Sec 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

- Recall the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are continuous functions on an open interval I.

- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the method of undetermined coefficients to solve the non-homogeneous equation, which relies on knowing solutions to the homogeneous equation.

Theorem 3.5.1

- If Y_1 and Y_2 are solutions of the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

then Y_1 - Y_2 is a solution of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

- If, in addition, $\{y_1, y_2\}$ forms a fundamental solution set of the homogeneous equation, then there exist constants c_1 and c_2 such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem 3.5.2 (General Solution)

- The general solution of the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 form a fundamental solution set for the homogeneous equation, c_1 and c_2 are arbitrary constants, and $Y(t)$ is a specific solution to the nonhomogeneous equation.

Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

- In this section we use the method of **undetermined coefficients** to find a particular solution Y to the nonhomogeneous equation, assuming we can find solutions y_1, y_2 for the homogeneous case.

- The method of undetermined coefficients is usually limited to when p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

(Example 1) Find a general solution of the non-homogeneous ODE.

(1) $$y'' - 3y' - 4y = 2t + 3$$

(2) $$y'' - 3y' - 4y = t^2$$

Example 1: $g(t)$ Polynomial

- Consider the non-homogeneous equation

$$y'' - 3y' - 4y = 2t + 3$$

- We seek Y satisfying this equation. Since the derivative of a polynomial is also a polynomial, a good start for Y is:

$$Y(t) = At + B \quad \Rightarrow \quad Y'(t) = A, \quad Y''(t) = 0$$

- Substituting these derivatives into the differential equation,

$$0 - 3A - 4(At + B) = 2t + 3$$

$$\Leftrightarrow \quad -4At - 3A - 4B = 2t + 3 \quad \Leftrightarrow \quad A = -1/2, \quad B = -3/8$$

- Thus a particular solution to the non-homogeneous ODE is

$$Y(t) = -\frac{1}{2}t - \frac{3}{8}$$

(Example 2) Find a general solution of the non-homogeneous ODE.

(1) $$y'' - 3y' - 4y = 3e^{2t}$$

(2) $$y'' - 3y' - 4y = (-2)e^{2t} + 5e^{-3t}$$

Example 1: Exponential $g(t)$

- Consider the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}$$

- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:

$$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t},\ Y''(t) = 4Ae^{2t}$$

- Substituting these derivatives into the differential equation,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Leftrightarrow \quad -6Ae^{2t} = 3e^{2t} \quad \Leftrightarrow \quad A = -1/2$$

- Thus a particular solution to the non-homogeneous ODE is

$$Y(t) = -\frac{1}{2}e^{2t}$$

(Example 3) Find a general solution of the non-homogeneous ODE.

(1) $$y'' - 3y' - 4y = 2\sin t$$

(2) $$y'' - 3y' - 4y = -8e^{t}\cos(2t)$$

Example 2: Sine $g(t)$, First Attempt (1 of 2)

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin t$$

- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$

- Substituting these derivatives into the differential equation,

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t$$

$$\Leftrightarrow \ (2 + 5A)\sin t + 3A\cos t = 0$$

$$\Leftrightarrow \ c_1 \sin t + c_2 \cos t = 0$$

- Since sin(x) and cos(x) are not multiples of each other, we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 3A = 0$, which is impossible.

$$y'' - 3y' - 4y = 2\sin t$$

Example 2: Sine *g*(*t*), Particular Solution (2 of 2)

- Our next attempt at finding a Y is

$$Y(t) = A\sin t + B\cos t$$
$$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$

- Substituting these derivatives into ODE, we obtain

$$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$$
$$\Leftrightarrow\ (-5A + 3B)\sin t + (-3A - 5B)\cos t = 2\sin t$$
$$\Leftrightarrow -5A + 3B = 2,\ \ -3A - 5B = 0$$
$$\Leftrightarrow\ A = -5/17,\ B = 3/17$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$$

Example 3: Product $g(t)$

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

- We seek Y satisfying this equation, as follows:

$$\begin{aligned}
Y(t) &= Ae^t \cos 2t + Be^t \sin 2t \\
Y'(t) &= Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t \\
&= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t \\
Y''(t) &= (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t + (-2A + B)e^t \sin 2t \\
&\qquad + 2(-2A + B)e^t \cos 2t \\
&= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t
\end{aligned}$$

- Substituting these into the ODE and solving for A and B:

$$A = \frac{10}{13},\ B = \frac{2}{13} \quad \Rightarrow Y(t) = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t$$

Discussion: Sum $g(t)$

- Consider again our general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Suppose that $g(t)$ is sum of functions:

$$g(t) = g_1(t) + g_2(t)$$

- If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$
$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

(Example 4) Find a general solution of the non-homogeneous ODE.

