7.6: Complex Eigenvalues

- We consider again a homogeneous system of n first order linear equations with constant, real coefficients,

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned}$$

and thus the system can be written as **x**' = **Ax**, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Conjugate Eigenvalues and Eigenvectors

- We know that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, provided r is an eigenvalue and $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$.

- The eigenvalues $r_1, \ldots, r_n$ are the roots of $\det(\mathbf{A} - r\mathbf{I}) = 0$, and the corresponding eigenvectors satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$.

- If $\mathbf{A}$ is real, then the coefficients in the polynomial equation $\det(\mathbf{A} - r\mathbf{I}) = 0$ are real, and hence any complex eigenvalues must occur in conjugate pairs. Thus if $r_1 = \lambda + i\mu$ **is an eigenvalue, then the second solution is** $r_2 = \lambda - i\mu$.

- The corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ are conjugates also.
  To see this, recall $\mathbf{A}$ and $\mathbf{I}$ have real entries, and hence

$$\left(\mathbf{A} - r_1\mathbf{I}\right)\boldsymbol{\xi}^{(1)} = \mathbf{0} \Rightarrow \left(\mathbf{A} - \bar{r}_1\mathbf{I}\right)\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{0} \Rightarrow \left(\mathbf{A} - r_2\mathbf{I}\right)\boldsymbol{\xi}^{(2)} = \mathbf{0}$$

Conjugate Solutions

- It follows from the previous slide that the solutions

$$\mathbf{x}^{(1)} = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \quad \mathbf{x}^{(2)} = \boldsymbol{\xi}^{(2)} e^{r_2 t}$$

corresponding to these eigenvalues and eigenvectors are conjugates as well, since

$$\mathbf{x}^{(2)} = \boldsymbol{\xi}^{(2)} e^{r_2 t} = \overline{\boldsymbol{\xi}}^{(1)} e^{\overline{r}_2 t} = \overline{\mathbf{x}}^{(1)}$$

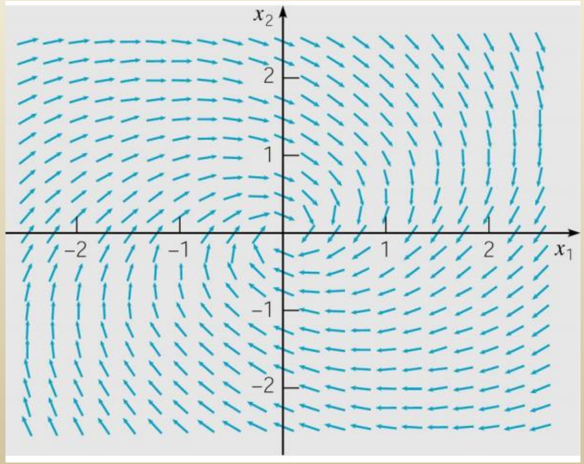
Example 1: Direction Field (1 of 7)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below:

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for $\mathbf{x}$, and rewriting system as $(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 1: Complex Eigenvalues (2 of 7)

- We determine r by solving det(**A**-r**I**) = 0. Now

$$\begin{vmatrix} -1/2-r & 1 \\ -1 & -1/2-r \end{vmatrix} = \left(r+1/2\right)^2 + 1 = r^2 + r + \frac{5}{4}$$

- Thus

$$r = \frac{-1 \pm \sqrt{1^2 - 4(5/4)}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

- Therefore the eigenvalues are $r_1 = -1/2 + i$ and $r_2 = -1/2 - i$.

Example 1: First Eigenvector (3 of 7)

- Eigenvector for r_1 = -1/2 + i: Solve

$$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi}=\mathbf{0} \Leftrightarrow \begin{pmatrix} -1/2-r & 1 \\ -1 & -1/2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & i & 0 \\ -1 & -i & 0 \end{pmatrix} \to \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}^{(1)}=\begin{pmatrix} -i\xi_2 \\ \xi_2 \end{pmatrix} \to \text{choose } \boldsymbol{\xi}^{(1)}=\begin{pmatrix} 1 \\ i \end{pmatrix}$$

- Thus

$$\boldsymbol{\xi}^{(1)}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}+i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example 1: Second Eigenvector (4 of 7)

- Eigenvector for r_1 = -1/2 - i: Solve

$$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi}=\mathbf{0} \Leftrightarrow \begin{pmatrix} -1/2-r & 1 \\ -1 & -1/2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & -i & 0 \\ -1 & i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{\xi}^{(2)}=\begin{pmatrix} i\xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \boldsymbol{\xi}^{(2)}=\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- Thus

$$\boldsymbol{\xi}^{(2)}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}+i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Real valued solutions

