6.6: The Convolution Integral

- Sometimes it is possible to write a Laplace transform $H(s)$ as $H(s) = F(s)G(s)$, where $F(s)$ and $G(s)$ are the transforms of known functions f and g, respectively.

- In this case we might expect $H(s)$ to be the transform of the product of f and g. That is, does $H(s) = F(s)G(s) = L\{f\}L\{g\} = L\{f\,g\}$?

(Ex) $H(s) = \dfrac{6}{s^4}: \quad F(s) = \dfrac{6}{s^2}, \; G(s) = \dfrac{1}{s^2}?$

- On the next slide we give an example that shows that this equality does not hold, and hence the Laplace transform cannot in general be commuted with ordinary multiplication.

- In this section we examine the **convolution** of f and g, which can be viewed as a generalized product, and one for which the Laplace transform does commute.

Observation

- Let $f(t) = 1$ and $g(t) = \sin(t)$. Recall that the Laplace Transforms of f and g are

$$L\{f(t)\} = L\{1\} = \frac{1}{s}, \quad L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

- Thus $$L\{f(t)g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

and $$L\{f(t)\}L\{g(t)\} = \frac{1}{s(s^2+1)}$$

- Therefore for these functions it follows that

$$L\{f(t)g(t)\} \neq L\{f(t)\}L\{g(t)\}$$

Theorem 6.6.1

- Suppose $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$. Then $H(s) = F(s)G(s) = L\{h(t)\}$ for $s > a$, where

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(t)g(t-\tau)d\tau \equiv f * g(t)$$

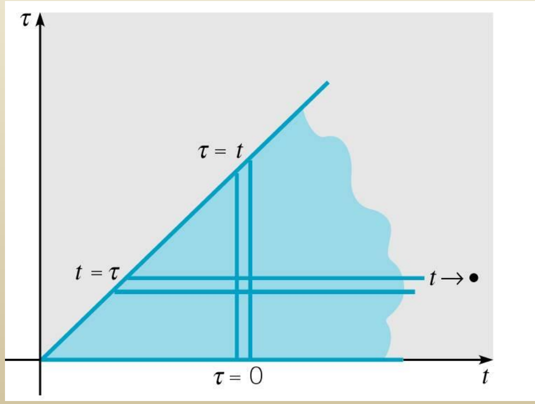
- The function $h(t)$ is known as the **convolution** of f and g and the integrals above are known as **convolution integrals**.

- Note that the equality of the two convolution integrals can be seen by making the substitution $u = t - \tau$.

- The convolution integral defines a "generalized product" and can be written as $h(t) = (f*g)(t)$. See text for more details.

Theorem 6.6.1 Proof Outline

$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty e^{-su} f(u)du \int_0^\infty e^{-s\tau} g(\tau)d\tau \\
&= \int_0^\infty g(\tau)\int_0^\infty e^{-s(\tau+u)} f(u)du\, d\tau \\
&= \int_0^\infty g(\tau)\int_\tau^\infty e^{-st} f(t-\tau)dt\, d\tau \qquad (t=\tau+u) \\
&= \int_0^\infty \int_\tau^\infty e^{-st} g(\tau) f(t-\tau)dt d\tau \\
&= \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)d\tau\, dt \\
&= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt \\
&= L\{h(t)\}
\end{aligned}
$$



Example 1: Find Inverse Transform (1 of 2)

- Find the inverse Laplace Transform of $H(s)$, given below.

$$H(s) = \frac{a}{s^2(s^2 + a^2)}$$

- Solution: Let $F(s) = 1/s^2$ and $G(s) = a/(s^2 + a^2)$, with

$$f(t) = L^{-1}\{F(s)\} = t$$
$$g(t) = L^{-1}\{G(s)\} = \sin(at)$$

- Thus by Theorem 6.6.1,

$$L^{-1}\{H(s)\} = h(t) = \int_0^t (t-\tau)\sin(a\tau)\,d\tau$$

$$\int_0^t u(\tau)v'(\tau)d\tau = \left[u(\tau)v(\tau)\right]_0^t - \int_0^t u'(\tau)v(\tau)d\tau \qquad L^{-1}\{H(s)\} = h(t) = \int_0^t (t-\tau)\sin(a\tau)d\tau$$

Example 1: Solution *h*(*t*) (2 of 2)

- We can integrate to simplify $h(t)$, as follows.

$$h(t) = \int_0^t (t-\tau)\sin(a\tau)d\tau = t\int_0^t \sin(a\tau)d\tau - \int_0^t \tau\sin(a\tau)d\tau$$

$$= -\frac{1}{a}t\cos(a\tau)\Big|_0^t - \left[-\frac{1}{a}\tau\cos(a\tau)\Big|_0^t + \frac{1}{a}\int_0^t \cos(a\tau)d\tau\right]$$

$$= -\frac{1}{a}t\left[\cos(at)-1\right] - \left[-\frac{1}{a}t\left[\cos(at)\right] + \frac{1}{a^2}\left[\sin(at)\right]\right]$$

$$= \frac{1}{a}t - \frac{1}{a^2}\sin(at)$$

$$= \frac{at - \sin(at)}{a^2}$$

(Example 2) Find the solutions of IVP:

