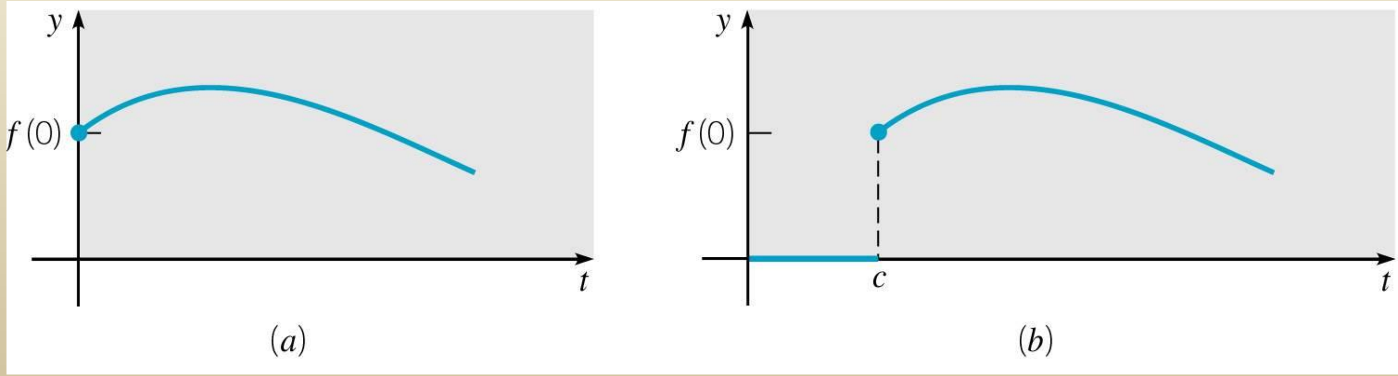
Ch 6.3: Step Functions

- Some of the most interesting elementary applications of the Laplace Transform method occur in the solution of linear equations with discontinuous or impulsive forcing functions.

- In this section, we will assume that all functions considered are piecewise continuous and **of exponential order**, so that their Laplace Transforms all exist, for *s* large enough:

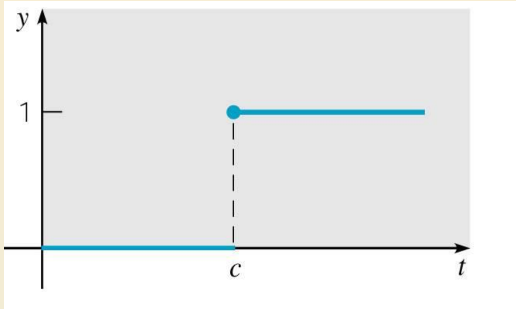
  $|f(t)| \le Ke^{at}, \ |f'(t)| \le Ke^{at}, \ \ldots, \ |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$, for constants a, K, M with $K, M > 0$.



Step Function definition

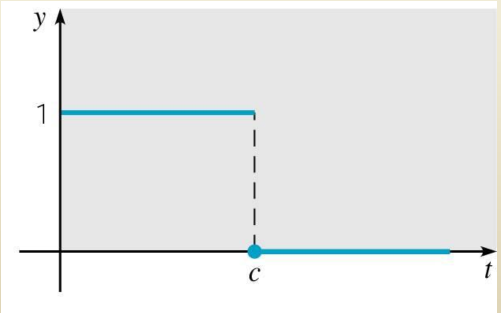
- Let $c \geq 0$. The **unit step function**, or Heaviside function, is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



- A negative step can be represented by

$$y(t) = 1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$



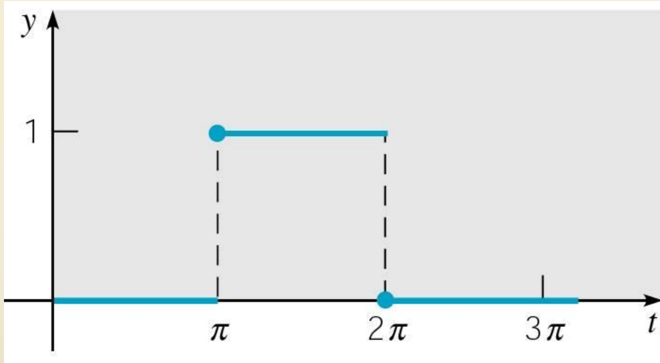
- Sketch the graph of $h(t) = u_\pi(t) - u_{2\pi}(t), \quad t \geq 0$

Example 1

- Sketch the graph of $$h(t) = u_\pi(t) - u_{2\pi}(t), \quad t \geq 0$$

- Solution: Recall that $u_c(t)$ is defined by $$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- Thus $$h(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t < \infty \end{cases}$$



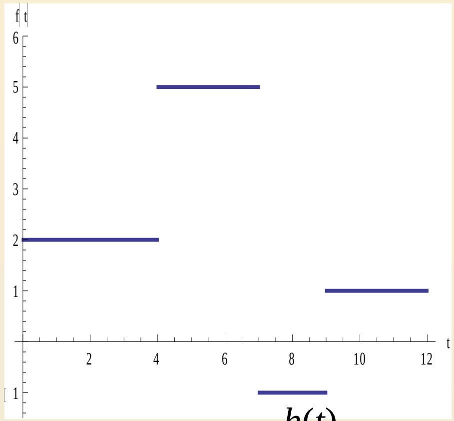
and hence the graph of $h(t)$ is a rectangular pulse.

