6.1: Definition of Laplace Transform

- Many practical engineering problems involve mechanical or electrical systems acted upon by discontinuous or impulsive forcing terms.

- For such problems the methods described in Chapter 3 are difficult to apply.

- In this chapter we use the **Laplace transform** to convert a problem for an unknown function *f* into a simpler problem for *F*, solve for *F*, and then recover *f* from its transform *F*.

- Given a known function *K*(*s*,*t*), an **integral transform** of a function *f* is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t) dt, \quad \infty \leq \alpha < \beta \leq \infty$$

(Question) How do we find a general solution of the ODE? How do we use Rule 1 and 2?

$$y'' + 3y' + 2y = g(t)$$

where $g(t) = 1$ for $0 \le t < 2,$ and $g(t) = e^t$ for $t \ge 2$

Improper Integrals

- The Laplace transform will involve an integral from zero to infinity. Such an integral is a type of improper integral.

- An improper integral over an unbounded interval is defined as the limit of an integral over a finite interval

$$\int_a^{\infty} f(t)dt = \lim_{A\to\infty} \int_a^{A} f(t)dt$$

  where A is a positive real number.

- If the integral from a to A exists for each $A > a$ and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise, the integral is said to **diverge** or fail to exist.

Example 1

- Consider the following improper integral. $\int_0^\infty e^{ct}\,dt$

- We can evaluate this integral as follows:

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\int_0^A e^{ct}\,dt = \lim_{A\to\infty}\frac{1}{c}\left(e^{cA}-1\right)$$

- Note that if $c = 0$, then $e^{ct} = 1$. Thus the following two cases hold:

$$\int_0^\infty e^{ct}\,dt = -\frac{1}{c},\ \text{if } c<0;\ \text{and}$$

$$\int_0^\infty e^{ct}\,dt\ \ \text{diverges, if } c\geq 0.$$

Example 2

- Consider the following improper integral.

$$\int_1^\infty 1/t \; dt$$

- We can evaluate this integral as follows:

$$\int_1^\infty 1/t \, dt = \lim_{A\to\infty} \int_1^A 1/t \, dt = \lim_{A\to\infty} \left(\ln A\right) \to \infty$$

- Therefore, the improper integral diverges.

Example 3

- Consider the following improper integral: $\int_1^{\infty} t^{-p}\ dt$

- From Example 2, this integral diverges at $p = 1$.

- We can evaluate this integral for $p \neq 1$ as follows:

$$\int_1^{\infty} t^{-p}\ dt = \lim_{A\to\infty} \int_1^{A} t^{-p}\ dt = \lim_{A\to\infty} \frac{1}{1-p}\left(A^{1-p} - 1\right)$$

- The improper integral diverges at $p = 1$ and

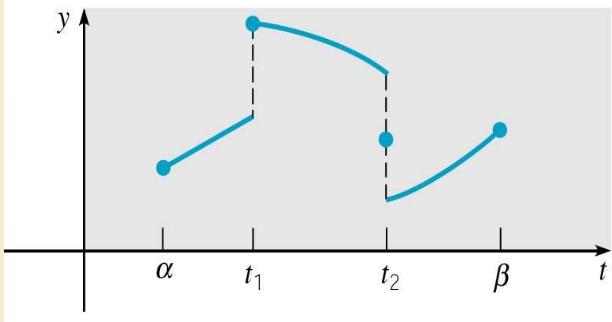
$$\text{If } p > 1,\ \lim_{A\to\infty} \frac{1}{1-p}\left(A^{1-p} - 1\right) = \frac{1}{p-1}$$

$$\text{If } p < 1,\ \lim_{A\to\infty} \frac{1}{1-p}\left(A^{1-p} - 1\right) \to \infty$$

Piecewise Continuous Functions

- A function f is **piecewise continuous** on an interval $[a, b]$ if this interval can be partitioned by a finite number of points $a = t_0 < t_1 < \ldots < t_n = b$ such that

  (1) f is continuous on each (t_k, t_{k+1})



- In other words, f is piecewise continuous on $[a, b]$ if it is continuous there except for a finite number of jump discontinuities.

$$(2)\ \left|\lim_{t \to t_k^+} f(t)\right| < \infty, \quad k = 0, \ldots, n-1$$

$$(3)\ \left|\lim_{t \to t_{k+1}^-} f(t)\right| < \infty, \quad k = 1, \ldots, n$$

The Laplace Transform

- Let f be a function defined for $t \geq 0$, and satisfies certain conditions to be named later.
- The **Laplace Transform of *f*** is defined as an **integral transform**:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

- The kernel function is $K(s,t) = e^{-st}$.
- Since solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.
- Note that the Laplace Transform is defined by an improper integral, and thus must be checked for convergence.
- On the next few slides, we review examples of improper integrals and piecewise continuous functions.

Theorem 6.1.2

- Suppose that f is a function for which the following hold:

  (1) f is piecewise continuous on $[0, b]$ for all $b > 0$.

  (2) A function f is said to have exponential order as $t \to \infty$ if $|f(t)| \le Ke^{at}$ when $t \ge M$, for constants a, K, M, with $K, M > 0$.

- Then the Laplace Transform of f exists for $s > a$.

