3.6 Variation of Parameters

- Recall the non-homogeneous equation $$y'' + p(t)y' + q(t)y = g(t)$$

  where p, q, g are continuous functions on an open interval I.

- The associated homogeneous equation is $$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the **variation of parameters** method to solve the non-homogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to the homogeneous equation.

- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.

(Example 1) Consider $y'' - 4y = 3e^{-2t}$

(1) Find a general solution (common solution) of the homogeneous equation.

(2) Find a particular solution of the nonhomogeneous equation by using the method of variation of parameters: $Y(t) = Ate^{-2t}$

(Example 2) Consider $t^2 y'' - 6y = t^2$

(1) Find a general solution (common solution) of the homogeneous equation.

(2) Find a particular solution of the nonhomogeneous equation.

(Example 3) Consider $y'' + 4y = 3\csc(t)$

(1) Find a general solution (common solution) of the homogeneous equation.

(2) Find a particular solution of the nonhomogeneous equation.

Example 1: Variation of Parameters (1 of 6)

- We seek a particular solution to the equation below.

$$y'' + 4y = 3\csc(t)$$

- We cannot use the undetermined coefficients method since $g(t)$ is a quotient of $\sin(t)$ or $\cos(t)$, instead of a sum or product.

- Recall that the solution to the homogeneous equation is

$$y_C(t) = c_1\cos(2t) + c_2\sin(2t)$$

- To find a particular solution to the non-homogeneous equation, we begin with the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

- Then $y'(t) = u_1'(t)\cos(2t) - 2u_1(t)\sin(2t) + u_2'(t)\sin(2t) + 2u_2(t)\cos(2t)$

- or $y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$

Example 1: Derivatives, 2nd Equation (2 of 6)

- From the previous slide,

$$y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t$$

- Note that we need two equations to solve for u_1 and u_2. The first equation is the differential equation. To get a second equation, we will require

- Then $u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$

- Next, $y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t$

$$y''(t) = -2u_1'(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 4u_2(t)\sin 2t$$

Example 1: Two Equations (3 of 6)

- Recall that our differential equation is $y'' + 4y = 3\csc t$

- Substituting y'' and y into this equation, we obtain

$$-2u_1'(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 4u_2(t)\sin 2t$$
$$+4\big(u_1(t)\cos 2t + u_2(t)\sin 2t\big) = 3\csc t$$

- This equation simplifies to $-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$

- Thus, to solve for u_1 and u_2, we have the two equations:

$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$
$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$

Example 1: Solve for u_1' (4 of 6)

- To find u_1 and u_2 , we first need to solve for u_1' and u_2'

$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$
$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$

- From second equation, $u_2'(t) = -u_1'(t)\dfrac{\cos 2t}{\sin 2t}$

- Substituting this into the first equation,

$$-2u_1'(t)\sin 2t + 2\left[-u_1'(t)\frac{\cos 2t}{\sin 2t}\right]\cos 2t = 3\csc t$$
$$-2u_1'(t)\sin^2(2t) - 2u_1'(t)\cos^2(2t) = 3\csc t \sin 2t$$
$$-2u_1'(t)\left[\sin^2(2t) + \cos^2(2t)\right] = 3\left[\frac{2\sin t\cos t}{\sin t}\right]$$
$$u_1'(t) = -3\cos t$$

Example 1: Solve for u_1 and u_2 (5 of 6)

- From the previous slide,

$$u_1'(t) = -3\cos t, \quad u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$

- Then

$$u_2'(t) = 3\cos t\left[\frac{\cos 2t}{\sin 2t}\right] = 3\cos t\left[\frac{1-2\sin^2 t}{2\sin t\cos t}\right] = 3\left[\frac{1-2\sin^2 t}{2\sin t}\right]$$

$$= 3\left[\frac{1}{2\sin t} - \frac{2\sin^2 t}{2\sin t}\right] = \frac{3}{2}\csc t - 3\sin t$$

- Thus

$$u_1(t) = \int u_1'(t)dt = \int -3\cos t dt = -3\sin t + c_1$$

$$u_2(t) = \int u_2'(t)dt = \int\left(\frac{3}{2}\csc t - 3\sin t\right)dt = -\frac{3}{2}\ln\left|\csc t + \cot t\right| + 3\cos t + c_2$$

Example 1: General Solution (6 of 6)

- Recall our equation and homogeneous solution y_C:

$$y'' + 4y = 3\csc t, \quad y_C(t) = c_1 \cos 2t + c_2 \sin 2t$$

- Using the expressions for u_1 and u_2 on the previous slide, the general solution to the differential equation is

$$\begin{aligned}
y(t) &= u_1(t)\cos 2t + u_2(t)\sin 2t + y_C(t) \\
&= -3\sin t\cos 2t - \frac{3}{2}\ln\left|\csc t + \cot t\right|\sin 2t + 3\cos t\sin 2t + y_C(t) \\
&= 3\left[\cos t\sin 2t - \sin t\cos 2t\right] - \frac{3}{2}\ln\left|\csc t + \cot t\right|\sin 2t + y_C(t) \\
&= 3\left[2\sin t\cos^2 t - \sin t\left(2\cos^2 t - 1\right)\right] - \frac{3}{2}\ln\left|\csc t + \cot t\right|\sin 2t + y_C(t) \\
&= 3\sin t - \frac{3}{2}\ln\left|\csc t + \cot t\right|\sin 2t + c_1\cos 2t + c_2\sin 2t
\end{aligned}$$

Summary

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Suppose y_1, y_2 are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on y'' is 1.

- To find u_1 and u_2, we need to solve the equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

- Doing so, and using the Wronskian, we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

- Thus

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_2$$

Theorem 3.6.1

- Consider the equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

- If the functions p, q and g are continuous on an open interval I, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt$$

and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

(Quiz) Find general solutions of the ODEs

(1) $$2y'' + 2y' + 3y = 0$$

(2) $$y'' - 8y' + 16y = 0$$