(Ex) Write h(t) in terms of $u_c(t)$: $$h(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 4, & 1 \leq t < 3 \\ -1, & 3 \leq t < \infty \end{cases}$$

Example 2

- For the function

$$h(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & t \ge 9 \end{cases}$$



$h(t)$

whose graph is shown,
write $h(t)$ in terms of $u_c(t)$.

- (Hint) we will need $u_4(t)$, $u_7(t)$, and $u_9(t)$.
 We begin with the 2, then add 3 to get 5, then subtract 6 to get -1, and finally add 2 to get 1 – each quantity is multiplied by the appropriate $u_c(t)$

$$h(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t), \quad t \ge 0$$

Laplace Transform of Step Function

- The Laplace Transform of $u_c(t)$ is

$$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s} e^{-st} \Big|_c^b \right]$$

$$= \lim_{b\to\infty}\left[-\frac{e^{-bs}}{s} + \frac{e^{-cs}}{s} \right]$$

$$= \frac{e^{-cs}}{s}$$

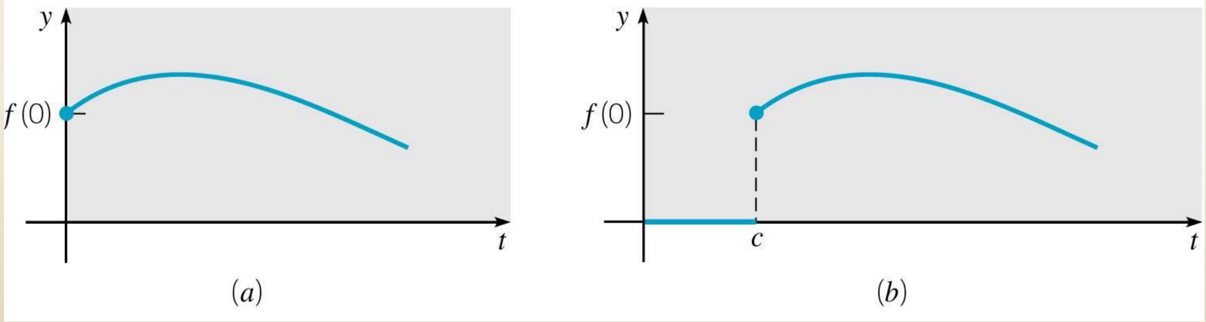
Translated Functions

- Given a function $f(t)$ defined for $t \geq 0$, we will often want to consider the related function $g(t) = u_c(t) f(t - c)$:

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

- Thus g represents a **translation of f** a distance c in the positive t direction.

- In the figure below, the graph of f is given on the left, and the graph of g on the right.



- Question: What is the Laplace transform $L\{g(t)\} = L\{u_c(t) f(t - c)\}$?

Theorem 6.3.1

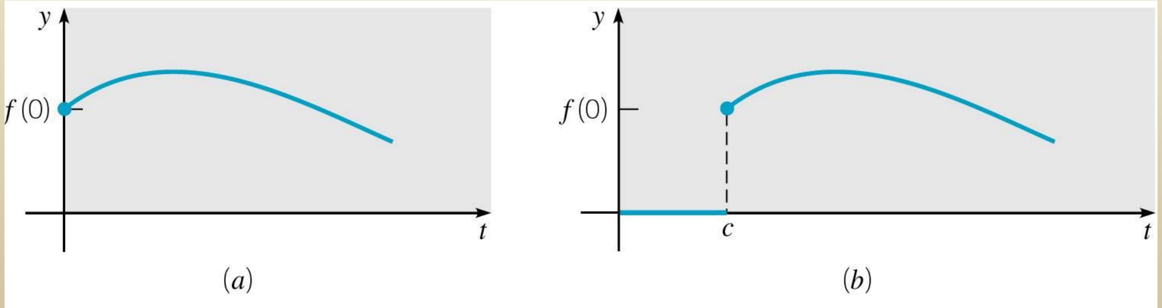
- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if $c > 0$, then

$$L\{u_c(t)f(t-c)\} = e^{-cs}L\{f(t)\} = e^{-cs}F(s)$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = L^{-1}\{e^{-cs}F(s)\}$$

- Thus the translation of $f(t)$ a distance c in the positive t direction corresponds to a multiplication of $F(s)$ by e^{-cs}.



