Ch 3.7: Mechanical & Electrical Vibrations

- Two important areas of application for second order linear equations with constant coefficients are in modeling mechanical and electrical oscillations.
- We will study the motion of a mass on a spring in detail.
- An understanding of the behavior of this simple system is the first step in investigation of more complex vibrating systems.



Spring – Mass System

- Suppose a mass *m* hangs from a vertical spring of original length *l*. The mass causes an elongation *L* of the spring.
- The force F_G of gravity pulls the mass down. This force has magnitude mg, where g is acceleration due to gravity.
- The force F_s of the spring stiffness pulls the mass up. For small elongations *L*, this force is proportional to *L*. That is, $F_s = -kL$ (Hooke's Law).
- When the mass is in equilibrium, $w + F_s = 0$, the forces balance each other :



Spring Model



- We will study the motion of a mass when it is acted on by an external force (forcing function) and/or is initially displaced.
- Let u(t) denote the displacement of the mass from its equilibrium position at ٠ time t, measured downward.
- Let f be the net force acting on the mass. We will use Newton's 2^{nd} Law: mu''(t) = f(t)
- In determining *f*, there are four separate forces to consider:
 - Weight: (downward force) w = mg
 - Spring force: $F_s = -k(L+u)$ (up or down force, see next slide)
 - Damping force: $F_d(t) = -\gamma u'(t)$
 - External force: F(t)

- (up or down, see following slide) (up or down force, see text)

Spring Model: Spring Force Details



- The spring force F_s acts to restore a spring to the natural position, and is proportional to L + u. If L + u > 0, then the spring is extended and the spring force acts upward. In this case $F_s = -k(L+u), L+u > 0$.
- If L + u < 0, then spring is compressed a distance of |L + u|, and the spring force acts downward. In this case $F_s = k|L+u| = k[-(L+u)] = -k(L+u)$
- In either case, $F_s = -k(L+u)$

Spring Model: Damping Force Details



- The damping or resistive force F_d acts in the opposite direction as the motion of the mass. This can be complicated to model. F_d may be due to air resistance, internal energy dissipation due to action of spring, friction between the mass and guides, or a mechanical device (dashpot) imparting a resistive force to the mass.
- We simplify this and assume F_d is proportional to the velocity.
- In particular, we find that
 - If u' > 0, then *u* is increasing, so the mass is moving downward. Thus F_d acts upward and hence $F_d = -\gamma u'$, where $\gamma > 0$.
 - If u' < 0, then *u* is decreasing, so the mass is moving upward. Thus F_d acts downward and hence $F_d = -\gamma u', \gamma > 0$.
- In either case, $F_d(t) = -\gamma u'(t), \quad \gamma > 0.$

Spring Model: Differential Equation



• Taking into account these forces, Newton's Law becomes:

 $mu''(t) = mg + F_s(t) + F_d(t) + F(t) = mg - k [L + u(t)] - \gamma u'(t) + F(t)$

- Recalling that mg = kL, this equation reduces to $mu''(t) + \gamma u'(t) + ku(t) = F(t)$ where the constants *m*, γ , and *k* are positive.
- We can prescribe initial conditions also: $u(0) = u_0, u'(0) = v_0$
- It follows from Theorem 3.2.1 that there is a unique solution to this initial value problem. Physically, if the mass is set in motion with a given initial displacement and velocity, then its position is uniquely determined at all future times.

(Question) How do we compute the constants m, γ , and k?

(Example) An object weighing 4 lb-force stretches a spring 2". The mass is displaced an additional 6" and then released; and is in a medium that exerts a viscous resistance of 6 lb-force when the mass has a velocity of 3 ft/sec. Formulate the initial value problem (IVP) that governs the motion of this object.

-
$$w = mg$$
 $g = 32$ ft/sec²

- $F_s = -kL$ $F_d(t) = -\gamma u'(t), \quad \gamma > 0.$
- Initial conditions ?

