MOTIVIC STRUCTURES ON HIGHER HOMOTOPY GROUPS OF HYPERPLANE ARRANGEMENTS

DEEPAM PATEL

Abstract. In this note, we show the existence of motivic structures on certain objects arising from the higher (rational) homotopy groups of non-nilpotent spaces. Examples of such spaces include several families of hyperplane arrangements. In particular, we construct an object in Nori's category of motives whose realization is a certain completion of $\pi_n(\mathbb{P}^n \setminus \{L_1, \ldots, L_{n+2}\})$ where the L_i are hyperplanes in general position. Similar results are shown to hold in Vovoedsky's setting of mixed motives.

Contents

1. Introduction

In his seminal paper on the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ ([4]), Deligne constructed a motivic structure on the Malcev Lie algebra and the Malcev completion of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x)$, where x is some fixed base point or, more generally, a tangential base point. Furthermore, he showed that the corresponding motives give rise to extensions of mixed Tate motives, and that their periods give rise to values of the Riemann zeta function at positive integer values.

Since then several authors have studied the motivic nature of homotopy theoretic invariants associated to algebraic varieties. In ([6]), Deligne and Goncharov recast Deligne's theory in Voevodsky's triangulated category of motives. The resulting motives were used to study the Tannakian fundamental group of various categories of mixed Tate motives over rings of integers of number fields ([5], [6]). For example, it was conjectured by Deligne (and proved recently by Francis Brown [3]), that the category of mixed Tate motives over $\mathbb{Z}$ (upto isogeny) is generated (in an appropriate Tannakian sense) by motives arising from the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. On the other hand, it is known that the analogous result is false for the category of mixed Tate motives over rings of integers in the cyclotomic extension given by N-th roots of unity (outside a finite set of values for N). More precisely, the corresponding mixed Tate category is not generated by the motives arising from the fundamental group of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$, where μ_N is the set of N-th roots of unity.

The main goal of this article is to generalize the constructions of mixed motives attached to fundamental groups to higher homotopy groups. As a result, we shall obtain a whole new class of examples of (extensions of) mixed Tate motives which one might use to understand the structure of mixed Tate motives over arbitrary rings of integers. Note that the existence of a motivic structure implies, in particular, that one has an associated Hodge structure. In the setting of nilpotent spaces, such Hodge theoretic results were obtained previously by Hain and Morgan. In ([8]), Hain constructed mixed Hodge structures on the higher homotopy groups of nilpotent spaces.[1] A similar result was proved by Morgan ([16]) in the case of smooth complex algebraic varieties.

Given a pointed complex algebraic variety (X, x), let $\mathfrak{g}_*(X, x)$ denote the homotopy (graded) Lie algebra of X. In particular, $\mathfrak{g}_i(X, x) = \pi_{i+1}(X, x)$ and the Lie product is given by the Whitehead product. If (X, x) is a nilpotent space, then it follows from the results of Hain ([8]) that the rational homotopy Lie algebra $\mathfrak{g}_*(\mathbb{Q}X, x)$ has a mixed Hodge structure. Here $\mathbb{Q}X$ is the $\mathbb{Q}$-completion of X (cf. section 2). For nilpotent spaces, one has:

$$\mathfrak{g}_*(\mathbb{Q}X, x) = \begin{cases} \mathbb{Q}\text{-Malcev completion of } \pi_1(X, x) & \text{if } * = 0 \\ \pi_{*+1}(X, x) \otimes \mathbb{Q} & \text{if } * > 0 \end{cases}$$

Hain's proof begins with the observation that the Chen-Sullivan de Rham theorem for the fundamental group (and higher homotopy of nilpotent spaces) allows one to compute the rational homotopy theory of nilpotent spaces through various differentially graded algebras such as Sullivan's minimal model or Chen's complex of iterated integrals. Hain shows that the corresponding complexes of iterated integrals have a natural structure of a mixed Hodge complex and, therefore, gives rise to a mixed Hodge structure on the rational homotopy theory of complex algebraic varieties whose underlying complex analytic space is a nilpotent space. A motivic version of this construction was given by K. Gartz ([7]). In particular, Gartz constructed a differentially graded Lie algebra (associated to any pointed variety) in Nori's abelian category of motives whose Hodge realization, in

[1] A pointed topological space is *nilpotent* if $\pi_1(X, x)$ is nilpotent, and acts nilpotently on all higher homotopy groups.

case of simply connected nilpotent spaces, gives Hain's mixed Hodge structure on the rational homotopy Lie algebra.

Unfortunately, from the point of view of constructing interesting extensions of mixed Tate motives, there do not seem to be any interesting nilpotent spaces. However, there are many interesting M-nilpotent spaces.[2] Our primary example of an M-nilpotent space is $\mathbb{P}^M \setminus \{L_1, \dots, L_{M+2}\}$, where the L_i's are hyperplanes in general position ([10]). In this case, an easy application of the homotopy group version of the weak Lefschetz theorem shows that $\pi_i(\mathbb{P}^M \setminus \{L_1, \dots, L_{M+2}\}) = 0$ for all $1 < i < M$ and therefore trivially M-nilpotent. More generally, results of Papadima and Suciu ([18]) show that the class of hypersolvable arrangements gives rise to M-nilpotent spaces for some M depending on the intersection lattice of the corresponding hyperplane arrangement. In this article, we show that a certain completion of the M-th homotopy group of any M-nilpotent space also has a motivic structure. This was suggested as a problem by Madhav Nori.