Theorem 6.3.1: Proof Outline

- We need to show $L\left\{u_c(t)f(t-c)\right\}=e^{-cs}F(s)$

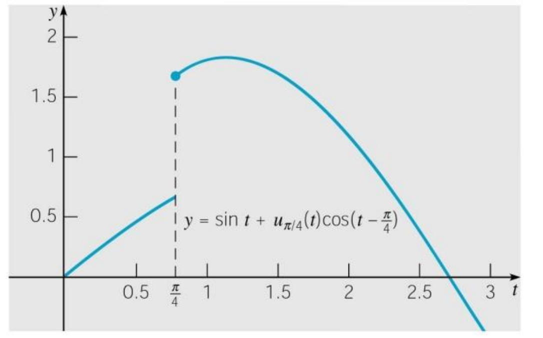
- Using the definition of the Laplace Transform, we have

$$
\begin{aligned}
L\left\{u_c(t)f(t-c)\right\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^\infty e^{-st} f(t-c)dt \\
&\overset{u=t-c}{=} \int_0^\infty e^{-s(u+c)} f(u)du \\
&= e^{-cs}\int_0^\infty e^{-su} f(u)du \\
&= e^{-cs}F(s)
\end{aligned}
$$

Example 3

- Find $L\{ g(t)\}$, where g is defined by

$$g(t) = \begin{cases} \sin t, & 0 \le t < \pi/4 \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4 \end{cases}$$



- Note that $g(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4)$

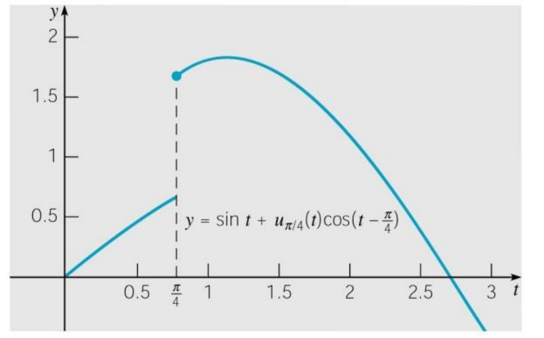
- Find the Laplace transform $L\{g\}$

$$L\{u_c(t) f(t-c)\} = e^{-cs} L\{f(t)\} = e^{-cs} F(s)$$

Example 3

- Find $L\{f(t)\}$, where f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4 \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4 \end{cases}$$



- Note that $f(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4)$, and

$$\begin{aligned} L\{f(t)\} &= L\{\sin t\} + L\{u_{\pi/4}(t)\cos(t - \pi/4)\} \\ &= L\{\sin t\} + e^{-\pi s/4} L\{\cos t\} \\ &= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} \\ &= \frac{1 + s e^{-\pi s/4}}{s^2+1} \end{aligned}$$

Example 4

- Find $L^{-1}\{F(s)\}$, where $F(s) = \dfrac{1 - e^{-2s}}{s^2}$

- Solution: $f(t) = L^{-1}\left\{\dfrac{1}{s^2}\right\} - L^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

Example 4

- Find $L^{-1}\{F(s)\}$, where $$F(s) = \frac{1-e^{-2s}}{s^2}$$

- Solution: $$f(t) = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$ $$= t - (t-2)u_2(t)$$

- The function may also be written as $$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$$

- Question: If $F(s) = L\{f(t)\}$ exists for $s > a \ge 0$, and if c is a constant, then $$L\left\{ e^{ct} f(t) \right\} = \ ?$$

Theorem 6.3.2

- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$L\left\{ e^{ct} f(t) \right\} = F(s-c), \quad s > a + c$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then $e^{ct} f(t) = L^{-1}\left\{ F(s-c) \right\}$

*** Translation ←→ Multiplying e^{ct} or e^{-cs}

- Thus, multiplication $f(t)$ by e^{ct} results in translating $F(s)$ a distance c in the positive t direction, and conversely.

- Proof Outline: $L\left\{ e^{ct} f(t) \right\} = \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt = F(s-c)$

(Example) Find the inverse transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$

Example 5

- To find the inverse transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$

- We first complete the square:

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{\left(s^2 - 4s + 4\right) + 1} = \frac{1}{\left(s-2\right)^2 + 1} = F(s-2)$$

- Since $L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$ and $L^{-1}\{F(s-2)\} = e^{2t} f(t)$

it follows that $g(t) = L^{-1}\{G(s)\} = e^{2t} \sin t$

(Ex 6) $G(s) = \dfrac{s}{s^2 - 4s + 5}$: $L\{G\} = ?$

Example 7

- Find the inverse Laplace transform of

$$G(s) = \frac{s+1}{s^2 - 4s + 5}$$