Example 1: Find Coefficients (1 of 2)



• A mass weighing 4 lb-force stretches a spring 2". The mass is displaced an additional 6" and then released; and is in a medium that exerts a viscous resistance of 6 lb-force when the object has a velocity of 3 ft/sec. Formulate the IVP that governs the motion of this object:

 $mu''(t) + \gamma u'(t) + ku(t) = F(t), \quad u(0) = u_0, \ u'(0) = v_0$

• Find
$$m: \quad w = mg \implies m = \frac{w}{g} \implies m = \frac{4 \text{ lb}}{32 \text{ ft} / \sec^2} \implies m = \frac{1}{8} \frac{\text{ lb sec}^2}{\text{ ft}}$$

• Find
$$\gamma: \quad \gamma = -\frac{F_d}{u'} \implies \gamma = \frac{61b}{3ft/sec} \implies \gamma = 2\frac{1bsec}{ft}$$

• Find k: $k = -\frac{F_s}{L}$, $F_s = -w \implies k = \frac{4 \text{lb}}{2 \text{in}} \implies k = \frac{4 \text{lb}}{1/6 \text{ft}} \implies k = 24 \frac{\text{lb}}{\text{ft}}$

Example 1: Find IVP (2 of 2)

• Thus our differential equation becomes

$$\frac{1}{8}u''(t) + 2u'(t) + 24u(t) = 0$$

and hence the initial value problem can be written as (unit: ft)

$$u''(t) + 16u'(t) + 192u(t) = 0$$

$$u(0) = \frac{1}{2}, \quad u'(0) = 0$$



Example 1: Find IVP (2 of 2)



• Thus our differential equation becomes

$$\frac{1}{8}u''(t) + 2u'(t) + 24u(t) = 0$$

and hence the initial value problem can be written as

$$u''(t) + 16u'(t) + 192u(t) = 0$$

$$u(0) = \frac{1}{2}, \quad u'(0) = 0$$

• This problem can be solved using the methods of Chapter 3.3 and yields the solution:

$$u(t) = \frac{1}{2}e^{-8t}\cos(8\sqrt{2}t) + \frac{1}{2\sqrt{2}}e^{-8t}\sin(8\sqrt{2}t)$$

(Example 2) A mass weighing 10 lb-force stretches a spring 2". The mass is displaced an additional 2" and then set in motion with an initial upward velocity of 1 ft/sec. Determine the position of the mass at any later time, and find the period, amplitude, and phase of the motion.

Example 2: Find IVP (1 of 2)

• A mass weighing 10 lb-force stretches a spring 2". The mass is displaced an additional 2" and then set in motion with an initial upward velocity of 1 ft/sec. Determine the position of the mass at any later time, and find the period, amplitude, and phase of the motion: $\gamma = 0$

$$mu''(t) + ku(t) = 0, \ u(0) = u_0, \ u'(0) = v_0$$

• Find *m*: $w = mg \implies m = \frac{w}{g} \implies m = \frac{101b}{32 \text{ft/sec}^2} \implies m = \frac{5}{16} \frac{10 \text{ sec}^2}{\text{ft}}$

• Find k:
$$F_s = -k L \Rightarrow k = \frac{101b}{2in} \Rightarrow k = \frac{101b}{1/6ft} \Rightarrow k = 60\frac{1b}{ft}$$

• Thus our IVP is

5/16u''(t) + 60u(t) = 0, u(0) = 1/6, u'(t) = -1

Example 2: Find Solution (2 of 2)

• Simplifying, we obtain u''(t) + 192u(t) = 0, u(0) = 1/6, u'(0) = -1

• To solve, use methods of Ch 3.3 to obtain

$$u(t) = \frac{1}{6} \cos \sqrt{192} \ t - \frac{1}{\sqrt{192}} \sin \sqrt{192} \ t$$

or

$$u(t) = \frac{1}{6}\cos(8\sqrt{3}) t - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}) t$$



Spring Model: Undamped Free Vibrations (1 of 4)

• Recall our differential equation for spring motion:

 $mu''(t) + \gamma u'(t) + ku(t) = F(t)$

• Suppose there is no external driving force and **no damping**. Then F(t) = 0and $\gamma = 0$, and our equation becomes mu''(t) + ku(t) = 0

• The general solution to this equation is

 $u(t) = A\cos\omega_0 t + B\sin\omega_0 t,$ where

$$\omega_0^2 = k / m$$



Spring Model: Undamped Free Vibrations (2 of 4)

• Using trigonometric identities, the solution

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t, \quad \omega_0^2 = k / m$$

can be rewritten as follows:

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \iff u(t) = R\cos(\omega_0 t - \delta)$$
$$\Leftrightarrow u(t) = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t,$$

where

$$A = R\cos\delta, \ B = R\sin\delta \implies R = \sqrt{A^2 + B^2}, \ \tan\delta = \frac{B}{A}$$

• Note that in finding δ , we must be careful to choose the correct quadrant. This is done using the signs of $\cos \delta$ and $\sin \delta$.