Our first main observation is that, for a M-nilpotent space, standard methods of rational homotopy theory (Chen's iterated integrals, Sullivan minimal models, and the Bar construction) can be used to compute the nilpotent completion of the M-th homotopy group. This result can be viewed as a direct generalization of the Chen-Sullivan π_1-de Rham theorem to the case of M-nilpotent spaces. The proof of this theorem is based on the observation that, while traditionally rational homotopy theory deals only with the class of nilpotent spaces, its methods and results extend to the slightly more general class of M-nilpotent spaces. We believe that this result is of independent topological interest.

Theorem 1.1. *Let X be a M-nilpotent space of finite $\mathbb{Q}$-type. Then we have the following:*

(1) $\pi_1(\mathbb{Q}X)$ *is the Malcev completion of* $\pi_1(X)$.

(2) *There are natural isomorphisms*

$$\pi_k(\mathbb{Q}X) \to \pi_k(X) \otimes \mathbb{Q}$$

for all $1 < k < M$.

(3) *There is a natural isomorphism*

$$\pi_M(\mathbb{Q}X) \to \varprojlim \pi_M(X) I^c \pi_M(X) \otimes \mathbb{Q}.$$

In particular, one has isomorphisms

$$\pi_M(\mathbb{Q}X)/I^c\pi_M(\mathbb{Q}X) \to \pi_M(X)/I^c\pi_M(X) \otimes \mathbb{Q}.$$

Furthermore, $\pi_M(X)/I^c\pi_M(X) \otimes \mathbb{Q}$ *is a finite dimensional vector space.*

The results of Hain-Sullivan now give the following corollary, which is a direct generalization of Deligne's theorem on existence of a Hodge structure on the unipotent completion of the fundamental group of the projective line minus three points.

[2] A space is *M-nilpotent* if $\pi_1(X, x)$ is nilpotent and acts nilpotently on $\pi_n(X)$ for all $n < M$.

Corollary 1.2. *Let $X = \mathbb{P}^M \setminus \{L_1, \dots, L_{M+2}\}$ denote the complement of a generic hyperplane arrangement. Then the nilpotent completion*

$$\varprojlim \pi_M(X) I^c \pi_M(X) \otimes \mathbb{Q}$$

has a natural mixed Hodge structure.

In fact, the above corollary holds more generally for varieties whose associated complex analytic spaces satisfy the conditions of the previous theorem. The second main result of this article is a motivic generalization of the previous corollary.

Theorem 1.3. *Let $X = \mathbb{P}^M \setminus \{L_1, \cdots, L_{M+2}\}$. Then there is an object $\mathcal{M}^{M,k}_{(X,x)}$ in Nori's category of motives whose Betti realization is $\pi_M(X,x)/I^k\pi_M(X,x) \otimes \mathbb{Q}$.*

In fact, we prove analogs of the previous theorem more generally for any M-nilpotent variety. Furthermore, we construct motivic differentially graded algebras whose betti realizations are appropriate nilpotent completions of the rational homotopy Lie algebra. Note that $\pi_M(X,x)/I^k\pi_M(X,x) \otimes \mathbb{Q}$ is a module over $\mathbb{Q}[\pi_1(X(\mathbb{C}),x)]/I^k$. We show more generally that this module structure also lifts to the category of motives. We refer the reader to section 4 for the precise statements. Finally, we also prove an analogous statement in the Deligne-Goncharov category of integral mixed Tate motives.

Theorem 1.4. *Let $X = \mathbb{P}^M_k \setminus \{L_1, \cdots, L_{M+2}\}$ where k is now a number field. Then there is an object $\mathcal{M}^{M,k}_{(X,x)}$ in the Deligne-Goncharov category of mixed Tate motives whose Betti realization is $\pi_M(X,x)/I^k\pi_M(X,x) \otimes \mathbb{Q}$.These motives can be lifted to motives over the ring of integers localized away from points of bad reduction.*

The objects $\mathcal{M}^{M,k}_{(X,x)}$ give rise to interesting extensions of mixed Tate motives. We believe that these should give rise to interesting periods. For example, we hope that by taking different types of hyperplane arrangements one can realize special values of hypergeometric functions as well as higher Aomoto logarithms as periods of these motives. Furthermore, these motives should lead to a better understanding of the category of mixed Tate motives. We do not study any of these questions here, and plan to pursue them elsewhere.

We now give a brief outline of article. In the second section, we review some results from rational homotopy theory, and, in particular, compute the homotopy groups of the $\mathbb{Q}$-completion of an M-nilpotent space. In the third section, we recall Gartz's construction of various dga's (differentially graded algebras) and dgla's (differentially graded Lie algebra's) in a general $\mathbb{Q}$-Karoubian category. In section 4 we construct motives in Nori's category arising form higher homotopy of M-nilpotent spaces. Finally, in the last section we recast these results in the Deligne-Goncharov category of mixed Tate motives.

Acknowledgements It will be clear the debt this article owes to the ideas of Professor Nori as well as K. Gartz. The author would like to thank Professor Nori for suggesting the above problem and patiently explaining his ideas.

2. Rational Homotopy Theory

The main goal of this section (cf. Theorem 2.10) is to compute the M-th homotopy group of the $\mathbb{Q}$-completion of M-nilpotent spaces. In the following, by a space we will mean a pointed path-connected topological space. However, we will drop the base point from the notation.We shall also assume that our spaces have the homotopy type of a CW complex. In this article, we are only interested in algebraic varieties, and so these assumptions will always be satisfied.

