7.6: Complex Eigenvalues

- We consider again a homogeneous system of $n$ first order linear equations with constant, real coefficients,

\[
x'_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n
\]
\[
x'_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n
\]
\[
\vdots
\]
\[
x'_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
\]

and thus the system can be written as $x' = Ax$, where

\[
x(t) = \begin{pmatrix}
  x_1(t) \\
  x_2(t) \\
  \vdots \\
  x_n(t)
\end{pmatrix}, \quad A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]
Conjugate Eigenvalues and Eigenvectors

- We know that \( x = \xi e^{rt} \) is a solution of \( x' = Ax \), provided \( r \) is an eigenvalue and \( \xi \) is an eigenvector of \( A \).

- The eigenvalues \( r_1, \ldots, r_n \) are the roots of \( \det(A-rI) = 0 \), and the corresponding eigenvectors satisfy \( (A-rI)\xi = 0 \).

- If \( A \) is real, then the coefficients in the polynomial equation \( \det(A-rI) = 0 \) are real, and hence any complex eigenvalues must occur in conjugate pairs. Thus if \( r_1 = \lambda + i\mu \) is an eigenvalue, then the second solution is \( r_2 = \lambda - i\mu \).

- The corresponding eigenvectors \( \xi^{(1)}, \xi^{(2)} \) are conjugates also. To see this, recall \( A \) and \( I \) have real entries, and hence

\[
(A - r_1 I)\xi^{(1)} = 0 \Rightarrow (A - \overline{r_1} I)\overline{\xi^{(1)}} = 0 \Rightarrow (A - r_2 I)\xi^{(2)} = 0
\]
Conjugate Solutions

• It follows from the previous slide that the solutions

\[ \mathbf{x}^{(1)} = \xi^{(1)} e^{r_1 t}, \quad \mathbf{x}^{(2)} = \xi^{(2)} e^{r_2 t} \]

corresponding to these eigenvalues and eigenvectors are conjugates as well, since

\[ \mathbf{x}^{(2)} = \xi^{(2)} e^{r_2 t} = \bar{\xi}^{(1)} e^{\bar{r}_2 t} = \bar{\mathbf{x}}^{(1)} \]
Example 1: Direction Field  (1 of 7)

• Consider the homogeneous equation \( x' = Ax \) below:

\[
x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x
\]

• A direction field for this system is given below.
• Substituting \( x = \xi e^{rt} \) in for \( x \), and rewriting system as \((A-rI)\xi = 0\), we obtain

\[
\begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
Example 1: Complex Eigenvalues  (2 of 7)

• We determine $r$ by solving $\det(A - rI) = 0$. Now

$$\begin{vmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{vmatrix} = (r + 1/2)^2 + 1 = r^2 + r + \frac{5}{4}$$

• Thus

$$r = \frac{-1 \pm \sqrt{1^2 - 4(5/4)}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

• Therefore the eigenvalues are $r_1 = -1/2 + i$ and $r_2 = -1/2 - i$. 
Example 1: First Eigenvector  (3 of 7)

- Eigenvector for $r_1 = -1/2 + i$: Solve

$$\begin{align*}
(A - rI)\xi &= 0 \\
\Leftrightarrow \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}$$

by row reducing the augmented matrix:

$$
\begin{pmatrix} 1 & i & 0 \\ -1 & -i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -i \xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \begin{pmatrix} -i \\ \xi \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}
$$

- Thus \( \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)
Example 1: Second Eigenvector  

- Eigenvector for $r_1 = -1/2 - i$: Solve

$$ (A - rI)\xi = 0 \iff \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} $$

$$ \iff \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} $$

by row reducing the augmented matrix:

$$ \begin{pmatrix} 1 & -i & 0 \\ -1 & i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(2)} = \begin{pmatrix} i \xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} $$

- Thus

$$ \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} $$
Real valued solutions

- The two solutions:
  \[ X^{(1)}(t) = e^{-t/2} \left( \begin{array}{c} 1 \\ i \end{array} \right) e^{it}, \quad X^{(2)}(t) = e^{-t/2} \left( \begin{array}{c} 1 \\ -i \end{array} \right) e^{-it} \]

- Remember \( e^{it} = \cos t + i \sin t \)