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}$$

- The two solutions: $X^{(1)}(t) = e^{-t/2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}, \qquad X^{(2)}(t) = e^{-t/2}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$
- Remember $e^{it} = \cos t + i \sin t$
- Then $X^{(1)}(t) = e^{-t/2}\begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix} \qquad X^{(2)}(t) = e^{-t/2}\begin{pmatrix} \cos t - i \sin t \\ -\sin t - i \cos t \end{pmatrix}$

$$\frac{1}{2}\left(X^{(1)}(t) + X^{(2)}(t)\right) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \frac{1}{2i}\left(X^{(1)}(t) - X^{(2)}(t)\right) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

- The two real valued solutions:

$$u(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \mathrm{v}(t) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Example 1: General Solution (5 of 7)

- The corresponding solutions $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{u}(t) = e^{-t/2}\left[\begin{pmatrix}1\\0\end{pmatrix}\cos t - \begin{pmatrix}0\\1\end{pmatrix}\sin t\right] = e^{-t/2}\begin{pmatrix}\cos t\\-\sin t\end{pmatrix}$$

$$\mathbf{v}(t) = e^{-t/2}\left[\begin{pmatrix}1\\0\end{pmatrix}\sin t + \begin{pmatrix}0\\1\end{pmatrix}\cos t\right] = e^{-t/2}\begin{pmatrix}\sin t\\\cos t\end{pmatrix}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} = e^{-t} \neq 0$$

- Thus $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are real-valued fundamental solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with general solution $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$.

Example 1: Phase Plane (6 of 7)

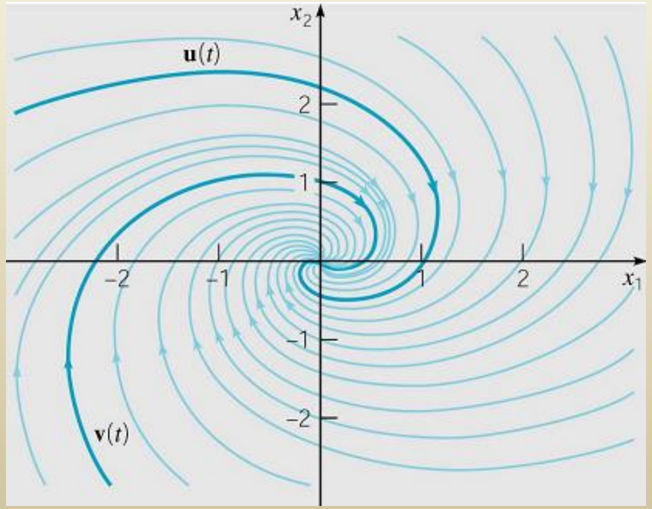
- Given below is the phase plane plot for solutions **x**, with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$$

- Each solution trajectory approaches **origin** along a spiral path as $t \to \infty$, since coordinates are products of decaying exponential and sine or cosine factors.

- The graph of **u** passes through (1,0), since u(0) = (1,0). Similarly, the graph of **v** passes through (0,1).

- The origin is a **spiral point**, and is asymptotically stable.

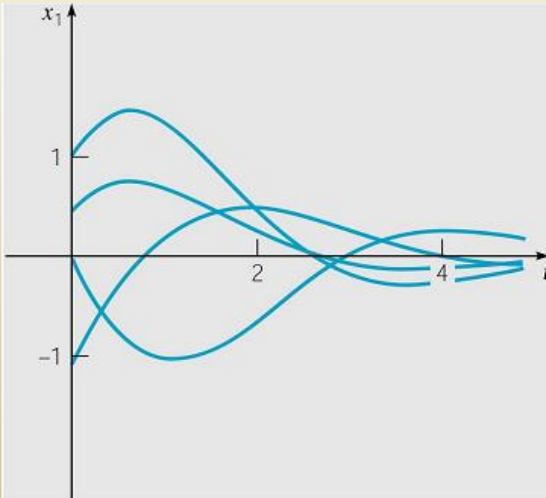


Example 1: Time Plots (7 of 7)

- The general solution is $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$:

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t \\ -c_1 e^{-t/2}\sin t + c_2 e^{-t/2}\cos t \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below, each one a decaying oscillation as $t \to \infty$.