Remark 2.1. In the following, we shall refer to several results of ([1]) and ([2]). In loc. cit., the authors work in the setting of simplicial sets. However, given a topological space X, the singular chains give rise to a fibrant simplicial set $Sing(X)$. On the other hand, taking the geometric realization of a simplicial set gives a topological space. This gives rise to a pair of adjoint functors between simplicial sets and topological spaces which induce an equivalence on the corresponding homotopy categories. As a result, the results in loc. cit. are also applicable to our setting. We refer the reader to ([2], Chapter 8) for the details.

In ([2]), Bousfield–Kan (functorially) associate to X a tower of fibrations $\{\mathbb{Q}_s X\}$. The resulting inverse limit, denoted $\mathbb{Q}X$, is called the $\mathbb{Q}$-*completion* of X. If X is a nilpotent space, then $\pi_i(\mathbb{Q}X) = \pi_i(X) \otimes \mathbb{Q}$ ([2], Chapter 5, Proposition 4.2).

Remark 2.2. ([1], pg. 51) A nilpotent Kan complex is of finite $\mathbb{Q}$-type (i.e. has finite dimensional rational homology) if and only if $\mathrm{H}_1(X, \mathbb{Q})$ is finite dimensional and $\pi_n(X) \otimes \mathbb{Q}$ is finite dimensional for all $n \geq 2$.

The following lemma will be the main tool used to compute the homotopy groups of the $\mathbb{Q}$-completion in the case of M-nilpotent spaces.

Lemma 2.3. *Let X be a space as above. Consider a commutative diagram:*

$$\begin{array}{ccccccc} X & \longleftarrow & X & \longleftarrow & X & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ N_1 & \longleftarrow & N_2 & \longleftarrow & N_3 & \longleftarrow & \cdots \end{array}$$

such that:

1. *Each N_i is a nilpotent space.*
2. *The bottom row is a tower of fibrations.*
3. *The natural morphism $\mathrm{H}_i(X, \mathbb{Q}) \to \mathrm{H}_i(\varprojlim N_r, \mathbb{Q})$ is an isomorphism for all $i \leq M+1$ and a surjection for $i = M+2$.*

Then, then natural morphism $\mathbb{Q}X \to \mathbb{Q}\varprojlim N_r$ induces an isomorphism $\pi_i(\mathbb{Q}) \to \pi_i(\mathbb{Q}\varprojlim N_r)$ for all $i \leq M$.

Proof. Let $N_\infty := \varprojlim N_r$. By ([2], Ch. 1, 6.2), under our hypothesis, one has $\pi_i(\mathbb{Q}X) \cong \pi_i(\mathbb{Q}N_\infty)$ for all $i \leq M$. □

In order to apply the previous lemma, we shall now construct good towers, satisfying the conditions of the previous lemma, for M-nilpotent spaces. We begin by recording the following consequence of the Leray-Serre spectral sequence for future use.

Lemma 2.4. *Let $X \to Y$ be a fibration with fiber given by a $K(G,n)$ and I denote the augmentation ideal of $\mathbb{Z}[\pi_1(Y)]$. Then there is a long exact sequence:*

$$\mathrm{H}_{n+2}(X) \to \mathrm{H}_{n+2}(Y) \to \mathrm{H}_1(\pi_1(Y), G) \to \mathrm{H}_{n+1}(X) \to \\ \mathrm{H}_{n+1}(X) \to G/IG \to \mathrm{H}_n(X) \to \mathrm{H}_n(Y) \to 0.$$

Proof. The proof is exactly the same as that of ([15], Lemma 8^{bis}.23). □

Given a space X, let $\{X^{(k)}, p_k\}$ denote the Moore–Postnikov tower of X. In particular, each $p_k : X^{(k)} \to X^{(k-1)}$ is a fibration with fiber a $K(\pi_n, n+1)$; furthermore, the maps $f_k : X \to X^{(k)}$ induce isomorphisms on π_i for all $i \leq k$, $\pi_i(X^{(k)}) = 0$ for all $i > k$, and $f_k = p_k \circ f_{k+1}$.

Remark 2.5. Since the natural map $X \to X^{(k)}$ induces an isomorphism $\pi_i(X) \to \pi_i(X^{(k)})$ for all $i \leq k$ and a surjection for $i = k+1$, it follows by the Whitehead theorem ([15], Theorem 4.5) that the induced map $\mathrm{H}_i(X) \to \mathrm{H}_i(X^{(k)})$ is an isomorphism for all $i \leq k$ and a surjection for $i = k+1$. It follows, via an application of the universal coefficients exact sequence, that $\mathrm{H}_i(X, \mathbb{Q}) \to \mathrm{H}_i(X^{(k)}, \mathbb{Q})$ is an isomorphism for all $i \leq k$ and a surjection for $i = k+1$.

