7.5: Homogeneous Linear Systems with Constant Coefficients

- We consider here a homogeneous system of n first order linear equations with constant, real coefficients:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n
\end{aligned}
$$

- This system can be written as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$
\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}
$$

Equilibrium Solutions

- Note that if $n = 1$, then the system reduces to

$$x' = ax \quad \Rightarrow \quad x(t) = e^{at}$$

- Recall that $x = 0$ is the only equilibrium solution if $a \neq 0$.

- Further, $x = 0$ is an asymptotically stable solution if $a < 0$, since other solutions approach $x = 0$ in this case.

- Also, $x = 0$ is an unstable solution if $a > 0$, since other solutions depart from $x = 0$ in this case.

- For $n > 1$, equilibrium solutions are similarly found by solving $\mathbf{Ax} = \mathbf{0}$. We assume $\det\mathbf{A} \neq 0$, so that $\mathbf{x} = \mathbf{0}$ is the only solution. Determining whether $\mathbf{x} = \mathbf{0}$ is asymptotically stable or unstable is an important question here as well.

Phase Plane

- When $n = 2$, then the system reduces to
$$x_1' = a_{11}x_1 + a_{12}x_2$$
$$x_2' = a_{21}x_1 + a_{22}x_2$$

- This case can be visualized in the x_1x_2-plane, which is called the **phase plane**.

- In the phase plane, a direction field can be obtained by evaluating **Ax** at many points and plotting the resulting vectors, which will be tangent to solution vectors.

- A plot that shows representative solution trajectories is called a **phase portrait**.

- Examples of phase planes, directions fields, and phase portraits will be given later in this section.

Solving Homogeneous System

- To construct a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, assume a solution of the form,

  $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, where the exponent r and the constant vector $\boldsymbol{\xi}$ are to be determined.

- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$r\boldsymbol{\xi} e^{rt} = \mathbf{A}\boldsymbol{\xi} e^{rt} \quad \Leftrightarrow \quad r\boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi} \quad \Leftrightarrow \quad (\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$

- Thus to solve the homogeneous system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we must find the eigenvalues and eigenvectors of $\mathbf{A}$.

- Therefore $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ provided that r is an eigenvalue and $\boldsymbol{\xi}$ is an eigenvector of the coefficient matrix $\mathbf{A}$.

Example 1 (1 of 2)

- Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\mathbf{x}, \quad \left[\begin{array}{c} x_1' = 2x_1 \\ x_2' = -3x_2 \end{array} \right]$

- The most important feature of this system is that the coefficient matrix is a diagonal matrix. Thus, by writing the system in scalar form, we obtain

$$x'_1 = 2x_{1,}\ x'_2 = -3x_2$$

- Each of these equations involves only one of the unknown variables, so we can solve the two equations separately. In this way we find that

$$x_1(t) = c_1 e^{2t}, \qquad x_2(t) = c_2 e^{-3t}$$

where c_1 and c_2 are arbitrary constants.

Example 1 (2 of 2)

- Then, by writing the solution in vector form, we have

$$X = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

- Now we define the two solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ so that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \ \mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

- The Wronskian of these solutions is

$$W\left[\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}\right](\mathrm{t}) = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t}$$

which is never zero. Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions.

Example 2: Direction Field

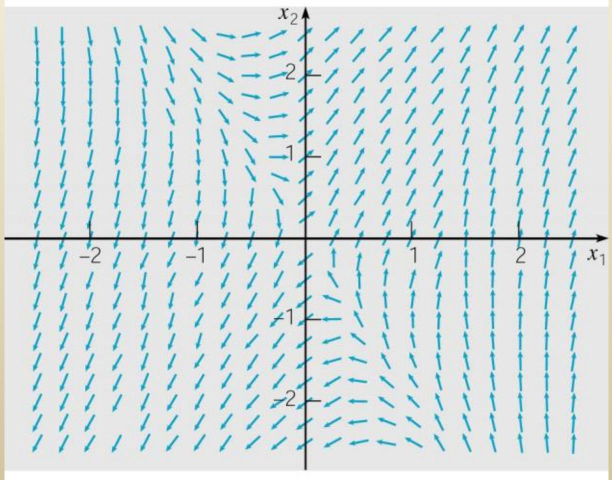
- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$$

- A direction field for this system is given below.

- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for $\mathbf{x}$, and rewriting system as $(\mathbf{A}\text{-}r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 2: Eigenvalues (2 of 9)

- Our solution has the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, where r and $\boldsymbol{\xi}$ are found by solving

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving $\det(\mathbf{A}-r\mathbf{I}) = 0$:

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1)$$

- Thus $r_1 = 3$ and $r_2 = -1$.

Example 2: First Eigenvector (3 of 9)

- Eigenvector for $r_1 = 3$: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{matrix} 1\xi_1 & -1/2\xi_2 & = 0 \\ & 0\xi_2 & = 0 \end{matrix}$$

$$\to \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1/2\xi_2 \\ \xi_2 \end{pmatrix} = c\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \ c \text{ arbitrary} \to \text{choose } \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Example 2: Second Eigenvector (4 of 9)

- Eigenvector for r_2 = -1: Solve

$$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi}=\mathbf{0} \Leftrightarrow \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 \\ 4 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rr} 1\xi_1 \quad +1/2\xi_2 & =0 \\ 0\xi_2 & =0 \end{array}$$

$$\to \boldsymbol{\xi}^{(2)}=\begin{pmatrix} -1/2\xi_2 \\ \xi_2 \end{pmatrix}=c\begin{pmatrix} -1/2 \\ 1 \end{pmatrix},\ c \text{ arbitrary} \to \text{choose } \boldsymbol{\xi}^{(2)}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Example 2: General Solution (5 of 9)

- The corresponding solutions $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{-2t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

(Question) How do we sketch the solution $\mathbf{x}^{(1)}$ in the phase plane?

Example 2: Phase Plane for $\mathbf{x}^{(1)}$ (6 of 9)

- To visualize solution, consider first $\mathbf{x} = c_1\mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \quad \Leftrightarrow \quad x_1 = c_1 e^{3t},\ x_2 = 2c_1 e^{3t}$$

- Now $\quad x_1 = c_1 e^{3t},\ x_2 = 2c_1 e^{3t} \Leftrightarrow e^{3t} = \dfrac{x_1}{c_1} = \dfrac{x_2}{2c_1} \Leftrightarrow x_2 = 2x_1$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2x_1$, which is the line through origin in direction of first eigenvector $\boldsymbol{\xi}^{(1)}$

- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.

- In either case, particle moves away from origin as t increases.

Example 2: Phase Plane for $\mathbf{x}^{(2)}$ (7 of 9)

- Next, consider $\mathbf{x} = c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \quad \Leftrightarrow \quad x_1 = c_2 e^{-t}, \;\; x_2 = -2c_2 e^{-t}$$

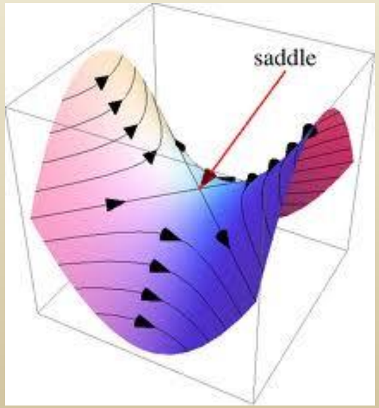
- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2x_1$, which is the line through origin in direction of 2nd eigenvector $\boldsymbol{\xi}^{(2)}$

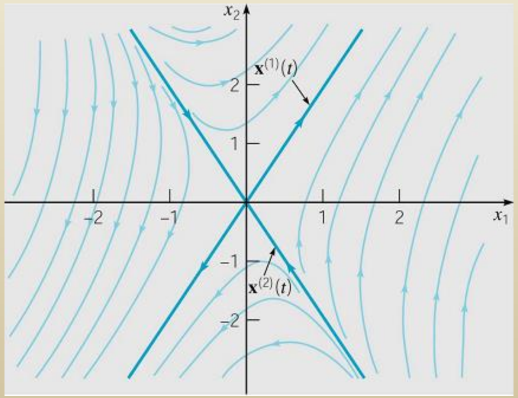
- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.

- In either case, particle moves towards origin as t increases.