General Solution

- To summarize, suppose $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, and that $r_3, \ldots, r_n$ are all real and distinct eigenvalues of **A**. Let the corresponding eigenvectors be

$$\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b},\ \boldsymbol{\xi}^{(2)} = \mathbf{a} - i\mathbf{b},\ \boldsymbol{\xi}^{(3)}, \boldsymbol{\xi}^{(4)}, \ldots, \boldsymbol{\xi}^{(n)}$$

- Then the general solution of **x**' = **Ax** is

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\boldsymbol{\xi}^{(3)}e^{r_3 t} + \ldots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}$$

where

$$\mathbf{u}(t) = e^{\lambda t}\left(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t\right),\ \mathbf{v}(t) = e^{\lambda t}\left(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t\right)$$

Real-Valued Solutions

- Thus for complex conjugate eigenvalues r_1 and r_2, the corresponding solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are conjugates also.

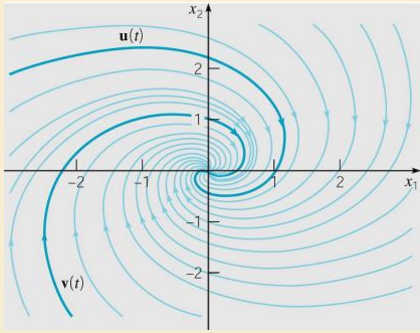
- To obtain real-valued solutions, use real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. To see this, let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\,\mathbf{b}$. Then

$$\begin{aligned}
\mathbf{x}^{(1)} &= \boldsymbol{\xi}^{(1)} e^{(\lambda + i\mu)t} = (\mathbf{a} + i\mathbf{b}) e^{\lambda t}(\cos \mu t + i \sin \mu t) \\
&= e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) \\
&= \mathbf{u}(t) + i\,\mathbf{v}(t)
\end{aligned}$$

where $\quad \mathbf{u}(t) = e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t), \; \mathbf{v}(t) = e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t),$

are real valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and can be shown to be linearly independent.

Spiral Points, Centers, Eigenvalues, and Trajectories



- In previous example, general solution was

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$$

- The origin was a **spiral point**, and was **asymptotically stable**.
- If real part of complex eigenvalues is positive, then trajectories spiral away, unbounded, from origin, and hence **origin would be an unstable spiral point**.
- If real part of complex eigenvalues is zero, then trajectories circle origin, neither approaching nor departing. Then origin is called a **center** and is stable, but not asymptotically stable. Trajectories periodic in time.
- The direction of trajectory motion depends on entries in **A**.

Example 2: Second Order System with Parameter (1 of 2)

- The system **x**' = **Ax** below contains a parameter α :

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix}\mathbf{x}$$

- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for **x** and rewriting system as $(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} \alpha - r & 2 \\ -2 & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Next, solve for r in terms of α :

$$\begin{vmatrix} \alpha - r & 2 \\ -2 & -r \end{vmatrix} = r(r-\alpha) + 4 = r^2 - \alpha r + 4 \Rightarrow r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

Example 2: Eigenvalue Analysis (2 of 2)

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

- The eigenvalues are given by the quadratic formula above.

- For $\alpha < -4$, both eigenvalues are real and negative, and hence origin is asymptotically stable node.
- For $\alpha > 4$, both eigenvalues are real and positive, and hence the origin is an unstable node.
- For $-4 < \alpha < 0$, eigenvalues are complex with a **negative real part**, and hence origin is asymptotically stable spiral point.
- For $0 < \alpha < 4$, eigenvalues are complex with a **positive real part**, and the origin is an unstable spiral point.

- For $\alpha = 0$, eigenvalues are purely imaginary, **origin is a center**. Trajectories closed curves about origin & periodic.

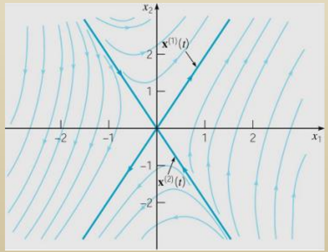
- For $\alpha = \pm 4$, eigenvalues real & equal, origin is a node (Ch 7.8)

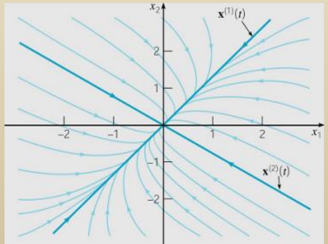
Second Order Solution Behavior and Eigenvalues: Three Main Cases

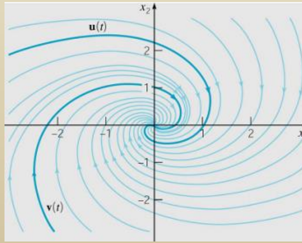
- For second order systems, the three main cases are:
  - Eigenvalues are real and have opposite signs; **x** = **0** is a saddle point.
  - Eigenvalues are real, distinct and have same sign; **x** = **0** is a node.
  - Eigenvalues are complex with nonzero real part; **x** = **0** a spiral point.