(1) $$y'' - 3y' - 4y = 3e^{2t} + 2\sin t$$

(2) $$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

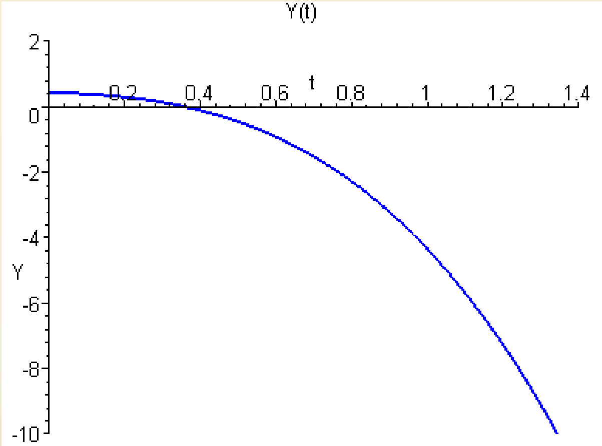
Example 4: Sum $g(t)$

- Consider the equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

- Our equations to solve individually are

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$



- Our particular solution is then

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

So far we have discussed

RULE 1

NEXT: RULE 2

(Example 5) Find a general solution of the non-homogeneous ODE.

(1) $y'' - 3y' - 4y = 2e^{-t}$

(2) $y'' - 9y = e^{-t} - 2e^{3t}$

(3) $y'' + y = \sin(t)$

Example 5: First Attempt (1 of 3)

- Consider the non-homogeneous equation $y'' - 3y' - 4y = 2e^{-t}$

- We seek Y satisfying this equation. We begin with

$$Y(t) = Ae^{-t} \Rightarrow Y'(t) = -Ae^{-t},\ Y''(t) = Ae^{-t}$$

- Substituting these derivatives into differential equation,

$$(A + 3A - 4A)e^{-t} = 2e^{-t}$$

- Since the left side of the above equation is always 0, no value of A can be found to make $Y(t) = Ae^{-t}$ a solution to the nonhomogeneous equation.

- To understand why this happens, we will look at the solution of the corresponding homogeneous differential equation

Example 5: Homogeneous Solution (2 of 3)

- To solve the corresponding homogeneous equation:

$$y'' - 3y' - 4y = 0$$

- We use the techniques from Section 3.1 and get

$$y_1(t) = e^{-t} \text{and } y_2(t) = e^{4t}$$

- Thus our assumed particular solution $Y(t) = Ae^{-t}$ solves the homogeneous equation instead of the nonhomogeneous equation.

- So we need another form for $Y(t)$ to arrive at the general solution of the form:

$$y(t) = c_1 e^{-t} + c_2 e^{4t} + Y(t)$$

$$y'' - 3y' - 4y = 2e^{-t}$$

Example 5: Particular Solution (3 of 3)

- Our next attempt at finding a $Y(t)$ is:

$$Y(t) = Ate^{-t}$$
$$Y'(t) = Ae^{-t} - Ate^{-t}$$
$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = Ate^{-t} - 2Ae^{-t}$$

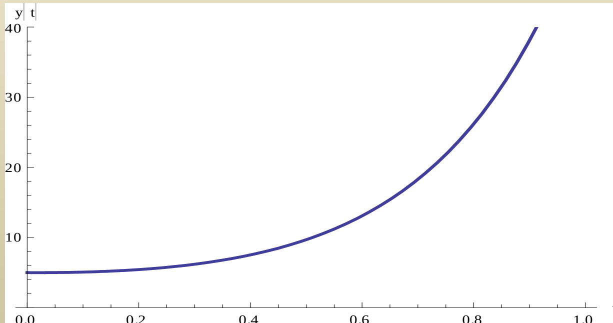
- Substituting these into the ODE,

$$Ate^{-t} - 2Ae^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$$
$$0 \cdot Ate^{-t} - 5Ate^{-t} = -5Ate^{-t} = 2e^{-t}$$
$$\Rightarrow A = -2/5$$
$$\Rightarrow Y(t) = -\frac{2}{5}te^{-t}$$

- So the general solution to the IVP is

$$y(t) = c_1 e^{-t} + c_2 e^{4t} - 2/5\, t e^{-t}$$

$$y(t) = 4e^{-t} + e^{4t} - 2/5\, t e^{-t}$$



Summary – Undetermined Coefficients (1 of 2)

- For the differential equation

$$ay'' + by' + cy = g(t)$$

where a, b, and c are constants, if $g(t)$ belongs to the class of functions discussed in this section (involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of these), the method of undetermined coefficients may be used to find a particular solution to the nonhomogeneous equation.

- The **first step** is to find the general solution for the corresponding homogeneous equation with $g(t) = 0$.

$$y_C(t) = c_1 y_1(t) + c_2 y_2(t)$$

Summary – Undetermined Coefficients (2 of 2)

- The **second step** is to select an appropriate form for the particular solution, $Y(t)$, to the non-homogeneous equation and determine the derivatives of that function.
- After substituting $Y(t)$, $Y'(t)$, and $Y''(t)$ into the non-homogeneous differential equation, if the form for $Y(t)$ is correct, all the coefficients in $Y(t)$ can be determined.
- Finally, the general solution to the non-homogeneous differential equation can be written as

$$y_{gen}(t) = y_C(t) + Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

(Example 6) Find particular solutions of the ODEs.

(1) $$y'' + 4y' + 13y = e^{-2t}\cos(3t)$$

(2) $$y'' - 4y' + 4y = e^{2t}$$