A *principal fibration* $p : E \to B$ is a fibration which is the pull back of the path-loop fibration of a space C with respect to a map $\theta : B \to C$. If X is simply connected, then each p_k is a *principal fibration.* In particular, one has mappings $g_k : X^{(k-1)} \to K(\pi_k(X), k+1)$ such that p_k is the pullback of the canonical path fibration $PK(\pi_k(X), k+1) \to K(\pi_k(X), k+1)$ along g_k. The g_k are referred to as the *Postnikov invariants* of the Postnikov tower. A fibration $p_k : X^{(k)} \to X^{(k-1)}$ is said to have a *principal refinement* if it can be written as a sequence

$$X^{(k)} = X^{(k,c)} \xrightarrow{p_{(k,c)}} X^{(k,c-1)} \to \cdots \to X^{(k,1)} = X^{(k-1)}$$

where each $p_{(k,r)}$ is a principal fibration. One has the following standard result:

Theorem 2.6. *(*[15]*, Theorem 8^{bis}.29) The following are equivalent:*

(1) *X is a nilpotent space.*
(2) *Every stage of the Postnikov tower of X admits a principal refinement.*

One has an easy generalization of the previous result to the case of M-nilpotent spaces. Recall that a space X is *M-nilpotent* if $M \geq 2$, $\pi_1(X)$ is nilpotent and $\pi_1(X)$ acts nilpotently on $\pi_i(X)$ for all $i < M$.

Proposition 2.7. *The following are equivalent:*

(1) *X is an M-nilpotent space.*
(2) *If $k < M$, then the k-th stage, p_k, of the Postnikov tower of X admits a principal refinement.*

Furthermore, one can construct a commutative diagram of fibrations:

$$\begin{array}{ccccccccccc}
 & & X^{(M)} & \longleftarrow & X^{(M)} & \longleftarrow & \cdots & \longleftarrow & X^{(M)} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
X^{(M-1)} & \underset{=}{\longleftarrow} & X^{(M,0)} & \longleftarrow & X^{(M,1)} & \longleftarrow & \cdots & \longleftarrow & X^{(M,c)} & \longleftarrow & \cdots
\end{array}$$

such that $\pi_i(X^{(M,c)}) = \pi_i(X)$ for $i < M$ and $\pi_M(X^{(M,c)}) = \pi_M(X)/I^c\pi_M(X)$, where $I \subset \mathbb{Z}[\pi_1(X)]$ is the augmentation ideal. Finally, if X is, in addition, of finite $\mathbb{Q}$-type then so are the $X^{(k)}$ for all $k < M$ and $X^{(M,c)}$ for all c.

Proof. One proceeds as in the proof of the previous theorem ([15], Theorem 8^{bis}.29). One need only note that the theorem is proved separately at each stage of the Postnikov tower. At the each stage of the Postnikov tower, one constructs inductively a tower $\{X^{(k,c)}\}$ such that $\pi_i(X^{(k,c)}) = \pi_i(X)$ for all $i < k$ and $\pi_k(X^{(k,c)}) = \pi_k(X)/I^c\pi_k(X)$. If $k < M$, then the M-nilpotency hypothesis ensures that $X^{(k,c)} = X^{(k)}$ for large enough c. At the M-th stage, this gives an infinite tower with the required properties. It only remains to show the last statement on finiteness. Therefore, suppose now that X is of finite $\mathbb{Q}$-type. Note that $X^{(1)}$ is a $K(\pi_1(X), 1)$ and the hypotheses on $\pi_1(X)$ and X ensure that this space is also of finite $\mathbb{Q}$-type. Now consider the fibration $X^{(2,1)} \to X^{(1)}$ with fiber $K(\pi_2(X)/I\pi_2(X), 2)$. An application of Lemma 2.4 to this fibration gives an exact sequence:

$$\mathrm{H}_3(X^{(1)}) \to \pi_2(X)/I\pi_2(X) \to \mathrm{H}_2(X^{(2,1)}) \to \mathrm{H}_2(X^{(1)}) \to 0.$$

Since X is of $\mathbb{Q}$-finite type, arguing as in Remark 2.5 shows that the $\mathrm{H}_2(X^{(2,c)}) \otimes \mathbb{Q}$ is finite dimensional. Furthermore, the right-most and left-most terms tensored with $\mathbb{Q}$ are also finite dimensional since $X^{(1)}$ is of finite $\mathbb{Q}$-type. It follows that $\pi_2(X)/I\pi_2(X) \otimes \mathbb{Q}$ is also finite dimensional. By Remark 2.2, it follows that $X^{(2,1)}$ is of finite $\mathbb{Q}$-type. A similar application of Lemma 2.4 to the fibration $X^{(2)} \to X^{(2,c)}$ gives an exact sequence:

$$\mathrm{H}_3(X^{(2,1)}) \to I\pi_2(X)/I^2\pi_2(X) \to \mathrm{H}_2(X^{(2)}) \to \mathrm{H}_2(X^{(2,1)}) \to 0.$$

Arguing as above, we conclude that $I\pi_2(X)/I^2\pi_2(X) \otimes \mathbb{Q}$ is finite dimensional, and therefore, $\pi_2(X)/I^2\pi_2(X) \otimes \mathbb{Q}$ is also finite dimensional. Therefore, by Remark $2.2, X^{(2,2)}$ is of finite $\mathbb{Q}$-type. An inductive argument shows that $X^{(2,c)}$ is of finite $\mathbb{Q}$ type for all c. If $2 < M$, then $X^{(2,c)} = X^{(2)}$ for large enough c, and therefore $X^{(2)}$ if of finite $\mathbb{Q}$-type. We can now again argue as above to conclude that $X^{(k)}$ is of finite $\mathbb{Q}$-type for all $k < M$ and $X^{(M,c)}$ is of finite $\mathbb{Q}$-type for all c. □

Remark 2.8. The previous proposition shows that if X is a M-nilpotent space of finite $\mathbb{Q}$-type then, $\pi_M(X)/I^c\pi_M(X) \otimes \mathbb{Q}$ $(c > 0)$ is a finite dimensional $\mathbb{Q}$-vector space.

