4.2: Homogeneous Equations with Constant Coefficients

- Consider the nth order linear homogeneous differential equation with constant, real coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- As with second order linear equations with constant coefficients, $y = e^{rt}$ is a solution for values of r that make characteristic polynomial $Z(r)$ zero:

$$L[e^{rt}] = e^{rt} \underbrace{\left[a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right]}_{\text{characteristic polynomial } Z(r)} = 0$$

- By the fundamental theorem of algebra, a polynomial of degree n has n roots $r_1, r_2, \ldots, r_n$, and hence

$$Z(r) = a_0 (r - r_1)(r - r_2) \cdots (r - r_n)$$

Real and Unequal Roots (case 1)

- If roots of characteristic polynomial $Z(r)$ are real and unequal, then there are n distinct solutions of the differential equation:

$$e^{r_1 t},\ e^{r_2 t},\ \ldots,\ e^{r_n t}$$

- If these functions are linearly independent, then general solution of differential equation is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$$

- The Wronskian can be used to determine linear independence of solutions.

(Example) Find general solutions of the ODE

(1) $$y''' - 3y'' + 2y' = 0$$

(2) $$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Example 1: Distinct Real Roots (1 of 3)

- Consider the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$
$$y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1$$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 + r^3 - 7r^2 - r + 6 = 0$$
$$\Leftrightarrow \quad (r-1)(r+1)(r-2)(r+3) = 0$$

- Thus the general solution is

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}$$

Example 1: Solution (2 of 3)

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_3 e^{-3t}$$

- The initial conditions $y(0) = 1, \; y'(0) = 0, \; y''(0) = -2, \; y'''(0) = -1$

  yield

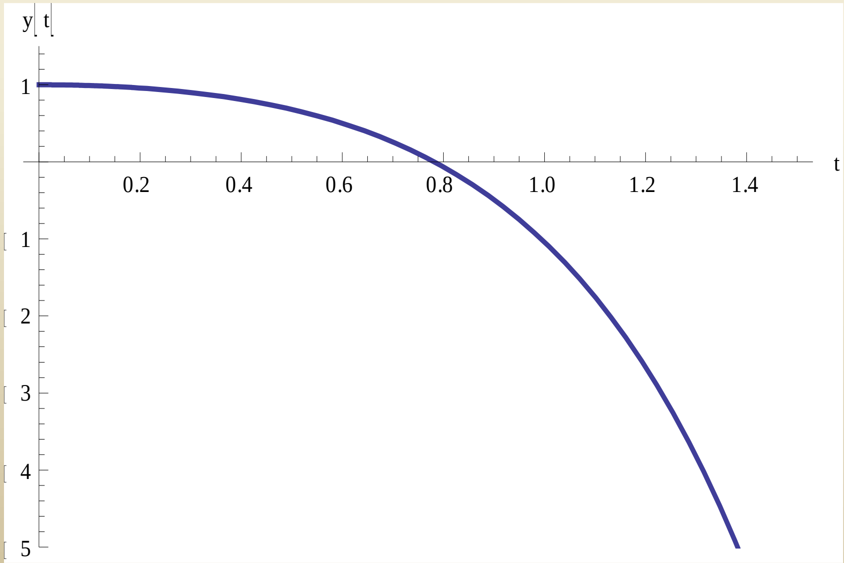
$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0 \\ c_1 + c_2 + 4c_3 + 9c_4 &= -2 \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1 \end{aligned}$$

- Solving, $c_1 = \frac{11}{8}, \; c_2 = \frac{5}{12}, \; c_3 = -\frac{2}{3}, \; c_4 = -\frac{1}{8}$
- Hence $y(t) = \frac{11}{8} e^{t} + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t}$

Example 1: Graph of Solution (3 of 3)

- The graph of the solution is given below. Note the effect of the largest root of the characteristic equation.

$$y(t) = \frac{11}{8}e^{t} + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}$$



Complex Roots

- If the characteristic polynomial $Z(r)$ has complex roots, then they must occur in conjugate pairs, $\lambda \pm i\mu$.

- Note that not all the roots need be complex.

- Solutions corresponding to complex roots have the form

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$$
$$e^{(\lambda-i\mu)t} = e^{\lambda t}\cos\mu t - ie^{\lambda t}\sin\mu t$$

- As in Chapter 3.4, we use the real-valued solutions: $e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t$

(Ex) $y^{(4)} - y = 0,\ \ y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$

Example 2: Complex Roots (1 of 2)

- Consider the initial value problem

$$y^{(4)} - y = 0,\ \ y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$$

- Then $\quad y(t) = e^{rt} \quad \Rightarrow \quad r^4 - 1 = 0 \Leftrightarrow \left(r^2 - 1\right)\left(r^2 + 1\right) = 0$

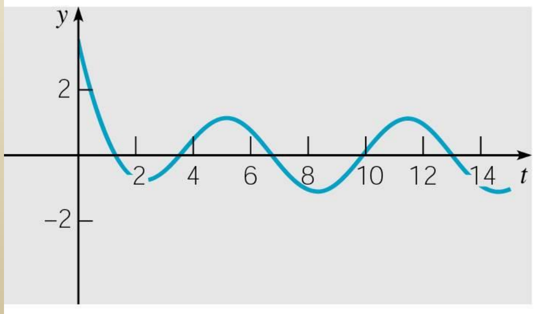
- The roots are 1, -1, *i*, -*i*. Thus the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

- Using the initial conditions, we obtain

$$y(t) = 0e^t + 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$

- The graph of solution is given on right.



