6.2: Solution of Initial Value Problems

- The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782.

- The techniques described in this chapter were developed primarily by Oliver **Heaviside** (1850-1925), an English electrical engineer.

- In this section we see how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients.

- The Laplace transform is useful in solving these differential equations because the transform of f' is related in a simple way to the transform of f, as stated in Theorem 6.2.1.

(Question) $L\{f'\} = ?$

Theorem 6.2.1

- Suppose that f is a function for which the following hold:
  (1) f is continuous and f' is piecewise continuous on $[0, b]$ for all $b > 0$.
  (2) $|f(t)| \leq K e^{at}$ when $t \geq M$, for constants a, K, M, with $K, M > 0$.

- Then the Laplace Transform of f' exists for $s > a$, with

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- **Proof** (outline): For f and f' continuous on $[0, b]$, we have

$$\begin{aligned}\lim_{b\to\infty}\int_0^b e^{-st} f'(t)dt &= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^b - \int_0^b (-s)e^{-st} f(t)dt \right] \\ &= \lim_{b\to\infty}\left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt \right]\end{aligned}$$

- Similarly for f' piecewise continuous on $[0, b]$, see text.

The Laplace Transform of f'

- Thus if f and f' satisfy the hypotheses of Theorem 6.2.1, then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- Now suppose f' and f'' satisfy the conditions specified for f and f' of Theorem 6.2.1. We then obtain

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s\left[sL\{f(t)\} - f(0)\right] - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

- Similarly, we can derive an expression for $L\{f^{(n)}\}$, provided f and its derivatives satisfy suitable conditions. This result is given in Corollary 6.2.2

Corollary 6.2.2

- Suppose that f is a function for which the following hold:

  (1) $f, f', f'', \ldots, f^{(n-1)}$ are continuous, and $f^{(n)}$ piecewise continuous on $[0, b]$ for all $b > 0$.

  (2) $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, $\ldots$, $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$, for constants a, K, M, with $K, M > 0$.

  Then the Laplace Transform of $f^{(n)}$ exists for $s > a$, with

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Example 1: Chapter 3 Method (1 of 4)

- Consider the initial value problem $y'' - y' - 2y = 0, \quad y(0) = 1, \ y'(0) = 0$
- Recall from Section 3.1: $y(t) = e^{rt} \Rightarrow r^2 - r - 2 = 0 \Leftrightarrow (r-2)(r+1) = 0$
- Thus r_1 = -2 and r_2 = -3, and general solution has the form

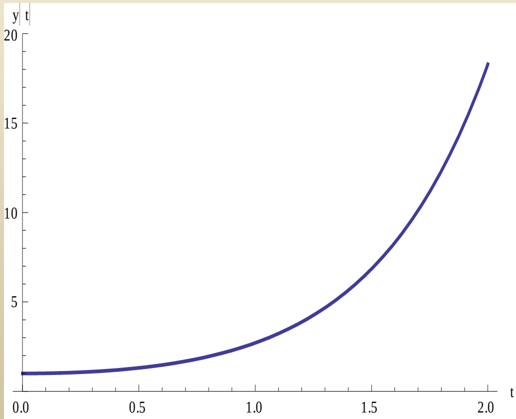
$$y(t) = c_1 e^{-t} + c_2 e^{2t}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + 2c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 2/3 \ , \ c_2 = 1/3$$

- Thus $y(t) = 2/3 \ e^{-t} + 1/3 \ e^{2t}$
- We now solve this problem using Laplace Transforms.

$y(t) = 2/3 \ e^{-t} + 1/3 \ e^{2t}$



(Example 1) Find the solution of the IVP by using Laplace transform

(1) $y'' + y = 0, \quad y(0) = 0, \; y'(0) = 1$

(2) $y'' - y' - 2y = 0, \quad y\left(0\right) = 1, \; y'\left(0\right) = 0$

Example 1: Laplace Transform Method (2 of 4)

$$y'' - y' - 2y = 0,$$
$$y(0) = 1, \;\; y'(0) = 0$$

- Assume that our IVP has a solution ϕ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y'' - y' - 2y\} = L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\} = 0$$

and hence $$\left[s^2 L\{y\} - sy(0) - y'(0)\right] - \left[sL\{y\} - y(0)\right] - 2L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have $\left(s^2 - s - 2\right)Y(s) - (s-1)y(0) - y'(0) = 0$

- Substituting in the initial conditions, we obtain $\left(s^2 - s - 2\right)Y(s) - (s-1) = 0$

- Thus $$L\{y\} = Y(s) = \frac{s-1}{(s-2)(s+1)}$$

Example 1: Partial Fractions (3 of 4)

- Using partial fraction decomposition, $Y(s)$ can be rewritten:

$$\frac{s-1}{(s-2)(s+1)} = \frac{a}{(s-2)} + \frac{b}{(s+1)}$$

$$s-1 = a(s+1) + b(s-2)$$

$$s-1 = (a+b)s + (a-2b)$$

$$a+b=1, \ \ a-2b=-1$$

$$a = 1/3 \ , \ b = 2/3$$

- Thus $$L\{y\} = Y(s) = \frac{1/3}{(s-2)} + \frac{2/3}{(s+1)}$$

Example 1: Solution (4 of 4)