Lemma 2.9. *Suppose X is an M-nilpotent space of finite $\mathbb{Q}$-type and $X^{(N,c)}$ is as in the proposition. Then* $\mathrm{H}_i(\varprojlim X^{(M,c)}) \cong \varprojlim \mathrm{H}_i(X^{(M,c)})$ *and* $\pi_i(\mathbb{Q}\varprojlim X^{(M,c)}) \cong \pi_i(\varprojlim \mathbb{Q}X^{(M,c)})$.

Proof. Let $X^{(M,\infty)} := \varprojlim X^{(M,c)}$. By ([2], Chapter 3, 3.4) and ([2], Chapter 3, 6.2), it is enough to show that $\{X^{(M,\infty)}\} \to \{X^{(M,c)}\}$ is a pro-weak homotopy equivalence (the left term is considered as a constant tower). For this it suffices to show that the morphism of towers of abelian groups $\{\pi_k(X^{(M,\infty)})\} \to \{\pi_k(X^{(M,c)})\}$ is a pro-isomorphism The standard exact sequence

$$0 \to R^1\varprojlim \pi_{k+1}(X^{(M,c)}) \to \pi_k(X^{(M,\infty)}) \to \varprojlim \pi_k(X^{(M,c)}) \to 0$$

and the fact that all the transition maps in the inverse system $\{\pi_k(X^{(M,c)})\}$ are surjective, shows that $\pi_k(X^{(M,\infty)}) \to \varprojlim \pi_k(X^{(M,c)})$ is an isomorphism. For $k \neq M$, the claim is clear. For $k = M$, we are reduced to showing that the morphism of towers $\{\varprojlim \pi_M(X^{(M)})/I^c\pi_M(X^{(M)})\} \to \{\pi_M(X^{(M)})/I^c\pi_k(X^{(M)})\}$ is a pro-isomorphism. The latter can be checked by hand. □

The main goal of this section is to prove the following theorem.

Theorem 2.10. *Let X be a M-nilpotent space of finite $\mathbb{Q}$-type. Then we have the following:*

(1) $\pi_1(\mathbb{Q}X)$ *is the Malcev completion of* $\pi_1(X)$.

(2) *There are natural isomorphisms*

$$\pi_k(\mathbb{Q}X) \to \pi_k(X) \otimes \mathbb{Q}$$

for all $1 < k < M$.

(3) *One has a natural isomorphism*

$$\pi_M(\mathbb{Q}X) \to \varprojlim(\pi_M(X)/I^c\pi_M(X) \otimes \mathbb{Q}).$$

Proof. The first part of the proposition is true for any space X by ([2]). Since

$$\pi_i(\mathbb{Q}X) = \pi_i(\mathbb{Q}X^{(M-1)})$$

for all $1 \leq i < M$ and $X^{(M-1)}$ is a nilpotent space, the second part follows. By the following lemma, one has an isomorphism

$$\pi_M(\mathbb{Q}X) \to \pi_M(\varprojlim \mathbb{Q}X^{(M,c)}).$$

Therefore, it suffices to compute the right hand side. One has an exact sequence:

$$0 \to R^1\varprojlim \pi_{M+1}(\mathbb{Q}X^{(M,c)}) \to \pi_M(\varprojlim \mathbb{Q}X^{(M,c)}) \to \varprojlim \pi_M(\mathbb{Q}X^{(M,c)}) \to 0.$$

Since $X^{(M,c)}$ is a nilpotent space, one has $\pi_i(\mathbb{Q}X^{(M,c)}) = \pi_i(X^{(M,c)}) \otimes \mathbb{Q}$ for all $i > 1$. It follows that $\pi_{M+1}(\mathbb{Q}X^{(M,c)}) = 0$ and $\pi_M(\mathbb{Q}X^{(M,c)}) = \pi_M(X)/I^c\pi_M(X) \otimes \mathbb{Q}$. The result now follows from the exact sequence above. □

Therefore, the proof of the theorem is reduced to the following lemma.

Lemma 2.11. *Let X be a M-nilpotent space of finite $\mathbb{Q}$-type. Then the natural morphism $\mathbb{Q}X \to \varprojlim \mathbb{Q}X^{(M,c)}$ induces an isomorphism*

$$\pi_M(\mathbb{Q}X) \to \pi_M(\varprojlim \mathbb{Q}X^{(M,c)}).$$

Proof. By ([2], Chapter 4, Proposition 5.1), $\pi_i(\mathbb{Q}X) = \pi_i(\mathbb{Q}X^{(M)})$ for all $k \leq M$. Therefore, we can may replace X by $X^{(M)}$. By Lemma 2.3, Proposition 2.7, and Lemma 2.9 it is enough to show that

$$\mathrm{H}_i(X^{(M)}, \mathbb{Q}) \to \mathrm{H}_i(\varprojlim X^{(M,c)}, \mathbb{Q}) = \varprojlim \mathrm{H}_i(X^{(M,c)}, \mathbb{Q})$$

is an isomorphism for all $i \leq M+1$ and surjective for $M+2$. An application of Lemma 2.4 to the fibration $X^{(M)} \to X^{(M,c)}$ gives an exact sequence

$$\mathrm{H}_{M+2}(X^{(M)}) \to \mathrm{H}_{M+2}(X^{(M,c)}) \to \mathrm{H}_1(\pi_1(X), I^c\pi_M(X)) \to \mathrm{H}_{M+1}(X^{(M)}) \to$$
$$\mathrm{H}_{M+1}(X^{(M,c)}) \to I^c\pi_M(X)/I^{c+1}\pi_M(X) \to \mathrm{H}_M(X^{(M)}) \to \mathrm{H}_M(X^{(M,c)}) \to 0.$$

Since $X^{(M,c)}$ is of finite $\mathbb{Q}$-type, tensoring this exact sequence with $\mathbb{Q}$, and taking the inverse limit over c gives the desired result.