Example 2: Small Change in an Initial Condition (2 of 2)

$$y(t) = 0e^{t} + 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$

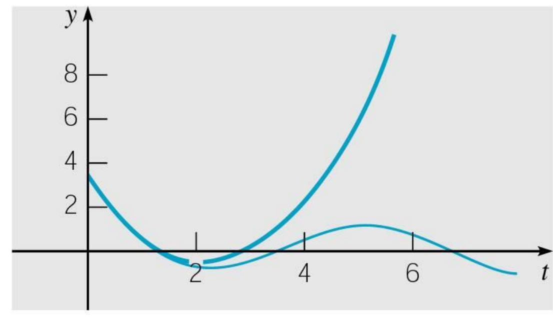
- Note that if one initial condition is slightly modified, then the solution can change significantly. For example, replace

$$y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$$

with $$y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -15/8$$

then $$y(t) = \frac{1}{32}e^{t} + \frac{95}{32}e^{-t} + \frac{1}{2}\cos(t) - \frac{17}{16}\sin(t)$$

- The graph of this solution and original solution are given below.



(EX) Find the general solution of the DE

$$y^{(4)} - 2y'' + y = 0$$

Repeated Roots

- Suppose a root r_k of characteristic polynomial $Z(r)$ is a repeated root with multiplicty s. Then linearly independent solutions corresponding to this repeated root have the form
$$e^{r_k t},\ te^{r_k t},\ t^2 e^{r_k t},\ \ldots,\ t^{s-1}e^{r_k t}$$

- If a complex root $\lambda + i\mu$ is repeated s times, then so is its conjugate $\lambda - i\mu$. There are $2s$ corresponding linearly independent solutions, derived from real and imaginary parts of
$$e^{(\lambda+iu)t},\ te^{(\lambda+iu)t},\ t^2 e^{(\lambda+iu)t},\ \ldots,\ t^{s-1}e^{(\lambda+iu)t}$$

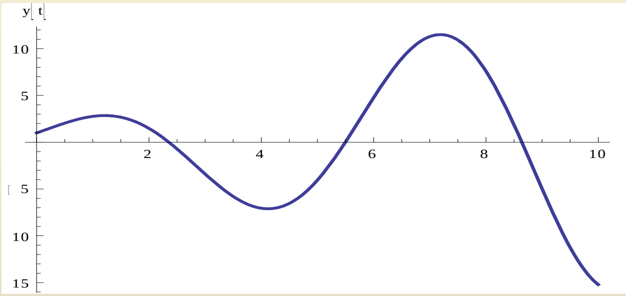
or
$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t,\ te^{\lambda t}\cos\mu t,\ te^{\lambda t}\sin\mu t, \ldots,$$
$$t^{s-1}e^{r_k t}\cos\mu t,\quad t^{s-1}e^{r_k t}e^{\lambda t}\sin\mu t,$$

(Ex) $\quad y^{(4)} + 2y'' + y = 0$

Example 4: Repeated Roots

- Consider the equation $$y^{(4)} + 2y'' + y = 0$$
- Then $$y(t) = e^{rt} \Rightarrow r^4 + 2r + 1 = 0 \Leftrightarrow (r^2+1)(r^2+1) = 0$$
- The roots are *i*, *i*, -*i*, -*i*. Thus the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t)$$



Sample Solution: *y*= (1 + *t*) Cos *t* + (1 + *t*) Sin *t*

(Ex) Find a general solution of the ODE:

$$y^{(4)} + y = 0$$

Example 4: Complex Roots of -1 (1 of 2)

- For the general solution of $y^{(4)} + y = 0$, the characteristic equation is

$$r^4 + 1 = 0$$

- To solve this equation, we need to use Euler's formula to find the four 4th roots of -1:

$$-1 = \cos\pi + i\sin\pi = e^{i\pi} \text{ or}$$

$$-1 = \cos(\pi + 2m\pi) + i\sin(\pi + 2m\pi) = e^{i(\pi + 2m\pi)} \text{ for any integer } m$$

$$(-1)^{1/4} = e^{i(\pi + 2m\pi)/4} = \cos(\pi/4 + m\pi/2) + i\sin(\pi/4 + m\pi/2)$$

- Letting m = 0, 1, 2, and 3, we get the roots:

$$\frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \text{ respectively.}$$

Example 4: Complex Roots of -1 (2 of 2)

$$r = \left\{ \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}} \right\}$$

- Given the four complex roots, extending the ideas from Chapter 4, we can form four linearly independent real solutions.

- For the complex conjugate pair $\frac{1 \pm i}{\sqrt{2}}$, we get the solutions

$$y_1 = e^{t/\sqrt{2}} \cos(t/\sqrt{2}),\ y_2 = e^{t/\sqrt{2}} \sin(t/\sqrt{2})$$

- For the complex conjugate pair $\frac{-1 \pm i}{\sqrt{2}}$, we get the solutions

$$y_3 = e^{-t/\sqrt{2}} \cos(t/\sqrt{2}),\ y_4 = e^{-t/\sqrt{2}} \sin(t/\sqrt{2})$$

- So the general solution can be written as $c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$