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt \quad \text{is finite.}$$

Example 4

- Let $f_1(t) = 1,$

$$f_2(t) = t,$$

$$f_3(t) = e^{2t},$$

$$f_4(\mathrm{t}) = \sin(3t)$$

$$f_5(\mathrm{t}) = \cos(kt)$$

Find the Laplace transform $F(s) = \mathrm{L}(f_k)$ of f_k

Example 4

- Let $f(t) = 1$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\{1\} = \int_0^\infty e^{-st}dt = \lim_{b\to\infty}\int_0^b e^{-st}dt = -\lim_{b\to\infty}\left.\frac{e^{-st}}{s}\right|_0^b$$

$$= \frac{1}{s}, \quad s > 0$$

Example 5

- Let $f(t) = e^{at}$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\,dt$$

$$= -\lim_{b\to\infty} \left.\frac{e^{-(s-a)t}}{s-a}\right|_0^b$$

$$= \frac{1}{s-a}, \quad s > a$$

Example 6

- Consider the following piecewise-defined function f:

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ k, & t = 1 \\ 0 & t > 1 \end{cases}$$

  where k is a constant. This represents a unit impulse.

- Noting that $f(t)$ is piecewise continuous, we can compute its Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{s}.$$

- Observe that this result does not depend on k, the function value at the point of discontinuity.

Example 7

- Let $f(t) = \sin(at)$ for $t \geq 0$. Using integration by parts twice, the Laplace transform $F(s)$ of f is found as follows:

$$F(s) = L\{\sin(at)\} = \int_0^\infty e^{-st} \sin at dt = \lim_{b\to\infty} \int_0^b e^{-st} \sin at dt$$

$$= \lim_{b\to\infty} \left[-(e^{-st} \cos at)/a \Big|_0^b - \frac{s}{a} \int_0^b e^{-st} \cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a} \lim_{b\to\infty} \left[\int_0^b e^{-st} \cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a} \lim_{b\to\infty} \left[(e^{-st} \sin at)/a \Big|_0^b + \frac{s}{a} \int_0^b e^{-st} \sin at \right]$$

$$= \frac{1}{a} - \frac{s^2}{a^2} F(s) \Rightarrow F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Linearity of the Laplace Transform

- Suppose f and g are functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively.

- Then, for s greater than the maximum of a_1 and a_2, the Laplace transform of $c_1 f(t) + c_2 g(t)$ exists. That is,

$$L\left\{c_1 f(t) + c_2 g(t)\right\} = \int_0^\infty e^{-st}\left[c_1 f(t) + c_2 g(t)\right]dt \quad \text{is finite}$$

with

$$\begin{aligned} L\left\{c_1 f(t) + c_2 g(t)\right\} &= c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_0^\infty e^{-st} g(t)dt \\ &= c_1 L\left\{f(t)\right\} + c_2 L\left\{g(t)\right\} \end{aligned}$$

Example 8

- Let $f(t) = 5e^{-2t} - 3\sin(4t)$ for $t \geq 0$.

- Then by linearity of the Laplace transform, and using results of previous examples, the Laplace transform $F(s)$ of f is:

$$\begin{aligned} F(s) &= L\{f(t)\} \\ &= L\left\{5e^{-2t} - 3\sin(4t)\right\} \\ &= 5L\left\{e^{-2t}\right\} - 3L\left\{\sin(4t)\right\} \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0 \end{aligned}$$

(Question) How do we compute the inverse Laplace transform ?

Inverse Problem

- The main difficulty in using the Laplace transform method is determining the function $y = \phi(t)$ such that $L\{\phi(t)\} = Y(s)$.
- This is an inverse problem, in which we try to find ϕ such that $\phi(t) = L^{-1}\{Y(s)\}$.
- There is a general formula for L^{-1}, but it requires knowledge of the theory of functions of a complex variable, and we do not consider it here.
- It can be shown that if f is continuous with $L\{f(t)\} = F(s)$, then f is the **unique** continuous function with $f(t) = L^{-1}\{F(s)\}$.
- Table 6.2.1 in the text lists many of the functions and their transforms that are encountered in this chapter.

Inverse Laplace Transform

(Formula)   The formula of the inverse Laplace transform involves "Contour integral" in Complex Analysis.

(Idea) We can use the list of the examples.

Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Inverse Laplace Transform

(Example) Find the inverse Laplace transforms of the functions

(1) $F(s) = \dfrac{5}{s^3}$

(2) $F(s) = \dfrac{3}{s+2}$

(3) $F(s) = \dfrac{3}{s^2+4}$

(4) $F(s) = \dfrac{1}{s^2-4}$

Linearity of the Inverse Transform

- Frequently a Laplace transform $F(s)$ can be expressed as

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

- Let $f_1(t) = L^{-1}\{F_1(s)\}, \ldots, f_n(t) = L^{-1}\{F_n(s)\}$

- Then the function $f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$

  has the Laplace transform $F(s)$, since L is linear.

- By the uniqueness result of the previous slide, no other continuous function f has the same transform $F(s)$.

- Thus L^{-1} is a linear operator with $f(t) = L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + \cdots + L^{-1}\{F_n(s)\}$