□

3. Differentially graded Lie algebras

In this section, we recall a construction due to K. Gartz of a differentially graded (Lie) algebra in a $\mathbb{Q}$-Karoubian symmetric monoidal category. We begin by recalling the basic construction of Gartz ([7]). Then we recall a theorem of Gartz relating this construction, in the case of certain differentially graded algebras, to the indecomposables in the Bar construction.

In the following, $\mathcal{C}$ will denote a symmetric monoidal category. We shall let $\otimes$ denote the monoidal structure in $\mathcal{C}$. Let Fin denote the category of finite sets, where morphisms are all set maps. We will view Fin as a symmetric monoidal category with monoidal structure given by the disjoint union. We will denote by $[n]$ the object $\{1, \ldots, n\}$ of Fin.[3] Suppose we have a functor $F : Fin^{op} \to \mathcal{C}$ and a natural transformation

$$N : \otimes \circ (F \times F) \to F \circ \coprod.$$

Furthermore, suppose that F and N satisfy the following properties:

(F1) If $S = \emptyset$, then $F(S) = 1_{\mathcal{C}}$.
(F2) For all $S \in Fin$,

$$F(\emptyset) \otimes F(S) \to F(S) \text{ and } F(S) \otimes F(\emptyset) \to F(S)$$

are isomorphisms.

[3]While this is contrary to standard notation, it is consistent with that of ([7]).

(F3) The following diagram commutes:

$$\begin{array}{ccc} F(S)\otimes F(T) & \xrightarrow{N(S,T)} & F(S\coprod T) \\ \downarrow & & \downarrow \\ F(T)\otimes F(S) & \xrightarrow{N(T,S)} & F(T\coprod S) \end{array}$$

(F4) The following diagram commutes:

$$\begin{array}{ccccc} (F(R)\otimes F(S))\otimes F(T) & \longrightarrow & F(R\coprod S)\otimes F(T) & & \\ & & & \searrow & \\ \downarrow & & & & F(R\coprod S\coprod T) \\ & & & \nearrow & \\ F(R)\otimes(F(S)\otimes F(T)) & \longrightarrow & F(R)\otimes F(S\coprod T) & & \end{array}$$

Given a category $\mathcal{A}$, let $\mathbb{Z}[\mathcal{A}]$ denote the category with the same objects as $\mathcal{A}$ and morphisms given by $\mathrm{Hom}_{\mathbb{Z}[\mathcal{A}]}(X,Y) := \mathbb{Z}[\mathrm{Hom}_{\mathcal{A}}(X,Y)]$. Since $\mathcal{C}$ is an additive category, any functor $F:\mathcal{A}\to\mathcal{C}$ extends to an additive functor $F:\mathbb{Z}[\mathcal{A}]\to\mathcal{C}$. This applies to Fin^{op} and F above.

Let $\mathbb{Z}[\Sigma_n]$ denote the group ring over the symmetric group. Then one has an inclusion (induced by the left action of Σ_n on $[n]$) $\mathbb{Z}[\Sigma_n] \hookrightarrow \mathrm{Hom}_{\mathbb{Z}[Fin]}([n],[n])$. Consider the following elements in $\mathbb{Z}[\sigma_n]$: $s_n := (1-\sigma_{(1,2)})\cdots(1-\sigma_{(12\dots n)})$ and $w_n := (1+\sigma_{(12)})\cdots(1+(-1)^n\sigma_{(12\cdots n)})$. By ([7], Proposition 2.2), one has $s_n^2 = ns_n$ and $w_n^2 = nw_n$. In the following, we will need to consider images under various morphisms in $\mathcal{C}$ induced via s_n and w_n. We recall the following definition.

Definition 3.1. An additive category $\mathcal{A}$ is $\mathbb{Q}$-Karoubian if for any morphism $f\in\mathrm{Hom}_{\mathcal{A}}(X,X)$ such that $f\circ f = nf$ for some integer n, the image of f exists in $\mathcal{A}$.

Remark 3.2. A $\mathbb{Q}$-linear Karoubian complete category is $\mathbb{Q}$-Karoubian. Given an endomorphism f such that $f\circ f = nf$, f/n is an idempotent and therefore has an image. It follows that f has an image.