(1) $$y'' + 9y = t, \quad y(0) = 1, \; y'(0) = 2$$

(2) $$y'' + 4y = g(t), \quad y(0) = 3, \; y'(0) = -1$$

Example 2: Initial Value Problem

- Find the solution to the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 3, \; y'(0) = -1$$

- Solution: $L\{y''\} + 4L\{y\} = L\{g(t)\}$

- or $\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = G(s)$

- Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$\left(s^2 + 4\right)Y(s) = 3s - 1 + G(s)$$

- Thus $Y(s) = \dfrac{3s-1}{s^2+4} + \dfrac{G(s)}{s^2+4}$

Example 2: Solution (2 of 4)

- We have
$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$
$$= 3\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\left[\frac{2}{s^2+4}\right]G(s)$$

- Thus
$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)d\tau$$

- Note that if $g(t)$ is given, then the convolution integral can be evaluated.

$$y'' + 4y = g(t), \quad y(0) = 3, \; y'(0) = -1$$

Example 2:
Laplace Transform of Solution (3 of 4)

- Recall that the Laplace Transform of the solution y is

$$Y(s) = \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4} = \Phi(s) + \Psi(s)$$

- Note $\Phi(s)$ depends only on system coefficients and initial conditions, while $\Psi(s)$ depends only on system coefficients and forcing function $g(t)$.

- Further, $\phi(t) = L^{-1}\{\Phi(s)\}$ solves the homogeneous IVP

$$y'' + 4y = 0, \quad y(0) = 3, \; y'(0) = -1$$

while $\psi(t) = L^{-1}\{\Psi(s)\}$ solves the nonhomogeneous IVP

$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

Example 2: Transfer Function (4 of 4)

- Examining $\Psi(s)$ more closely,

$$\Psi(s) = \frac{G(s)}{s^2+4} = H(s)G(s), \text{ where } H(s) = \frac{1}{s^2+4}$$

- The function $H(s)$ is known as the **transfer function**, and depends only on system coefficients.

- The function $G(s)$ depends only on external excitation $g(t)$ applied to system.

- If $G(s) = 1$, then $g(t) = \delta(t)$ and hence $h(t) = L^{-1}\{H(s)\}$ solves the nonhomogeneous initial value problem

$$y'' + 4y = \delta(t), \quad y(0) = 0, \; y'(0) = 0$$

- Thus $h(t)$ is response of system to unit impulse applied at $t = 0$, and hence $h(t)$ is called the **impulse response** of system.

Input-Output Problem (1 of 3)

- Consider the general initial value problem

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \; y'(0) = y_0'$$

- This IVP is often called an **input-output problem**. The coefficients a, b, c describe properties of physical system, and $g(t)$ is the input to system. The values y_0 and y_0' describe initial state, and solution y is the output at time t.

- Using the Laplace transform, we obtain

$$a\left[s^2Y(s) - sy(0) - y'(0)\right] + b\left[sY(s) - y(0)\right] + cY(s) = G(s)$$

or

$$Y(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c} = \Phi(s) + \Psi(s)$$

$$ay'' + by' + cy = g(t), \quad y(0) = y_0,\ y'(0) = y_0'$$

Laplace Transform of Solution (2 of 3)

- We have $$Y(s) = \frac{(as+b)y_0 + ay_0'}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c} = \Phi(s) + \Psi(s)$$

- As before, $\Phi(s)$ depends only on system coefficients and initial conditions, while $\Psi(s)$ depends only on system coefficients and forcing function $g(t)$.

- Further, $\phi(t) = L^{-1}\{\Phi(s)\}$ solves the homogeneous IVP

$$ay'' + by' + cy = 0, \quad y(0) = y_0,\ y'(0) = y_0'$$

while $\psi(t) = L^{-1}\{\Psi(s)\}$ solves the nonhomogeneous IVP

$$ay'' + by' + cy = g(t), \quad y(0) = 0,\ y'(0) = 0$$

Transfer Function (3 of 3)

- Examining $\Psi(s)$ more closely,

$$\Psi(s) = \frac{G(s)}{as^2 + bs + c} = H(s)G(s), \text{ where } H(s) = \frac{1}{as^2 + bs + c}$$

- As before, $H(s)$ is the **transfer function**, and depends only on system coefficients, while $G(s)$ depends only on external excitation $g(t)$ applied to system.

- Thus if $G(s) = 1$, then $g(t) = \delta(t)$ and hence $h(t) = L^{-1}\{H(s)\}$ solves the nonhomogeneous IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \; y'(0) = 0$$

- Thus $h(t)$ is response of system to unit impulse applied at $t = 0$, and hence $h(t)$ is called the **impulse response** of system, with

$$\psi(t) = L^{-1}\left\{H(s)G(s)\right\} = \int_0^t h(t-\tau)g(\tau)\,d\tau$$