7.1: Introduction to Systems of First Order Linear Equations

- A system of simultaneous first order ordinary differential equations has the general form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

  where each x_k is a function of t. If each F_k is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations is said to be **linear**, otherwise it is **nonlinear**.
- Systems of higher order differential equations can similarly be defined.

(Question) Can the DE $u''(t) + 0.125u'(t) + u(t) = 0$ be transformed to a system of first order ODE ?

Example 1

- The motion of a certain spring-mass system from Section 3.7 was described by the differential equation

$$u''(t) + 0.125u'(t) + u(t) = 0$$

- This second order equation can be converted into a system of first order equations by letting $x_1 = u$ and $x_2 = u'$. Thus

$$\begin{aligned} x_1' &= x_2 \\ x_2' + 0.125\,x_2 + x_1 &= 0 \end{aligned}$$

or

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 0.125\,x_2 \end{aligned}$$

*N*th Order ODEs and Linear 1st Order Systems

- The method illustrated in the previous example can be used to transform an arbitrary *n*th order equation

$$y^{(n)} = F\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

into a system of *n* first order equations, first by defining

$$x_1 = y,\; x_2 = y',\; x_3 = y'',\; \ldots, x_n = y^{(n-1)}$$

Then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, x_2, \ldots x_n) \end{aligned}$$

Solutions of First Order Systems

- A system of simultaneous first order ordinary differential equations has the general form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n). \end{aligned}$$

  It has a **solution** on $I: \alpha < t < \beta$ if there exists n functions

$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$

  that are differentiable on I and satisfy the system of equations at all points t in I.

- Initial conditions may also be prescribed to give an IVP:

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0, \ldots, x_n(t_0) = x_n^0$$

Theorem 7.1.1

- Suppose $F_1, \ldots, F_n$ and $\partial F_1/\partial x_1, \ldots, \partial F_1/\partial x_n, \ldots, \partial F_n/\partial x_1, \ldots, \partial F_n/\partial x_n$, are continuous in the region R of $t\, x_1\, x_2 \ldots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1, \ldots, \alpha_n < x_n < \beta_n$, and let the point $\left(t_0, x_1^0, x_2^0, \ldots, x_n^0\right)$ be contained in R.

Then in some interval $(t_0 - h, t_0 + h)$ there exists a unique solution

$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$

that satisfies the IVP:

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

Linear Systems

- If each F_k is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations has the general form

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

- If each of the $g_k(t)$ is zero on I, then the system is **homogeneous**, otherwise it is **nonhomogeneous**.

Theorem 7.1.2

- Suppose $p_{11}, p_{12}, \ldots, p_{nn}$, $g_1, \ldots, g_n$ are **continuous** on an interval I: $\alpha < t < \beta$ with t_0 in I, and let
$$x_1^0, x_2^0, \ldots, x_n^0$$
prescribe the initial conditions.

Then there exists a unique solution $x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$

that satisfies the IVP, and exists throughout I.

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$