From now on we shall assume that $\mathcal{C}$ is $\mathbb{Q}$-Karoubian. In the following, we shall denote $F([n])$ simply by $F(n)$. For each $i\in\{1,\cdots,n+1\}$, let $\delta_i\in\mathrm{Hom}_{Fin}([n+1],[n])$ denote the morphism given by $\delta_i(j) = j-1$ if $i<j$ and $\delta_i(j)=j$ if $i\geq j$. Then $f_n = \sum_{i=1}^{n}(-1)^{i-1}\delta_i$ satisfies the relation $f_n\circ f_{n+1} = 0$ when considered as an element of $\mathbb{Z}[Fin]$. In particular, this gives rise to a complex

$$F(1)\to F(2)\to\cdots$$

in $\mathcal{C}$ (here $F(i)$ is in degree i). On the other hand, $\mathcal{R}_F = \oplus F(n)$ is an associative graded algebra with the algebra structure coming from the monoidal structure. Furthermore, the differential from the complex induces a differential on $\mathcal{R}_F$. We can also define the truncated associative algebra $\mathcal{R}_F^N = \oplus_1^N F(n)$ with a differential. Let $\mathcal{P}_F = \oplus F(n)F(w_n)$ and its truncation $\mathcal{P}_F^N = \oplus_1^N F(n)F(w_n)$. Then the main result of ([7]) states the following:

Theorem 3.3. *(K. Gartz, [7])*

(1) *$\mathcal{R}_F$ and $\mathcal{R}_F^N$ are differentially graded algebras with differential and algebra structure given above. In particular, they acquire the structure of differentially graded Lie algebras.*
(2) *Both $\mathcal{P}_F$ and $\mathcal{P}_F^N$ are differentially graded Lie subalgebras of $\mathcal{R}_F$ and $\mathcal{R}_F^N$ (respectively).*

Note that $\mathcal{R}_F^N$ is naturally a quotient of $\mathcal{R}_F$. In particular, one has a natural inverse system of differentially graded algebras:

$$\mathcal{R}_F^1 \leftarrow \mathcal{R}_F^2 \leftarrow \cdots$$

Similar remarks also apply to $\mathcal{P}_F^N$. We list some examples of functors F and categories $\mathcal{C}$ to which one can apply the previous theorem, and will be useful in the following.

Example 3.4. Let $PVar_k$ denote the category of pairs of varieties over k. Let $\mathbb{Q}[PVar_k]$ denote the corresponding $\mathbb{Q}$-linear category and $\mathbb{Q}[PVar_k]^\kappa$ its Karoubian completion. By Remark 3.2, the latter category is $\mathbb{Q}$-Karoubian. Given a pointed variety (X, x), we can define a functor $F(X, x) : Fin^{op} \to \mathbb{Q}[PVar_k]^\kappa$ which sends $[n]$ to the pair $(X^n, x \times X^{n-1} \cup \ldots \cup X^{n-1} \times x)$.

Example 3.5. If $k \subset \mathbb{C}$, then we have a natural functor

$$\mathbb{Q}[PVar_k] \to Ch(Vect_{\mathbb{Q}})$$

given by sending a pair to the corresponding relative singular chain complex tensored with $\mathbb{Q}$. By composing with $F(X, x)$, this gives rise to a functor $S(X, x) : Fin^{op} \to Ch(Vect_{\mathbb{Q}})$ and therefore differentially graded Lie algebras $\mathcal{P}_S^N(X, x)$ in $Ch(Vect_{\mathbb{Q}})$. We can think of the latter as a double complex, and denote by $Tot(\mathcal{P}_S^N(X, x))$ the corresponding total complex.

We conclude this section by recalling a result of Gartz which, in the setting of the previous example, relates $Tot(\mathcal{P}_S^N(X, x))$ to the indecomposables in the Bar construction on the Sullivan polynomial de Rham complex of X. We begin by recalling the Bar construction.

Let A denote a commutative differentially graded algebra over a field k of characteristic 0. We will assume that A comes equipped with an augmentation denoted $\nu : A \to k$. We will also assume A is positively graded and that $A^0 = k$. The augmentation ideal IA is the kernel of the augmentation map. Then one can construct a (second quadrant) double complex $B^{-s,t} = [\otimes^s IA]^t$. The element $a_1 \otimes a_2 \otimes \cdots \otimes a_s$ of $B^{-s,t}$ is usually denoted by $[a_1|\cdots|a_s]$. This double complex comes equipped with two differentials, an internal one and an external one. Given a graded vector space V, let $J : V \to V$ denote the involution

$Jv = (-1)^{deg(v)}v$. Then the internal differential $d_I : B^{-s,t} \to B^{-s,t+1}$ is given by the formula:

$$d_I([a_1|\cdots|a_s]) = \sum_{i=1}^{s}(-1)^i[Ja_1|\cdots|Ja_{i-1}|da_i|a_{i+1}|\cdots|a_s]$$

The exterior differential $d_E : B^{-s,t} \to B^{-s+1,t}$ is given by the formula:

$$d_E([a_1|\cdots|a_s] = \sum_{i=1}^{s-1}(-1)^{i+1}[Ja_1|\cdots|Ja_{i-1}|Ja_i\cdots a_{i+1}|\cdots|a_s]$$

One has $d_I^2 = d_E^2 = 0$ and $d_Id_E + d_Ed_I = 0$. The Bar construction on A is the total complex $B(A)$ associated to this double complex with differential $d_I + d_E$. Then $B(A)$ has a dg coalgebra structure given by the tensor product and an algebra structure coming from the shuffle product. In particular, $B(A)$ is a differentially graded Hopf algebra. Furthermore, $B(A)$ comes equipped with the Bar filtration:

$$B(A) \supseteq \cdots \supseteq \mathfrak{B}^{-1} \supseteq \mathfrak{B}^0 = k,$$

where $\mathfrak{B}^{-s} = \oplus_{u\leq s}B^{-u,t}$.

