6.4: Differential Equations with Discontinuous Forcing Functions

• In this section focus on examples of nonhomogeneous initial value problems in which the forcing function is discontinuous.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y'_0$$



Initial Value Problem

(Ex 1) Find the solutions to the initial value problem

$$y'' + y = u_2(t),$$
 $y(0) = 0,$ $y'(0) = 0$

(Ex 2) Find the solutions to the initial value problem

$$y'' + 4y = g(t), y(0) = 0, y'(0) = 0$$

where

$$g(t) = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5\\ (t-5)/5 & 5 \le t < 10\\ 1, & t \ge 10 \end{cases}$$

Smoothness of Solution in General

• Consider a general second order linear equation

y'' + p(t)y' + q(t)y = g(t)

where p and q are continuous on some interval (a, b) but g is only piecewise continuous there.

- If $y = \psi(t)$ is a solution, then ψ and ψ' are continuous on (a, b) but ψ'' has jump discontinuities at the same points as g.
- Similarly for higher order equations, where the highest derivative of the solution has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous over (*a*, *b*).

Example 2: Initial Value Problem (1 of 12)

• Find the solution to the initial value problem

$$y'' + 4y = g(t), y(0) = 0, y'(0) = 0$$

where

$$g(t) = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5\\ (t-5)/5 & 5 \le t < 10\\ 1, & t \ge 10 \end{cases}$$

The graph of forcing function g(t) is given on right, and is known as ramp loading.



$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0) = 0, \ y'(0) = 0$$

Example 2: Laplace Transform (2 of 12)

• Assume that this ODE has a solution $y = \phi(t)$ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y''\} + 4L\{y\} = \left[L\{u_5(t)(t-5)\}\right]/5 - \left[L\{u_{10}(t)(t-10)\}\right]/5$$

or
$$\left[s^{2}L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = \frac{e^{-5s} - e^{-10s}}{5s^{2}}$$

• Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$(s^{2}+4)Y(s) = (e^{-5s}-e^{-10s})/5s^{2}$$

• Thus $Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2 + 4\right)}$

$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0) = 0, \quad y'(0) = 0$$

Example 2: Factoring Y(s) (3 of 12)

• We have
$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2 + 4\right)} = \frac{e^{-5s} - e^{-10s}}{5}H(s)$$

where $H(s) = \frac{1}{s^2\left(s^2 + 4\right)}$

• If we let $h(t) = L^{-1}{H(s)}$, then by Theorem 6.3.1,

$$y = \phi(t) = \frac{1}{5} \left[u_5(t)h(t-5) - u_{10}(t)h(t-10) \right]$$

$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0) = 0, \ y'(0) = 0$$

Example 2: Partial Fractions (4 of 12)

• Thus we examine H(s), as follows.

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

• This partial fraction expansion yields the equations

$$(A+C)s^{3} + (B+D)s^{2} + 4As + 4B = 1$$

 $\Rightarrow A = 0, B = 1/4, C = 0, D = -1/4$

• Thus $H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$

$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0) = 0, \ y'(0) = 0$$

Example 2: Solution (5 of 12)

• Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2 + 4}$$
$$= \frac{1}{4} \left[\frac{1}{s^2} \right] - \frac{1}{8} \left[\frac{2}{s^2 + 4} \right]$$

and hence
$$h(t) = L^{-1}{H(s)} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

• For *h*(*t*) as given above, and recalling our previous results, the solution to the initial value problem is then

$$y = \phi(t) = \frac{1}{5} \left[u_5(t)h(t-5) - u_{10}(t)h(t-10) \right]$$

Example 2: Graph of Solution (6 of 12)

• Thus the solution to the initial value problem is

$$\phi(t) = \frac{1}{5} \left[u_5(t)h(t-5) - u_{10}(t)h(t-10) \right], \text{ where}$$
$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

• The graph of this solution is given below.



Example 2: Composite IVPs (7 of 12)

• The solution to original IVP can be viewed as a composite of three separate solutions to three separate IVPs (discuss):

$$\begin{array}{ll} 0 \leq t < 5: & y_1'' + 4y_1 = 0, & y_1(0) = 0, \ y_1'(0) = 0 \\ 5 < t < 10: & y_2'' + 4y_2 = (t-5)/5, & y_2(5) = 0, \ y_2'(5) = 0 \\ t > 10: & y_3'' + 4y_3 = 1, & y_3(10) = y_2(10), \ y_3'(10) = y_2'(10) \end{array}$$



Example 2: First IVP (8 of 12)

• Consider the first initial value problem

$$y_1'' + 4y_1 = 0$$
, $y_1(0) = 0$, $y_1'(0) = 0$; $0 \le t < 5$

- From a physical point of view, the system is initially at rest, and since there is no external forcing, it remains at rest.
- Thus the solution over [0, 5) is $y_1 = 0$, and this can be verified analytically as well. See graphs below.





Example 2: Second IVP (9 of 12)

• Consider the second initial value problem

 $y_2'' + 4y_2 = (t-5)/5, \quad y_2(5) = 0, y_2'(5) = 0; \quad 5 < t < 10$

• Using methods of Chapter 3, the solution has the form

$$y_2 = c_1 \cos(2t) + c_2 \sin(2t) + t/20 - 1/4$$

• Thus the solution is an oscillation about the line (t-5)/20, over the time interval (5, 10). See graphs below.



Example 2: Third IVP (10 of 12)

• Consider the third initial value problem

$$y_3'' + 4y_3 = 1$$
, $y_3(10) = y_2(10)$, $y_3'(10) = y_2'(10)$; $t > 10$

• Using methods of Chapter 3, the solution has the form

$$y_3 = c_1 \cos(2t) + c_2 \sin(2t) + 1/4$$

• Thus the solution is an oscillation about y = 1/4, for t > 10. See graphs below.





Example 2: Amplitude (11 of 12)

• Recall that the solution to the initial value problem is

$$y = \phi(t) = \frac{1}{5} \left[u_5(t)h(t-5) - u_{10}(t)h(t-10) \right], \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- To find the amplitude of the eventual steady oscillation, we locate one of the maximum or minimum points for t > 10.
- Solving y' = 0, the first maximum is (10.642, 0.2979).
- Thus the amplitude of the oscillation is about 0.0479.



Example 2: Solution Smoothness (12 of 12)

• Our solution is
$$y = \phi(t) = \frac{1}{5} \left[u_5(t)h(t-5) - u_{10}(t)h(t-10) \right], \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- In this example, the forcing function g is continuous but g' is discontinuous at t = 5 and t = 10.
- It follows that ϕ and its first two derivatives are continuous everywhere, but ϕ''' has discontinuities at t = 5 and t = 10 that match the discontinuities of g' at t = 5 and t = 10.



