7.1: Introduction to Systems of First Order Linear Equations

- A system of simultaneous first order ordinary differential equations has the general form

\[
x_1' = F_1(t, x_1, x_2, \ldots, x_n) \\
x_2' = F_2(t, x_1, x_2, \ldots, x_n) \\
\vdots \\
x_n' = F_n(t, x_1, x_2, \ldots, x_n)
\]

where each \( x_k \) is a function of \( t \). If each \( F_k \) is a linear function of \( x_1, x_2, \ldots, x_n \), then the system of equations is said to be **linear**, otherwise it is **nonlinear**.

- Systems of higher order differential equations can similarly be defined.

(Question) Can the DE \( u''(t) + 0.125u'(t) + u(t) = 0 \) be transformed to a system of first order ODE?
Example 1

- The motion of a certain spring-mass system from Section 3.7 was described by the differential equation
  \[ u''(t) + 0.125u'(t) + u(t) = 0 \]

- This second order equation can be converted into a system of first order equations by letting \( x_1 = u \) and \( x_2 = u' \). Thus
  \[
  x_1' = x_2 \\
  x_2' + 0.125 x_2 + x_1 = 0 
  \]
  or
  \[
  x_1' = x_2 \\
  x_2' = -x_1 - 0.125 x_2 
  \]
The method illustrated in the previous example can be used to transform an arbitrary $n$th order equation

$$y^{(n)} = F(t, y, y', y'', \ldots, y^{(n-1)})$$

into a system of $n$ first order equations, first by defining

$$x_1 = y, x_2 = y', x_3 = y'', \ldots, x_n = y^{(n-1)}$$

Then

$$x_1' = x_2$$
$$x_2' = x_3$$
$$\vdots$$
$$x_{n-1}' = x_n$$
$$x_n' = F(t, x_1, x_2, \ldots, x_n)$$
Solutions of First Order Systems

• A system of simultaneous first order ordinary differential equations has the general form

\[ \begin{align*}
    x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\
    &\quad \vdots \\
    x_n' &= F_n(t, x_1, x_2, \ldots x_n).
\end{align*} \]

It has a solution on \( I: \alpha < t < \beta \) if there exists \( n \) functions

\[ x_1 = \phi_1(t), \ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t) \]

that are differentiable on \( I \) and satisfy the system of equations at all points \( t \) in \( I \).

• Initial conditions may also be prescribed to give an IVP:

\[ x_1(t_0) = x_1^0, \ x_2(t_0) = x_2^0, \ldots, x_n(t_0) = x_n^0 \]
Theorem 7.1.1

• Suppose \( F_1, \ldots, F_n \) and \( \partial F_1/\partial x_1, \ldots, \partial F_1/\partial x_n, \ldots, \partial F_n/\partial x_1, \ldots, \partial F_n/\partial x_n \), are continuous in the region \( R \) of \( t x_1 x_2 \ldots x_n \)-space defined by \( \alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \ldots, \alpha_n < x_n < \beta_n \), and let the point \( (t_0, x_1^0, x_2^0, \ldots, x_n^0) \) be contained in \( R \).

Then in some interval \((t_0 - h, t_0 + h)\) there exists a unique solution

\[
x_1 = \phi_1(t), \ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)
\]

that satisfies the IVP:

\[
x_1' = F_1(t, x_1, x_2, \ldots x_n) \\
x_2' = F_2(t, x_1, x_2, \ldots x_n) \\
\vdots \\
x_n' = F_n(t, x_1, x_2, \ldots x_n)
\]
Linear Systems

• If each $F_k$ is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations has the general form

\[
\begin{align*}
    x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
    x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
          &\vdots \\
    x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{align*}
\]

• If each of the $g_k(t)$ is zero on $I$, then the system is homogeneous, otherwise it is nonhomogeneous.
Theorem 7.1.2

- Suppose $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an interval $I: \alpha < t < \beta$ with $t_0$ in $I$, and let 
  \[ x_1^0, x_2^0, \ldots, x_n^0 \]
  
  prescribe the initial conditions.

Then there exists a unique solution 
  \[ x_1 = \phi_1(t), x_2 = \phi_2(t), \ldots, x_n = \phi_n(t) \]
  
  that satisfies the IVP, and exists throughout $I$.

\[
\begin{align*}
  x'_1 &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
  x'_2 &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
  &\vdots \\
  x'_n &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{align*}
\]