Example 2: Phase Plane for General Solution (8 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$: $\quad \mathbf{x}(t) = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-t}$
- As $t \to \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach the line $x_2 = 2x_1$ as $t \to \infty$.
- Similarly, for $c_2 \neq 0$, all solutions asymptotically approach the line $x_2 = -2x_1$ as $t \to -\infty$.
- The origin is a **saddle point**, and is unstable. See graphs.





Example 2:
Time Plots for General Solution (9 of 9)

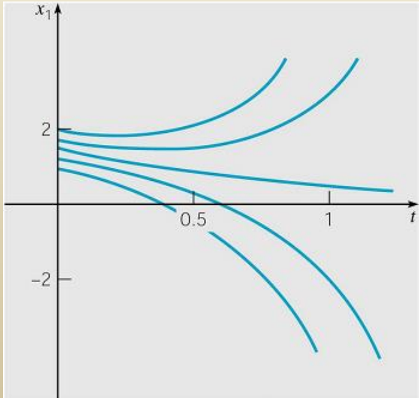
- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} \Leftrightarrow \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \begin{pmatrix}c_1e^{3t} + c_2e^{-t}\\2c_1e^{3t} - 2c_2e^{-t}\end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below.

- Note that when $c_1 = 0$, $x_1(t) = c_2e^{-t} \to 0$ as $t \to \infty$.
  Otherwise, $x_1(t) = c_1e^{3t} + c_2e^{-t}$ grows unbounded as $t \to \infty$.

- Graphs of x_2 are similarly obtained.



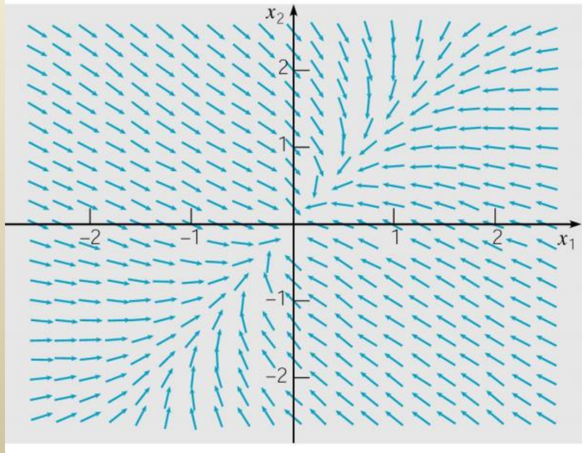
Example 3: Direction Field (1 of 9)

- Consider the homogeneous equation **x**' = **Ax** : $$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\mathbf{x}$$

- A direction field for this system is given below.

- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for $\mathbf{x}$, and rewriting system as $(\mathbf{A}\text{-}r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 3: Eigenvalues (2 of 9)

- Our solution has the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, where r and $\boldsymbol{\xi}$ are found by solving

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving det($\mathbf{A}$-$r\mathbf{I}$) = 0:

$$\begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = (-3-r)(-2-r)-2 = r^2+5r+4 = (r+1)(r+4)$$

- Thus r_1 = -1 and r_2 = -4.

Example 3: First Eigenvector (3 of 9)

- Eigenvector for r_1 = -1: Solve

$$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi}=\mathbf{0} \Leftrightarrow \begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \boldsymbol{\xi}^{(1)}=\begin{pmatrix} \sqrt{2}/2\,\xi_2 \\ \xi_2 \end{pmatrix} \to \text{choose } \boldsymbol{\xi}^{(1)}=\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

Example 3: Second Eigenvector (4 of 9)

- Eigenvector for r_2 = -4: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\sqrt{2}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$\to \text{choose } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Example 3: General Solution (5 of 9)

- The corresponding solutions $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{-t} & -\sqrt{2}e^{-4t} \\ \sqrt{2}e^{-t} & e^{-4t} \end{vmatrix} = 3e^{-5t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

Example 3: Phase Plane for $\mathbf{x}^{(1)}$ (6 of 9)

- To visualize solution, consider first $\mathbf{x} = c_1\mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} \quad \Leftrightarrow \quad x_1 = c_1 e^{-t}, \; x_2 = \sqrt{2} c_1 e^{-t}$$

- Now $\quad x_1 = c_1 e^{-t}, \; x_2 = \sqrt{2} c_1 e^{-t} \; \Leftrightarrow \; e^{-t} = \frac{x_1}{c_1} = \frac{x_2}{\sqrt{2} c_1} \; \Leftrightarrow \; x_2 = \sqrt{2} x_1$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2^{1/2} x_1$, which is the line through origin in direction of first eigenvector $\boldsymbol{\xi}^{(1)}$

- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.