- Then
  \[ X^{(1)}(t) = e^{-t/2} \left( \begin{array}{c} \cos t + i \sin t \\ -\sin t + i \cos t \end{array} \right), \quad X^{(2)}(t) = e^{-t/2} \left( \begin{array}{c} \cos t - i \sin t \\ -\sin t - i \cos t \end{array} \right) \]

\[
\frac{1}{2} \left( X^{(1)}(t) + X^{(2)}(t) \right) = e^{-t/2} \left( \begin{array}{c} \cos t \\ -\sin t \end{array} \right), \quad \frac{1}{2i} \left( X^{(1)}(t) - X^{(2)}(t) \right) = e^{-t/2} \left( \begin{array}{c} \sin t \\ \cos t \end{array} \right)
\]

- The two real valued solutions:
  \[ u(t) = e^{-t/2} \left( \begin{array}{c} \cos t \\ -\sin t \end{array} \right), \quad v(t) = e^{-t/2} \left( \begin{array}{c} \sin t \\ \cos t \end{array} \right) \]
Example 1: General Solution (5 of 7)

• The corresponding solutions \( x = \xi e^{rt} \) of \( x' = Ax \) are

\[
\begin{align*}
  u(t) &= e^{-t/2} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] = e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \\
  v(t) &= e^{-t/2} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right] = e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}
\end{align*}
\]

• The Wronskian of these two solutions is

\[
W[x^{(1)}, x^{(2)}](t) = \begin{vmatrix} e^{-t/2} \cos t & e^{-t/2} \sin t \\ -e^{-t/2} \sin t & e^{-t/2} \cos t \end{vmatrix} = e^{-t} \neq 0
\]

• Thus \( u(t) \) and \( v(t) \) are real-valued fundamental solutions of \( x' = Ax \), with general solution \( x = c_1 u + c_2 v \).
Example 1: Phase Plane (6 of 7)

- Given below is the phase plane plot for solutions $x$, with

$$ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2} \cos t \\ -e^{-t/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2} \sin t \\ e^{-t/2} \cos t \end{pmatrix} $$

- Each solution trajectory approaches origin along a spiral path as $t \to \infty$, since coordinates are products of decaying exponential and sine or cosine factors.

- The graph of $u$ passes through $(1,0)$, since $u(0) = (1,0)$. Similarly, the graph of $v$ passes through $(0,1)$.

- The origin is a spiral point, and is asymptotically stable.
Example 1: Time Plots (7 of 7)

• The general solution is $x = c_1 u + c_2 v$:

$$x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t \\ -c_1 e^{-t/2} \sin t + c_2 e^{-t/2} \cos t \end{pmatrix}$$

• As an alternative to phase plane plots, we can graph $x_1$ or $x_2$ as a function of $t$. A few plots of $x_1$ are given below, each one a decaying oscillation as $t \to \infty$. 
General Solution

• To summarize, suppose $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, and that $r_3, \ldots, r_n$ are all real and distinct eigenvalues of $A$. Let the corresponding eigenvectors be

$$
\xi^{(1)} = \mathbf{a} + i\mathbf{b}, \quad \xi^{(2)} = \mathbf{a} - i\mathbf{b}, \quad \xi^{(3)}, \xi^{(4)}, \ldots, \xi^{(n)}
$$

• Then the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$
\mathbf{x} = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) + c_3 \xi^{(3)} e^{r_3 t} + \ldots + c_n \xi^{(n)} e^{r_n t}
$$

where

$$
\mathbf{u}(t) = e^{\lambda t} \left( \mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t \right), \quad \mathbf{v}(t) = e^{\lambda t} \left( \mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t \right)
$$
Real-Valued Solutions

- Thus for complex conjugate eigenvalues $r_1$ and $r_2$, the corresponding solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are conjugates also.

- To obtain real-valued solutions, use real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. To see this, let $\xi^{(1)} = a + i b$. Then

\[
\mathbf{x}^{(1)} = \xi^{(1)} e^{(\lambda + i \mu)t} = (a + ib)e^{\lambda t} (\cos \mu t + i \sin \mu t)
\]
\[
= e^{\lambda t} (a \cos \mu t - b \sin \mu t) + ie^{\lambda t} (a \sin \mu t + b \cos \mu t)
\]
\[
= \mathbf{u}(t) + i \mathbf{v}(t)
\]

where

\[
\mathbf{u}(t) = e^{\lambda t} (a \cos \mu t - b \sin \mu t), \quad \mathbf{v}(t) = e^{\lambda t} (a \sin \mu t + b \cos \mu t),
\]

are real valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and can be shown to be linearly independent.
Spiral Points, Centers, Eigenvalues, and Trajectories

- In previous example, general solution was

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2} \cos t \\ -e^{-t/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2} \sin t \\ e^{-t/2} \cos t \end{pmatrix}
\]

- The origin was a **spiral point**, and was **asymptotically stable**.