•
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Spring Model: Undamped Free Vibrations (3 of 4)

• Thus our solution is $u(t) = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta)$

where
$$\omega_0 = \sqrt{k/m}$$

• The solution is a shifted cosine (or sine) curve, that describes simple harmonic motion, with period 2π \sqrt{m}

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}}$$

• The circular frequency ω_0 (radians/time) is the **natural frequency** of the vibration, *R* is the **amplitude** of the maximum displacement of mass from equilibrium, and δ is the **phase** or phase angle (dimensionless).

Spring Model: Undamped Free Vibrations (4 of 4)

• Note that our solution $u(t) = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta), \quad \omega_0 = \sqrt{k/m}$

is a shifted cosine (or sine) curve with period $T = 2\pi \sqrt{\frac{m}{k}}$

- Initial conditions determine *A* & *B*, hence also the amplitude *R*.
- The system always vibrates with the same frequency ω_0 , regardless of the initial conditions.
- The period *T* increases as *m* increases, so larger masses vibrate more slowly. However, *T* decreases as *k* increases, so stiffer springs cause a system to vibrate more rapidly.

Example 2: Find IVP (1 of 3)

• A mass weighing 10 lb-force stretches a spring 2". The mass is displaced an additional 2" and then set in motion with an initial upward velocity of 1 ft/sec. Determine the position of the mass at any later time, and find the period, amplitude, and phase of the motion: $\gamma = 0$

$$mu''(t) + ku(t) = 0, \ u(0) = u_0, \ u'(0) = v_0$$

• Find *m*: $w = mg \implies m = \frac{w}{g} \implies m = \frac{101b}{32 \text{ft/sec}^2} \implies m = \frac{5}{16} \frac{16 \text{sec}^2}{\text{ft}}$

• Find k:
$$F_s = -k L \Rightarrow k = \frac{101b}{2in} \Rightarrow k = \frac{101b}{1/6ft} \Rightarrow k = 60\frac{1b}{ft}$$

• Thus our IVP is

5/16u''(t) + 60u(t) = 0, u(0) = 1/6, u'(t) = -1

Example 2: Find Solution (2 of 3)

• Simplifying, we obtain u''(t) + 192u(t) = 0, u(0) = 1/6, u'(0) = -1

• To solve, use methods of Ch 3.3 to obtain

$$u(t) = \frac{1}{6} \cos \sqrt{192} \ t - \frac{1}{\sqrt{192}} \sin \sqrt{192} \ t$$

or

$$u(t) = \frac{1}{6}\cos(8\sqrt{3}) t - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}) t$$



Example 2: $u(t) = \frac{1}{6}\cos 8\sqrt{3}t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$ **Find Period, Amplitude, Phase (3 of 3)**

- The natural frequency is $\omega_0 = \sqrt{k/m} = \sqrt{192} = 8\sqrt{3} \approx 13.856 \text{ rad/sec}$
- The period is $T = 2\pi / \omega_0 \cong 0.45345 \text{sec}$
- The amplitude is $R = \sqrt{A^2 + B^2} \approx 0.18162$ ft
- Next, determine the phase δ :

 $A = R\cos\delta, B = R\sin\delta, \tan\delta = B/A$

$$\tan \delta = \frac{B}{A} \Rightarrow \tan \delta = \frac{-\sqrt{3}}{4} \Rightarrow \delta = \tan^{-1} \left(\frac{-\sqrt{3}}{4}\right) \approx -0.40864 \, \text{rad}$$

Thus $u(t) = 0.182 \cos(8\sqrt{3}t + 0.409)$



Spring Model: Damped Free Vibrations (1 of 8)

• Suppose there is damping but no external driving force F(t):

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

- What is effect of the damping coefficient γ on system?
- The characteristic equation is $r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 4mk}}{2m} = \frac{\gamma}{2m} \left[-1 \pm \sqrt{1 \frac{4mk}{\gamma^2}} \right]$
- Three cases for the solution:

$$\begin{aligned} \gamma^2 - 4mk > 0: \quad u(t) &= Ae^{r_1 t} + Be^{r_2 t}, \text{ where } r_1 < 0, r_2 < 0; \\ \gamma^2 - 4mk &= 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \text{ where } \gamma/2m > 0; \\ \gamma^2 - 4mk < 0: \quad u(t) &= e^{-\gamma t/2m} (A\cos\mu t + B\sin\mu t), \ \mu &= \frac{\sqrt{4mk - \gamma^2}}{2m} > 0. \end{aligned}$$

Note: In all three cases, $\lim_{t\to\infty} u(t) = 0$, as expected from the damping term.