- In either case, particle moves towards **origin** as t increases.

Example 3: Phase Plane for $\mathbf{x}^{(2)}$ (7 of 9)

- Next, consider $\mathbf{x} = c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \quad \Leftrightarrow \quad x_1 = -\sqrt{2}c_2e^{-4t}, \;\; x_2 = c_2e^{-4t}$$

- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2^{1/2} x_1$, which is the line through origin in direction of 2nd eigenvector $\boldsymbol{\xi}^{(2)}$

- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.

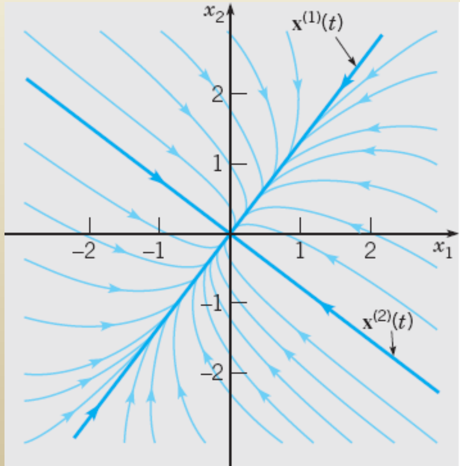
- In either case, particle moves towards **origin** as t increases.

Example 3:
Phase Plane for General Solution (8 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- As $t \to \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach origin along the line $x_2 = 2^{1/2} x_1$ as $t \to \infty$.
- Similarly, all solutions are unbounded as $t \to -\infty$.
- The origin is a **node**, and is asymptotically stable.



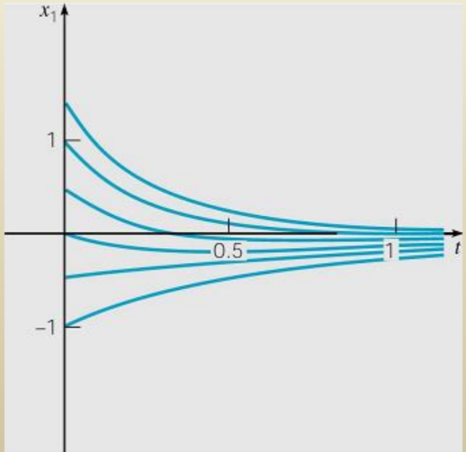
Example 3:
Time Plots for General Solution (9 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \quad \Leftrightarrow \quad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} - \sqrt{2} c_2 e^{-4t} \\ \sqrt{2} c_1 e^{-t} + c_2 e^{-4t} \end{pmatrix}$$

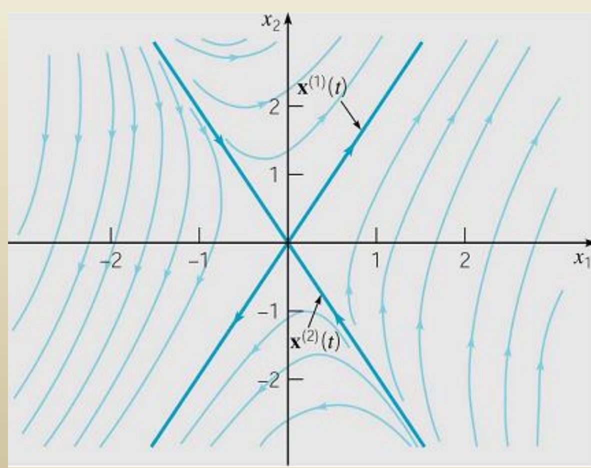
- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below.