- Other possibilities exist and occur as transitions between two of the cases listed above:
  - A zero eigenvalue occurs during transition between saddle point and node. Real and equal eigenvalues occur during transition between nodes and spiral points. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$







Example 3: Multiple Spring-Mass System (1 of 6)

- The equations for the system of two masses and three springs discussed in Section 7.1, assuming no external forces, can be expressed as:

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2 x_2 \quad \text{and} \quad m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)x_2$$

$$\text{or} \quad m_1 y_3' = -(k_1 + k_2)y_1 + k_2 y_2 \quad \text{and} \quad m_2 y_4' = k_2 y_1 - (k_2 + k_3)y_2$$

$$\text{where} \quad y_1 = x_1,\ y_2 = x_2,\ y_3 = x_1',\ \text{and} \quad y_4 = x_2'$$

- Given $m_1 = 2,\ m_2 = 9/4,\ k_1 = 1,\ k_2 = 3,\ \text{and}\ k_3 = 15/4,$ the equations become

$$y_1' = y_3,\ y_2' = y_4,\ y_3' = -2y_1 + 3/2\ y_2,\ \text{and}\ \ y_4' = 4/3\ y_1 - 3y_2$$

$y_1' = y_3,\ y_2' = y_4,\ y_3' = -2y_1 + 3/2\ y_2$, and $y_4' = 4/3\ y_1 - 3y_2$

Example 3: Multiple Spring-Mass System (2 of 6)

- Writing the system of equations in matrix form:

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}$$

- Assuming a solution of the form $\mathbf{y} = \boldsymbol{\xi} e^{rt}$, where r must be an eigenvalue of the matrix $\mathbf{A}$ and $\boldsymbol{\xi}$ is the corresponding eigenvector, the characteristic polynomial of $\mathbf{A}$ is

$$r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4)$$

yielding the eigenvalues: $r_1 = i,\ r_2 = -i,\ r_3 = 2i$, and $r_4 = -2i$

Example 3: Multiple Spring-Mass System (3 of 6)

$$y' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} y = Ay$$

- For the eigenvalues $r_1 = i,\ r_2 = -i,\ r_3 = 2i,$ and $r_4 = -2i$ the corresponding eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix},\ \xi^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix},\ \xi^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix},\ \text{and } \xi^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}$$

- The products $\boldsymbol{\xi}^{(1)}e^{it}$ and $\boldsymbol{\xi}^{(3)}e^{2it}$ yield the complex-valued solutions:

$$\xi^{(1)}e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}(\cos t + i\sin t) = \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i\begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i\,\mathbf{v}^{(1)}(t)$$

$$\xi^{(3)}e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}(\cos 2t + i\sin 2t) = \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i\begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i\,\mathbf{v}^{(2)}(t)$$

$y_1' = y_3,\ y_2' = y_4,\ y_3' = -2y_1 + 3/2\ y_2$, and $y_4' = 4/3\ y_1 - 3y_2$

Example 3: Multiple Spring-Mass System (4 of 6)

- After validating that $\mathbf{u}^{(1)}(t),\ \mathbf{v}^{(1)}(t),\ \mathbf{u}^{(2)}(t),\ \mathbf{v}^{(2)}(t)$ are linearly independent, the general solution of the system of equations can be written as

$$\boldsymbol{y} = c_1\begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2\begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3\begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4\begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix}$$

- where c_1, c_2, c_3, c_4 are arbitrary constants.

- Each solution will be periodic with period 2π, so each trajectory is a closed curve. The first two terms of the solution describe motions with frequency 1 and period 2π while the second two terms describe motions with frequency 2 and period π. The motions of the two masses will be different relative to one another for solutions involving only the first two terms or the second two terms.

y_1 and y_2 represent the motion of the masses and $y_3 = y_1', y_4 = y_1'$

Example 3: Multiple Spring-Mass System (5 of 6)

- To obtain the fundamental mode of vibration with frequency 1

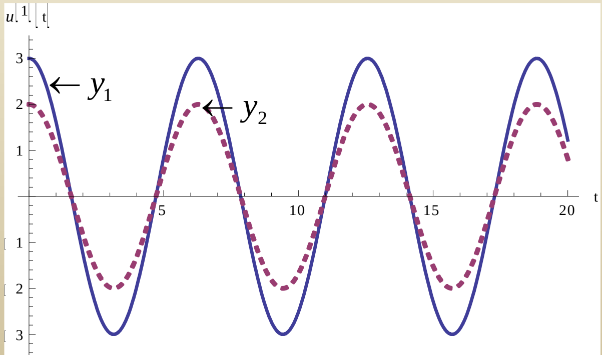
$$c_3 = c_4 = 0 \rightarrow \text{occurs when } 3y_2(0) = 2y_1(0) \text{ and } 3y_4(0) = 2y_3(0)$$