- If real part of complex eigenvalues is positive, then trajectories spiral away, unbounded, from origin, and hence origin would be an **unstable spiral point**.

- If real part of complex eigenvalues is zero, then trajectories circle origin, neither approaching nor departing. Then origin is called a **center** and is **stable**, but not asymptotically stable. Trajectories periodic in time.

- The direction of trajectory motion depends on entries in A.
Example 2: Second Order System with Parameter (1 of 2)

- The system $x' = Ax$ below contains a parameter $\alpha$:
  \[ x' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} x \]

- Substituting $x = \xi e^{rt}$ in for $x$ and rewriting system as $(A - rI)\xi = 0$, we obtain
  \[
  \begin{pmatrix}
  \alpha - r & 2 \\
  -2 & -r
  \end{pmatrix}
  \begin{pmatrix}
  \xi_1 \\
  \xi_1
  \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
  \]

- Next, solve for $r$ in terms of $\alpha$:
  \[
  \left| \begin{array}{cc}
  \alpha - r & 2 \\
  -2 & -r
  \end{array} \right| = r(r - \alpha) + 4 = r^2 - \alpha r + 4 \Rightarrow r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}
  \]
Example 2: Eigenvalue Analysis  (2 of 2)

• The eigenvalues are given by the quadratic formula above.

\[ r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2} \]

• For \( \alpha < -4 \), both eigenvalues are real and negative, and hence origin is asymptotically stable node.

• For \( \alpha > 4 \), both eigenvalues are real and positive, and hence the origin is an unstable node.

• For \(-4 < \alpha < 0\), eigenvalues are complex with a **negative real part**, and hence origin is asymptotically stable spiral point.

• For \( 0 < \alpha < 4 \), eigenvalues are complex with a **positive real part**, and the origin is an unstable spiral point.

• For \( \alpha = 0 \), eigenvalues are purely imaginary, origin is a center. Trajectories closed curves about origin & periodic.

• For \( \alpha = \pm 4 \), eigenvalues real & equal, origin is a node (Ch 7.8)
Second Order Solution Behavior and Eigenvalues: Three Main Cases

For second order systems, the three main cases are:

- Eigenvalues are real and have opposite signs; \( x = 0 \) is a saddle point.
- Eigenvalues are real, distinct and have same sign; \( x = 0 \) is a node.
- Eigenvalues are complex with nonzero real part; \( x = 0 \) a spiral point.

Other possibilities exist and occur as transitions between two of the cases listed above:

- A zero eigenvalue occurs during transition between saddle point and node. Real and equal eigenvalues occur during transition between nodes and spiral points. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points.

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Example 3: Multiple Spring-Mass System (1 of 6)

- The equations for the system of two masses and three springs discussed in Section 7.1, assuming no external forces, can be expressed as:

\[ m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2x_2 \quad \text{and} \quad m_2 \frac{d^2 x_2}{dt^2} = k_2x_1 - (k_2 + k_3)x_2 \]

or \( m_1 y_3' = -(k_1 + k_2)y_1 + k_2y_2 \) \quad \text{and} \quad \( m_2 y_4' = k_2y_1 - (k_2 + k_3)y_2 \)

where \( y_1 = x_1 \), \( y_2 = x_2 \), \( y_3 = x_1' \), and \( y_4 = x_2' \)

- Given \( m_1 = 2 \), \( m_2 = 9/4 \), \( k_1 = 1 \), \( k_2 = 3 \), and \( k_3 = 15/4 \), the equations become

\[ y_1' = y_3, \quad y_2' = y_4, \quad y_3' = -2y_1 + 3/2\ y_2, \quad \text{and} \quad y_4' = 4/3\ y_1 - 3y_2 \]
Example 3: Multiple Spring-Mass System (2 of 6)