- Recall from Section 6.1:

$$L\left\{e^{at}\right\} = F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad s > a$$

- Thus $$Y(s) = \frac{1/3}{(s-2)} + \frac{2/3}{(s+1)} = 1/3\ L\{e^{2t}\} + 2/3\ L\{e^{-t}\}, \ \ s > 2$$

- Recalling $Y(s) = L\{y\}$, we have $$L\{y\} = L\{2/3\ e^{-t} + 1/3\ e^{2t}\}$$

and hence $$y(t) = 2/3\ e^{-t} + 1/3\ e^{2t}$$

General Laplace Transform Method

- Consider the constant coefficient equation $ay'' + by' + cy = f(t)$
- Assume that this equation has a solution $y = \phi(t)$, and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{ay'' + by' + cy\} = aL\{y''\} + bL\{y'\} + cL\{y\} = L\{f(t)\}$$

- If we let $Y(s) = L\{y\}$ and $F(s) = L\{f\}$, then

$$a\left[s^2 L\{y\} - sy(0) - y'(0)\right] + b\left[sL\{y\} - y(0)\right] + cL\{y\} = F(s)$$

$$\left(as^2 + bs + c\right)Y(s) - \left(as + b\right)y(0) - ay'(0) = F(s)$$

$$Y(s) = \frac{\left(as + b\right)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

Algebraic Problem

- Thus the differential equation has been transformed into the algebraic equation

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

  for which we seek $y = \phi(t)$ such that $L\{\phi(t)\} = Y(s)$.

- Note that we do not need to solve the homogeneous and non-homogeneous equations separately, nor do we have a separate step for using the initial conditions to determine the values of the coefficients in the general solution.

Characteristic Polynomial

- Using the Laplace transform, our initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \; y'(0) = y_0'$$

becomes

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

- The polynomial in the denominator is the characteristic polynomial associated with the differential equation.

- The partial fraction expansion of $Y(s)$ used to determine ϕ requires us to find the roots of the characteristic equation.

- For higher order equations, this may be **difficult**, especially if the roots are irrational or complex.

Example 2: Non-homogeneous Problem (1 of 2)

- Consider the initial value problem $y'' + y = \sin 2t, \quad y(0) = 2, \; y'(0) = 1$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + L\{y\} = 2/(s^2+4)$$

- Letting $Y(s) = L\{y\}$, we have $\left(s^2+1\right)Y(s) - sy(0) - y'(0) = 2/(s^2+4)$

- Substituting in the initial conditions, we obtain

$$\left(s^2+1\right)Y(s) - 2s - 1 = 2/(s^2+4)$$

- Thus

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

Example 2: Solution (2 of 2)

- Using partial fractions,
$$Y(s)=\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

- Then
$$\begin{aligned}2s^3+s^2+8s+6&=(As+B)(s^2+4)+(Cs+D)(s^2+1)\\&=(A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D)\end{aligned}$$

- Solving, we obtain A = 2, B = 5/3, C = 0, and D = -2/3. Thus
$$Y(s)=\frac{2s}{s^2+1}+\frac{5/3}{s^2+1}-\frac{2/3}{s^2+4}$$

- Hence
$$y(t)=2\cos t+\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$$

Example 3: Solving a 4th Order IVP (1 of 2)

- Consider the initial value problem

$$y^{(4)} - y = 0, \qquad y(0) = 0,\ y'\left(0\right) = 1,\ y''\left(0\right) = 0,\ y'''(0) = 0$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)\right] - L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$ and substituting the initial values, we have

$$Y(s) = \frac{s^2}{(s^4 - 1)} = \frac{s^2}{(s^2 - 1)(s^2 + 1)}$$

- Using partial fractions $\quad Y(s) = \dfrac{s^2}{(s^2 - 1)(s^2 + 1)} = \dfrac{as + b}{(s^2 - 1)} + \dfrac{cs + d}{(s^2 + 1)}$

- Thus

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2$$

Example 3: Solving a 4^{th} Order IVP (2 of 2)

$$y^{(4)} - y = 0, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0$$

- In the expression: $(as+b)(s^2+1)+(cs+d)(s^2-1)=s^2$
- Setting s = 1 and s = -1 enables us to solve for *a* and *b*:

$$2(a+b)=1 \text{ and } 2(-a+b)=1 \quad \Rightarrow \quad a=0, b=1/2$$

- Setting s = 0, $b - d = 0$, so $d = 1/2$
- Equating the coefficients of s^3 in the first expression gives $a + c = 0$, so $c = 0$
- Thus $Y(s) = \dfrac{1/2}{(s^2-1)} + \dfrac{1/2}{(s^2+1)}$
- Using Table 6.2.1, the solution is $y(t) = \dfrac{\sinh t + \sin t}{2}$

$$y(t) = \frac{\sinh t + \sin t}{2}$$