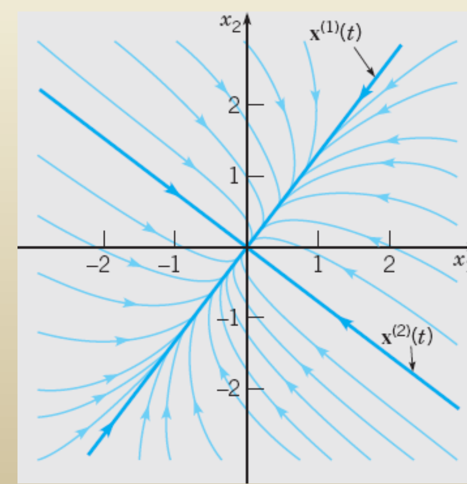
- Graphs of x_2 are similarly obtained.



2 x 2 Case:
Real Eigenvalues, Saddle Points and Nodes

- The previous two examples demonstrate the two main cases for a 2 x 2 real system with real and different eigenvalues:
  - Both eigenvalues have opposite signs, in which case origin is a **saddle point** and is unstable.
  - Both eigenvalues have the same sign, in which case origin is a **node**, and is asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.





Eigenvalues, Eigenvectors and Fundamental Solutions

- In general, for an n x n real linear system **x**' = **Ax**:
  - Case 1: All eigenvalues are real and different from each other.
  - Case 2: Some eigenvalues occur in complex conjugate pairs.
  - Case 3: Some eigenvalues are repeated.

- If eigenvalues $r_1,\ldots, r_n$ are real & different, then there are n corresponding linearly independent eigenvectors $\boldsymbol{\xi}^{(1)},\ldots, \boldsymbol{\xi}^{(n)}$. The associated solutions of **x**' = **Ax** are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

- Using Wronskian, it can be shown that these solutions are linearly independent, and hence form a fundamental set of solutions. Thus general solution is

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + \ldots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}$$

Hermitian Case: Eigenvalues, Eigenvectors & Fundamental Solutions

- If **A** is an *n* x *n* Hermitian matrix (real and symmetric), then all eigenvalues $r_1,\ldots, r_n$ are **real**, although some may repeat.

- In any case, there are *n* corresponding **linearly independent** and orthogonal eigenvectors $\boldsymbol{\xi}^{(1)},\ldots, \boldsymbol{\xi}^{(n)}$. The associated solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

and form a fundamental set of solutions.

Example 4: Hermitian Matrix (1 of 3)

- Consider the homogeneous equation **x**' = **Ax** below.

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x}$$

- The **eigenvalues** were found previously in Ch 7.3, and were:
  $r_1 = 2$, $r_2 = -1$ and $r_3 = -1$.

- Corresponding eigenvectors:

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Example 4: General Solution (2 of 3)

- The fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

with general solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

Example 4: General Solution Behavior (3 of 3)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)}$:

$$\mathbf{x} = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-t} + c_3\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-t}$$

- As $t \to \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$, $c_3\mathbf{x}^{(3)}$ become negligible.

- Thus, for $c_1 \neq 0$, all solutions $\mathbf{x}$ become unbounded as $t \to \infty$, while for $c_1 = 0$, all solutions $\mathbf{x} \to \mathbf{0}$ as $t \to \infty$.

- The initial points that cause $c_1 = 0$ are those that lie in plane determined by $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$. Thus solutions that start in this plane approach **origin** as $t \to \infty$.

Complex Eigenvalues and Fundamental Solutions

- If some of the eigenvalues $r_1, \ldots, r_n$ occur in complex conjugate pairs, but otherwise are different, then there are still n corresponding linearly independent solutions

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t},$$

  which form a fundamental set of solutions. Some may be complex-valued, but real-valued solutions may be derived from them. This situation will be examined in Ch 7.6.

- If the coefficient matrix **A** is complex, then complex eigenvalues need not occur in conjugate pairs, but solutions will still have the above form (if the eigenvalues are distinct) and these solutions may be complex-valued.

Repeated Eigenvalues and Fundamental Solutions

- If some of the eigenvalues $r_1, \ldots, r_n$ are repeated, then there may not be n corresponding linearly independent solutions of the form

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

- In order to obtain a fundamental set of solutions, it may be necessary to seek additional solutions of another form.

- This situation is analogous to that for an nth order linear equation with constant coefficients, in which case a repeated root gave rise solutions of the form

$$e^{rt}, te^{rt}, t^2 e^{rt}, \ldots$$

This case of repeated eigenvalues is examined in Section 7.8.