Damped Free Vibrations: Small Damping (2 of 8)

• Of the cases for solution form, the last is most important, which occurs when the damping is small: $\gamma^2 - 4mk > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$, $r_1 < 0$, $r_2 < 0$

$$\gamma^{2} - 4mk > 0: \quad u(t) = Ae^{r_{1}t} + Be^{r_{2}t}, \quad r_{1} < 0, \quad r_{2} < 0$$

$$\gamma^{2} - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma / 2m > 0$$

$$\gamma^{2} - 4mk < 0: \quad u(t) = e^{-\gamma t/2m} (A\cos\mu t + B\sin\mu t), \quad \mu > 0$$

• We examine this last case. Recall $A = R\cos\delta$, $B = R\sin\delta$

• Then
$$u(t) = R e^{-\gamma t/2m} \cos(\mu t - \delta)$$

and hence $|u(t)| \leq R e^{-\gamma t/2m}$

(damped oscillation)



Damped Free Vibrations: Quasi Frequency (3 of 8)

• Thus we have damped oscillations:

$$u(t) = R e^{-\gamma t/2m} \cos(\mu t - \delta) \implies |u(t)| \le R e^{-\gamma t/2m}$$

• The amplitude *R* depends on the initial conditions, since

$$u(t) = e^{-\gamma t/2m} (A\cos\mu t + B\sin\mu t), \ A = R\cos\delta, \ B = R\sin\delta$$

- Although the motion is not periodic, the parameter μ determines the mass oscillation frequency.
- Thus μ is called the **quasi frequency**.
- Recall

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$$



Damped Free Vibrations: Quasi Period (4 of 8)

• Compare μ with ω_0 , the frequency of undamped motion:

$$\frac{\mu}{\omega_0} = \frac{\sqrt{4km - \gamma^2}}{2m\sqrt{k/m}} = \frac{\sqrt{4km - \gamma^2}}{\sqrt{4m^2}\sqrt{k/m}} = \frac{\sqrt{4km - \gamma^2}}{\sqrt{4km}} = \sqrt{1 - \frac{\gamma^2}{4km}}$$

For small γ
$$\approx \sqrt{1 - \frac{\gamma^2}{4km} + \frac{\gamma^4}{64k^2m^2}} = \sqrt{\left(1 - \frac{\gamma^2}{8km}\right)^2} = 1 - \frac{\gamma^2}{8km}$$

- Thus, small damping reduces oscillation frequency slightly.
- Similarly, the **quasi period** is defined as $T_d = 2\pi/\mu$. Then

$$\frac{T_d}{T} = \frac{2\pi/\mu}{2\pi/\omega_0} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1 - \frac{\gamma^2}{8km}\right)^{-1} \cong 1 + \frac{\gamma^2}{8km}$$

• Thus, small damping increases quasi period.

Damped Free Vibrations: Neglecting Damping for Small $\gamma^2/4km$ (5 of 8)

• Consider again the comparisons between damped and undamped frequency and period: 1/2

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \ \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

- Thus it turns out that a small γ is not as telling as a small ratio $\gamma^2/4km$.
- For small $\gamma^{2/4km}$, we can neglect the effect of damping when calculating the quasi frequency and quasi period of motion. But if we want a detailed description of the motion of the mass, then we cannot neglect the damping force, no matter how small it is.

