4.3: Nonhomogeneous Equations: Method of Undetermined Coefficients

- The method of **undetermined coefficients** can be used to find a particular solution Y of an nth order linear, constant coefficient, nonhomogeneous ODE

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t),$$

  provided g is of an appropriate form.

- As with 2nd order equations, the method of undetermined coefficients is typically used when g is a sum or product of polynomial, exponential, and sine or cosine functions.

(Ex) Find a general solution of the ODE $y''' - 4y' = e^t$.

Example 1

- Consider the differential equation $y''' - 3y'' + 3y' - y = 4e^t$

- For the homogeneous case,

$$y(t) = e^{rt} \quad \Rightarrow \quad r^3 - 3r^2 + 3r - 1 = 0 \Leftrightarrow (r-1)^3 = 0$$

- Thus the general solution of homogeneous equation is

$$y_C(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

- For nonhomogeneous case, keep in mind the form of homogeneous solution. Thus begin with $Y(t) = At^3 e^{2t}$

- As in Chapter 3, it can be shown that

$$Y(t) = \frac{2}{3} t^3 e^{2t} \quad \Rightarrow \quad y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^{2t}$$

Example 2

- Consider the equation $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

- For the homogeneous case, $y(t) = e^{rt} \Rightarrow r^4 + 2r^2 + 1 = 0 \Leftrightarrow (r^2+1)(r^2+1) = 0$

- Thus the general solution of the homogeneous equation is

$$y_c(t) = c_1\cos t + c_2\sin t + c_3 t\cos(t) + c_4 t\sin(t)$$

- For the nonhomogeneous case, because of the form of the solution for the homogeneous equation, we need $Y(t) = t^2(A\sin t + B\cos t)$

- As in Chapter 3, it can be shown that $Y(t) = t^2\left(-\frac{3}{8}\sin t + \frac{5}{8}\cos t\right)$

- Thus, the general solution for the nonhomgeneous equation is

$$y(t) = y_c(t) + Y(t)$$

Example 3

- Consider the equation $y''' - 4y' = t + 3\cos t + e^{-2t}$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^3 - 4r = 0 \Leftrightarrow r\left(r^2 - 4\right) \Leftrightarrow r(r-2)(r+2) = 0$$

- Thus the general solution of homogeneous equation is $y_C(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$

- For nonhomogeneous case, keep in mind form of homogeneous solution. Thus we have two subcases:

$$Y_1(t) = (A + Bt)t,\ Y_2(t) = C\cos t + D\sin t,\ Y_3(t) = Ete^{2t},$$

- As in Chapter 3, can be shown that $Y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}t\,e^{-2t}$
- The general solution is

$$y(t) = y_c(t) + Y(t)$$