- To obtain the fundamental mode of vibration with frequency 2

$$c_1 = c_2 = 0 \rightarrow \text{occurs when } 3y_2(0) = -4y_1(0) \text{ and } 3y_4(0) = -4y_3(0)$$

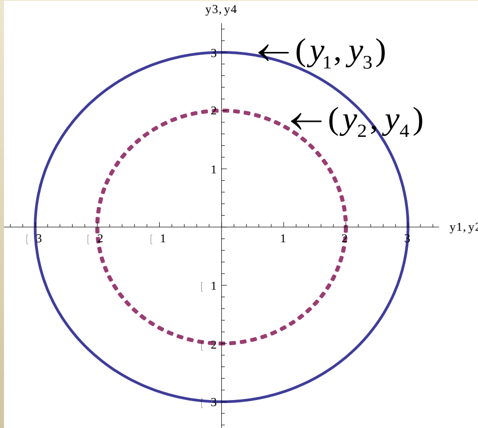
- Plots of y_1 and y_2 and parametric plots (y, y') are shown for a selected solution with frequency 1

Plots of the solutions as functions of time



$$y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

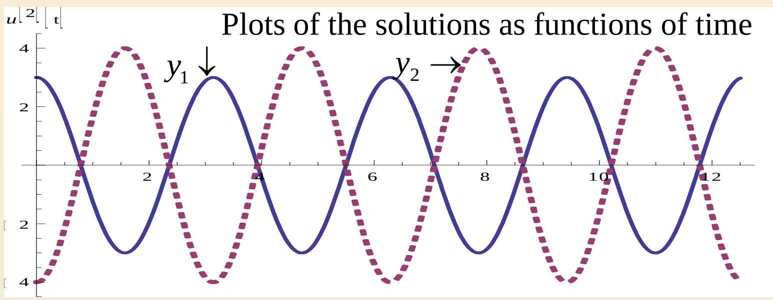
Phase plane plots



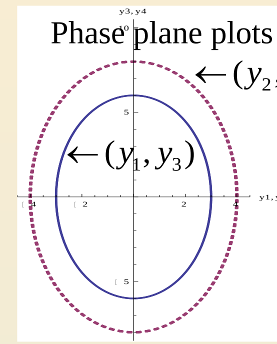
y_1 and y_2 represent the motion of the masses and $y_3 = y_1', y_4 = y_1'$

Example 3: Multiple Spring-Mass System (6 of 6)

- Plots of y_1 and y_2 and parametric plots (y, y') are shown for a selected solution with frequency 2

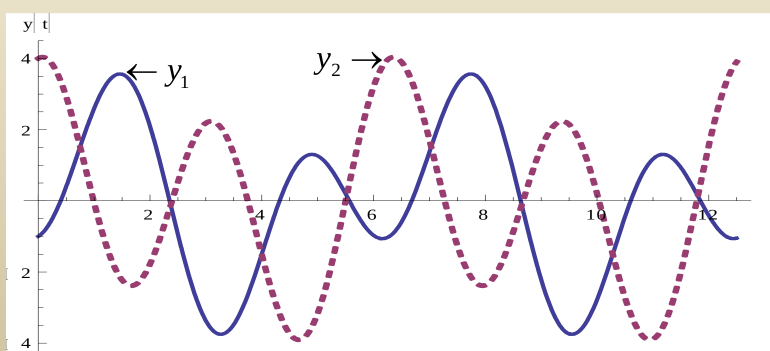


$$\boldsymbol{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix}$$

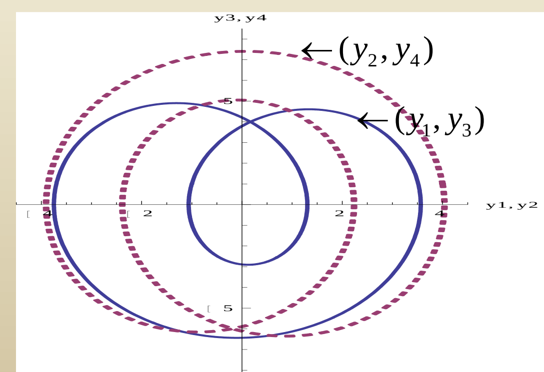


- Plots of y_1 and y_2 and parametric plots (y, y') are shown for a selected solution with mixed frequencies satisfying the initial condition stated

Plots of the solutions as functions of time



$$\boldsymbol{y}(0) = \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$



Phase plane plots