Damped Free Vibrations: Frequency, Period (6 of 8)

• Ratios of damped and undamped frequency, period:

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \ \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

• Thus
$$\lim_{\gamma \to 2\sqrt{km}} \mu = 0$$
 and $\lim_{\gamma \to 2\sqrt{km}} T_d = \infty$

• The importance of the relationship between γ^2 and 4km is supported by our previous equations:

$$\gamma^{2} - 4mk > 0: \quad u(t) = Ae^{r_{1}t} + Be^{r_{2}t}, \quad r_{1} < 0, \quad r_{2} < 0$$

$$\gamma^{2} - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma/2m > 0$$

$$\gamma^{2} - 4mk < 0: \quad u(t) = e^{-\gamma t/2m} (A\cos\mu t + B\sin\mu t), \quad \mu > 0$$

Damped Free Vibrations: Critical Damping Value (7 of 8)

- Thus the nature of the solution changes as γ passes through the value $2\sqrt{km}$.
- This value of γ is known as the critical damping value, and for larger values of γ the motion is said to be overdamped.
- Thus for the solutions given by these cases,

$$\gamma^2 - 4mk > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0, \quad r_2 < 0$$
 (1)

$$\gamma^2 - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma/2m > 0$$
 (2)

$$\gamma^2 - 4mk < 0: \ u(t) = e^{-\gamma t/2m} (A\cos\mu t + B\sin\mu t), \ \mu > 0$$
 (3)

we see that the mass creeps back to its equilibrium position for solutions (1) and (2), but does not oscillate about it, as it does for small γ in solution (3).

• Soln (1) is overdamped and soln (2) is critically damped.

Damped Free Vibrations: Characterization of Vibration (8 of 8)

• The mass creeps back to the equilibrium position for solutions (1) & (2), but does not oscillate about it, as it does for small γ in solution (3).

$$u^2 - 4mk > 0: u(t) = Ae^{r_1 t} + Be^{r_2 t}, r_1 < 0, r_2 < 0$$
 (Green) (1)

$$\gamma^2 - 4mk = 0: \ u(t) = (A + Bt)e^{-\gamma t/2m}, \ \gamma/2m > 0$$
 (Red, Black) (2)

$$\gamma^2 - 4mk < 0: \quad u(t) = e^{-\gamma t/2m} \left(A \cos \mu t + B \sin \mu t \right) \quad \text{(Blue)} \tag{3}$$

• Solution (1) is overdamped and

Y

- Solution (2) is critically damped.
- Solution (3) is underdamped



Example 3: Initial Value Problem (1 of 4)

- Suppose that the motion of a spring-mass system is governed by the initial value problem u'' + 0.125u' + u = 0, u(0) = 2, u'(0) = 0
- Find the following:

(a) quasi frequency and quasi period;
(b) time at which mass passes through equilibrium position;
(c) time *τ* such that |u(t)| < 0.1 for all t > τ.

• For Part (a), using methods of this chapter we obtain:

$$u(t) = e^{-t/16} \left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t \right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

where

$$\tan \delta = \frac{1}{\sqrt{255}} \Rightarrow \delta \cong 0.06254 \quad (\operatorname{recall} A = R\cos\delta, B = R\sin\delta)$$

Example 3: Quasi Frequency & Period (2 of 4)

• The solution to the initial value problem is:

$$u(t) = e^{-t/16} \left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t \right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

- The graph of this solution, along with solution to the corresponding undamped problem, is given below.
- The quasi frequency is $\mu = \sqrt{255}/16 \approx 0.998$

and quasi period is $T_d = 2\pi / \mu \cong 6.295$

• For the undamped case:

 $\omega_0 = 1, T = 2\pi \cong 6.283$



Example 3: Quasi Frequency & Period (3 of 4)

• The damping coefficient is $\gamma = 0.125 = 1/8$, and this is 1/16 of the critical value $2\sqrt{km} = 2$

- Thus damping is small relative to mass and spring stiffness. Nevertheless the oscillation amplitude diminishes quickly.
- Using a solver, we find that |u(t)| < 0.1for $t > \tau \approx 47.515$ sec



Example 3: Quasi Frequency & Period (4 of 4)

• To find the time at which the mass first passes through the equilibrium position, we must solve

$$u(t) = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16}t - \delta\right) = 0$$

• Or more simply, solve

$$\frac{\sqrt{255}}{16}t - \delta = \frac{\pi}{2}$$
$$\Rightarrow t = \frac{16}{\sqrt{255}} \left(\frac{\pi}{2} + \delta\right) \approx 1.637 \,\text{sec}$$





Electric Circuits

• The flow of current in certain basic electrical circuits is modeled by second order linear ODEs with constant coefficients:

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t)$$
$$I(0) = I_0, \ I'(0) = I'_0$$

- It is interesting that the flow of current in this circuit is mathematically equivalent to motion of spring-mass system.
- For more details, see text.

