

(Question) What kind of functions can be its solutions?

- Exponential function? What is the feature of the ODE?
How do we find its general solution?

(Example) Euler equations

(1) $t^2 y'' + ty' - y = 0$

(2) $2t^2 y'' - 3ty' - 3y = 0$

(3) $t^2 y'' - ty' + y = 0$

Reduction of Order

- The method used so far in this section also works for equations with nonconstant coefficients: $$y'' + p(t)y' + q(t)y = 0$$

- That is, given that y_1 is solution, try $y_2 = v(t)y_1$:

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2'(t) &= v'(t)y_1(t) + v(t)y_1'(t) \\ y_2''(t) &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \end{aligned}$$

- Substituting these into ODE and collecting terms,

$$y_1 v'' + \left(2y_1' + py_1\right)v' + \left(y_1'' + py_1' + qy_1\right)v = 0$$

- Since y_1 is a solution to the differential equation, this last equation reduces to a first order equation in v': $$y_1 v'' + \left(2y_1' + py_1\right)v' = 0$$

Example 4: Reduction of Order (1 of 3)

- Given the variable coefficient equation (Euler equation) and solution y_1,

$$t^2 y'' + 3ty' + y = 0, \quad t > 0; \qquad\qquad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution:

$$y_2(t) = v(t)\, t^{-1}$$
$$y_2'(t) = v'(t)\, t^{-1} - v(t)\, t^{-2}$$
$$y_2''(t) = v''(t)\, t^{-1} - 2v'(t)\, t^{-2} + 2v(t)\, t^{-3}$$

- Substituting these into the ODE and collecting terms,

$$t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) + vt^{-1} = 0$$
$$\Leftrightarrow v''t - 2v' + 2vt^{-1} + 3v' - 3vt^{-1} + vt^{-1} = 0$$
$$\Leftrightarrow \; tv'' + v' = 0$$
$$\Leftrightarrow \; tu' + u = 0, \text{ where } u(t) = v'(t)$$

Example 4: Finding *v*(*t*)

- To solve $tu' + u = 0, \quad u(t) = v'(t)$

  for *u*, we can use the separation of variables method:

$$t\frac{du}{dt} + u = 0 \quad \Leftrightarrow \quad \int \frac{du}{u} = -\int \frac{1}{t}\,dt \quad \Leftrightarrow \quad \ln|u| = -\ln|t| + C$$

$$\Leftrightarrow \quad |u| = |t|^{-1} e^{C} \quad \Leftrightarrow \quad u = ct^{-1}, \quad \text{since } t > 0.$$

- Thus $v' = ct^{-1}$

  and hence $v(t) = c\ln t + k$

Example 4: General Solution (3 of 3)

- Since $v(t) = c \ln t + k$,

$$y_2(t) = (c \ln t + k)t^{-1} = ct^{-1} \ln t + kt^{-1}$$

- Recall $y_1(t) = t^{-1}$

- So we can neglect the second term of y_2 to obtain $y_2(t) = t^{-1} \ln t$

- The Wronskian of $y_1(t)$ and $y_2(t)$ can be computed

$$W(y_1, y_2)(t) = 3/2\, t^{-3/2} \neq 0, \quad t > 0$$

- Hence the general solution to the differential equation is

$$y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t$$

Quiz

Find the solution of the initial value problem

$$y'' + 2y' - 15y = 0, \qquad y(0) = 0, \quad y'(0) = 4$$