MA 303 Systems of Differential Equations (Dr. Park)

Sec 6.2 Almost Linear systems

Theorem 2 **(Stability of almost linear system**

(H) Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix (: Jacobian matrix) of the linearized system.

(C) (1) (CASE 1) $\lambda_1, \lambda_2 \in \mathbb{R}$ $(\lambda_1 \neq \lambda_2)$ or (CASE 2) $\lambda_1 = \alpha + \beta i,\ \lambda_2 = \alpha - \beta i$ $(\alpha \neq 0)$:

the critical point $(0, 0)$ of the almost linear system is of the same type and stability as the critical point $(0, 0)$ of the linearized system.

(2) (CASE 2) $\lambda_1 = \beta i,\ \lambda_2 = -\beta i$:

the critical point $(0, 0)$ of the almost linear system is either a center or a spiral point. The point is either asymtotically stable, stable or unstable.

(3) (CASE 3) $\lambda_1, \lambda_2 \in \mathbb{R}$ $(\lambda_1 = \lambda_2)$:

the critical point $(0, 0)$ of the almost linear system is either a node or a spiral point.

(a) If $\lambda_1 > 0$, then the point is unstable.

(b) If $\lambda_1 < 0$, then the point is asymptotically stable.

Sec 6.3 Ecological models

1. Malthus (1798): Human population grows exponentially.

$$\frac{dx}{dt} = kx, \qquad k > 0$$

2. **Vito Volterra (1860 - 1940): Predator - Prey systems**

(**Assumption**)

(1) In the absence of predators, the prey population would grow at a natural rate: $\frac{dx}{dt} = ax, \qquad a > 0.$

(2) In the absence of preys, the predator population would decline at a natural rate: $\frac{dy}{dt} = -by, \qquad b > 0.$

(3) **Nature of interaction:** PREDATION

When both predators and preys are present, there occurs, in combination with these natural rate of growth and decline, a decline in the prey population and a growth in predator population, each at a rate of proportional to the frequency of encounters between individuals of two species.

We assume that the frequency is proportional to the product xy. The consumption of the preys by predators results in

(a) an interaction rate of decline: $-pxy$ in the prey population x.

(b) an interaction rate of growth: qxy in the predator population y.

$$\begin{aligned} \frac{dx}{dt} &= ax - pxy \\ \frac{dy}{dt} &= -by + qxy \end{aligned}$$

(EXAMPLE 1)

$$x' = 10x - xy, \qquad y' = -4y + 2xy$$

(1) Find critical points of the system

(2) Around each critical point linearize the system. Classify the critical point and determine its stability.

(3) Plot the phase portraits to analyze numerically the system.

3. **COMPETING SPECIES**:

Consider two species (of animals, or plants or bacteria) with populations $x(t)$ and $y(t)$ at time t that compete with each other for the food available in their common environment.

(**Assumptions**)

(1) Realistic model: we assume that each species has bounded (logistic) population:

$$x'(t) = a_1x - b_1x^2, \qquad y'(t) = a_2y - b_2y^2.$$

(2) Competition has the effect of a rate of decline in each population that is negatively proportional to the product xy.

Then, the competition system is

$$\begin{aligned} \frac{dx}{dt} &= a_1x - b_1x^2 - c_1xy = x(a_1 - b_1x - c_1y) \\ \frac{dy}{dt} &= a_2y - b_2y^2 - c_2xy = y(a_2 - b_2y - c_2x), \end{aligned}$$

where all the coefficients a_i, b_i and c_i, $i = 1, 2$ are positive.

(EXAMPLE 2)

$$x' = 4x - x^2 - xy, \qquad\qquad y' = 6y - 2y^2 - xy$$

(1) Find the critical points of the system.

(2) Classify the critical points and determine their stability.