- Writing the system of equations in matrix form:

\[
y' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} y = Ay
\]

- Assuming a solution of the form \( y = \xi e^{rt} \), where \( r \) must be an eigenvalue of the matrix \( A \) and \( \xi \) is the corresponding eigenvector, the characteristic polynomial of \( A \) is

\[
r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4)
\]

yielding the eigenvalues: \( r_1 = i, r_2 = -i, r_3 = 2i, \) and \( r_4 = -2i \)
Example 3: Multiple Spring-Mass System (3 of 6)

• For the eigenvalues \( r_1 = i, \ r_2 = -i, \ r_3 = 2i, \) and \( r_4 = -2i \) the corresponding eigenvectors are

\[
\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \quad \text{and} \quad \xi^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}
\]

• The products \( \xi^{(1)} e^{it} \) and \( \xi^{(3)} e^{2it} \) yield the complex-valued solutions:

\[
\xi^{(1)} e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 3 \cos t \\ 2 \cos t \\ -3 \sin t \\ -2 \sin t \end{pmatrix} + i \begin{pmatrix} 3 \sin t \\ 2 \sin t \\ 3 \cos t \\ 2 \cos t \end{pmatrix} = u^{(1)}(t) + i \ v^{(1)}(t)
\]

\[
\xi^{(3)} e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix} (\cos 2t + i \sin 2t) = \begin{pmatrix} 3 \cos 2t \\ -4 \cos 2t \\ -6 \sin 2t \\ 8 \sin 2t \end{pmatrix} + i \begin{pmatrix} 3 \sin 2t \\ -4 \sin 2t \\ 6 \cos 2t \\ -8 \cos 2t \end{pmatrix} = u^{(2)}(t) + i \ v^{(2)}(t)
\]
Example 3: Multiple Spring-Mass System (4 of 6)

- After validating that $u^{(1)}(t)$, $v^{(1)}(t)$, $u^{(2)}(t)$, $v^{(2)}(t)$ are linearly independent, the general solution of the system of equations can be written as

$$
\begin{align*}
\mathbf{y} &= c_1 \begin{pmatrix} 3 \cos t \\ 2 \cos t \\ -3 \sin t \\ -2 \sin t \end{pmatrix} + c_2 \begin{pmatrix} 3 \sin t \\ 2 \sin t \\ 3 \cos t \\ 2 \cos t \end{pmatrix} + c_3 \begin{pmatrix} 3 \cos 2t \\ -4 \cos 2t \\ -6 \sin 2t \\ 8 \sin 2t \end{pmatrix} + c_4 \begin{pmatrix} 3 \sin 2t \\ -4 \sin 2t \\ 6 \cos 2t \\ -8 \cos 2t \end{pmatrix} \\
\end{align*}
$$

- where $c_1$, $c_2$, $c_3$, $c_4$ are arbitrary constants.

- Each solution will be periodic with period $2\pi$, so each trajectory is a closed curve. The first two terms of the solution describe motions with frequency 1 and period $2\pi$ while the second two terms describe motions with frequency 2 and period $\pi$. The motions of the two masses will be different relative to one another for solutions involving only the first two terms or the second two terms.
Example 3: Multiple Spring-Mass System (5 of 6)

- To obtain the fundamental mode of vibration with frequency 1

\[
c_3 = c_4 = 0 \implies 3y_2(0) = 2y_1(0) \quad \text{and} \quad 3y_4(0) = 2y_3(0)
\]

- To obtain the fundamental mode of vibration with frequency 2

\[
c_1 = c_2 = 0 \implies 3y_2(0) = -4y_1(0) \quad \text{and} \quad 3y_4(0) = -4y_3(0)
\]

- Plots of \( y_1 \) and \( y_2 \) and parametric plots \((y, y')\) are shown for a selected solution with frequency 1

\[
y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}
\]

Plots of the solutions as functions of time

Phase plane plots
Example 3: Multiple Spring-Mass System (6 of 6)

- Plots of $y_1$ and $y_2$ and parametric plots ($y, y'$) are shown for a selected solution with frequency 2

\[ y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix} \]

- Plots of $y_1$ and $y_2$ and parametric plots ($y, y'$) are shown for a selected solution with mixed frequencies satisfying the initial condition stated

\[ y(0) = \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix} \]