4.1: Higher Order Linear ODEs: General Theory

- An ***n*th order ODE** has the general form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

- We assume that $P_0, \ldots, P_n$, and G are continuous real-valued functions on some interval $I = (\alpha, \beta)$, and that P_0 is nowhere zero on I.

- Dividing by P_0, the ODE becomes

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

- For an *n*th order ODE, there are typically n initial conditions:

$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Theorem 4.1.1

- Consider the *n*th order initial value problem

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$
$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

- If the functions $p_1, \ldots, p_n$, and g are continuous on an open interval I, then there exists exactly one solution $y = \phi(t)$ that satisfies the initial value problem. This solution exists throughout the interval I.

Homogeneous Equations (Superposition)

- As with 2nd order case, we begin with homogeneous ODE:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0$$

- If $y_1, \ldots, y_n$ are solutions to ODE, then so is linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Every solution can be expressed in this form, with coefficients determined by initial conditions, iff we can solve:

$$\begin{aligned}
c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\
c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\
&\vdots \\
c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)}
\end{aligned}$$

Homogeneous Equations & Wronskian

- The system of equations on the previous slide has a unique solution iff its determinant, or Wronskian, is nonzero at t_0:

$$W(y_1, y_2, \ldots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix}$$

- Since t_0 can be any point in the interval I, the Wronskian determinant needs to be nonzero at every point in I.

- As before, it turns out that the Wronskian is either zero for every point in I, or it is never zero on I.

Theorem 4.1.2

- Consider the nth order initial value problem

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0$$
$$y(t_0) = y_0,\; y'(t_0) = y_0',\; \ldots,\; y^{(n-1)}(t_0) = y^{(n-1)}$$

- If the functions $p_1, \ldots, p_n$ are continuous on an open interval I, and if $y_1, \ldots, y_n$ are solutions with $W(y_1, \ldots, y_n)(t) \neq 0$ for at least one t in I, then every solution y of the ODE can be expressed as a linear combination of $y_1, \ldots, y_n$:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

(Example) In each problems, determine intervals in which solutions are sure to exist.

(1) $2u'' + 10u' + 3u = 20t$

(2) $t^2u'' + tu' + t^3u = 20\sin(t)$

(3) $(t^2 - 1)u''' + tu' + 8u = 2e^{4t}$

(4) $tu^{(4)} + \sqrt{2t-4}\; u'' + t^3u = \cos(3t)$

(Question) How do we find general solutions of (3) & (4)?

Linear Dependence and Independence

- Two functions f and g are **linearly dependent** if there exist constants c_1 and c_2, not both zero, such that $c_1 f(t) + c_2 g(t) = 0$ for all t in I.

  Note that this reduces to determining whether f and g are multiples of each other.

- If the only solution to this equation is $c_1 = c_2 = 0$, then f and g are **linearly independent**.

- For example, let $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$, and consider the linear combination $c_1 \sin 2x + c_2 \sin x \cos x = 0$

  This equation is satisfied if we choose $c_1 = 1$, $c_2 = -2$, and hence f and g are linearly dependent.

- f and g are **linearly independent** if and only if $W(f, g) \neq 0$

Example 1

- Are the following functions linearly independent or dependent on the interval I:

$$f_1(t) = 1,\ f_2(t) = t,\ f_3(t) = t^2 \qquad 0 < t < \infty$$

- Form the linear combination and set it equal to zero: $k_1 + k_2 t + k_3 t^2 = 0$

- Evaluating this at t = 0, t = 1, and t = =1, we get

$$\begin{aligned} k_1 &= 0 \\ k_1 + k_2 + k_3 &= 0 \\ k_1 - k_2 + k_3 &= 0 \end{aligned} \qquad k_1 = k_2 = k_3 = 0$$

- Therefore, the given functions are linearly independent

- In other words,

$$W(1, t, t^2) = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

Example 2

- Are the following functions linearly independent or dependent on any interval I:

$$f_1(t) = 1,\ f_2(t) = 2 + t,\ f_3(t) = 3 - t^2,\ f_4(t) = 4t + t^2$$

- Form the linear combination and set it equal to zero

$$k_1 + k_2(2+t) + k_3(3-t^2) + k_4(4t+t^2) = 0$$

- Evaluating this at t = 0, t = 1, and t = -1, we get

$$k_1 + 2k_2 + k_3 = 0$$
$$k_2 + 4k_4 = 0$$
$$-k_3 + k_4 = 0$$

- There are many nonzero solutions to this system of equations

- Therefore, the given functions are linearly dependent:

$$W(f_1, f_2, f_3, f_4) = 0$$

Theorem 4.1.3

- If $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions of

$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

on an interval I, then $\{y_1, \ldots, y_n\}$ are linearly independent on that interval.

- Conversely, if $\{y_1, \ldots, y_n\}$ are linearly independent solutions to the above differential equation, then they form a fundamental set of solutions on the interval I

Fundamental Solutions & Linear Independence

- Consider the nth order ODE: $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$

- A set $\{y_1,\ldots, y_n\}$ of solutions with $W(y_1,\ldots, y_n) \neq 0$ on I is called a **fundamental set of solutions**.

- Since all solutions can be expressed as a linear combination of the fundamental set of solutions, the **general solution** is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- If $y_1,\ldots, y_n$ are fundamental solutions, then $W(y_1,\ldots, y_n) \neq 0$ on I. It can be shown that this is equivalent to saying that $y_1,\ldots, y_n$ are **linearly independent**:

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0 \text{ iff } c_1 = c_2 = \cdots = c_n = 0$$

Nonhomogeneous Equations

- Consider the nonhomogeneous equation:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t) y = g(t)$$

- If Y_1, Y_2 are solutions to nonhomogeneous equation, then Y_1 - Y_2 is a solution to the homogeneous equation:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$$

- Then there exist coefficients $c_1, \ldots, c_n$ such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Thus the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$$

where Y is any particular solution to nonhomogeneous ODE.