Given a cdga A, the homology of the Bar construction $\mathrm{H}(B(A))$ also has a Hopf algebra structure. Furthermore, the filtration $\mathfrak{B}^{-s}$ induces a filtration:

$$0 \subset k \subseteq \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathrm{H}(B(A))$$

Since $B(A)$ is an augmented dg Hopf algebra, we can define the Lie coalgebra of indecomposables in the following way. Let J denote the augmentation ideal of $B(A)$. Then the indecomposables $QB(A)$ are defined to be the cokernel of the multiplication map

$$J \otimes J \to J.$$

We also define $Q\mathfrak{B}^{-N}$ to be the cokernel of the multiplication

$$\mathfrak{B}^{-N} \otimes \mathfrak{B}^{-N} \to \mathfrak{B}^{-N}.$$

One has an induced filtration: $0 \subset k \subseteq Q\mathfrak{B}_0 \subseteq Q\mathfrak{B}_1 \subseteq \cdots \subseteq Q\mathrm{H}(B(A))$.[4]

Before stating Gartz's result, we recall some preliminaries on the Sullivan polynomial forms. Given a topological space X, Sullivan ([19]) constructed a complex of rational polynomial forms $A^{PL}(X)$. This gives a contravariant functor from topological spaces to the category of positively graded commutative dg $\mathbb{Q}$-algebras. In particular, a choice of a point $x \in X$ gives rise to an augmentation of cdga's:

$$A^{PL}(X) \to A^{PL}(x) = \mathbb{Q}.$$

In this case, one has the following theorem due to K. Gartz ([7]):

[4] One has $Q\mathfrak{B}^{-N} = \oplus_{-s>-N}QBar^{-s}(A)$.

Theorem 3.6. *(K. Gartz, [7]) The integration pairing induces a natural morphism of dg Lie coalgebras*

$$Q\mathfrak{B}^{-N}(A^{PL}(X)) \to (Tot(\mathcal{P}_S^N(X,x)))^\vee$$

such that the induced map

$$\mathrm{H}^*(Q\mathfrak{B}^{-N}(A^{PL}(X))) \to \mathrm{H}^*((Tot(\mathcal{P}_S^N(X,x)))^\vee)$$

is an isomorphism of Lie coalgebras.

4. Motivic structures on homotopy groups

4.1. **Nori's category of motives.** In this section, we recall some basic facts about Nori's category of motives ([17], [12], [14]). In the following, $k \subset \mathbb{C}$ will be a field of characteristic zero.[5] Nori has defined two categories of (effective) mixed motives, homological and cohomological, over a field of characteristics zero. In the following, we shall mostly work with the homological version.

In ([17]), Nori constructs an abelian category $\mathrm{EHM}(k)$ of effective homological motives. One can localize this category at the Tate object to get an abelian category of mixed motives $\mathcal{M}(k)$. These categories satisfy the following properties:

(1) There is a faithful exact functor $ff : \mathcal{M}(k) \to \mathbb{Z} - mod$. We shall refer to this as the 'realization functor'.
(2) For all pairs of varieties (X, Y), where Y is a closed subvariety of X, and non-negative q, there is an object $\mathrm{H}_q(X, Y)$ in $\mathcal{M}(k)$.
(3) $ff(\mathrm{H}_q(X, Y)) = \mathrm{H}_q(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$.
(4) There is a tensor functor $\otimes : \mathcal{M}(k) \times \mathcal{M}(k) \to \mathcal{M}(k)$ giving $\mathcal{M}(k)$ the structure of a symmetric monodical category. The identity is given by $\mathrm{H}_0(Spec(k), \emptyset)$.
(5) There is a functor $C_* : Aff_k \to \mathrm{Ch}(\mathrm{Ind}(\mathcal{M}(k)))$ such that $\mathrm{H}_q(ff(C_*(X))) = \mathrm{H}_q(X(\mathbb{C}))$. A closed imbedding $Y \subset X$ gives a monomorphism $C_*(Y) \to C_*(X)$. We denote the quotient by $C_*(X, Y)$.
(6) The tensor structure on $\mathcal{M}(k)$ extends to give one on $\mathrm{Ch}(\mathrm{Ind}(\mathcal{M}(k))$. Furthermore, one has a natural quasi-isomorphism:

$$C_*(X) \otimes C_*(Y) \to C_*(X \otimes Y).$$

This gives rise to a Σ_n-equivariant map

$$C_*(X)^{\otimes n} \to C_*(X^n).$$

Remark 4.1. We may replace $\mathcal{M}(k)$ with $\mathrm{EHM}(k)$ in the above statements. There is an analogous category $\mathrm{ECM}(k)$ of effective cohomological motives. Then one has analogs of the previous properties, where the realization functors will now compute cohomology. Furthermore, taking duals gives rise to natural duality (contravariant) functors $\mathrm{EHM}(k) \to \mathrm{ECM}(k)$.

[5] In the following, we fix such an embedding. However, the construction of Nori's category of motives does not depend on the choice of embedding.

Given a pointed algebraic variety (X, x) over k, let $F_{mot}(X, x) : Fin^{op} \to \mathrm{Ch}(\mathrm{Ind}(\mathrm{EHM}(k)))$ denote the following functor:

$$[n] \to C_*(X^n, x \times X^{n-1} \cup \ldots \cup X^{n-1} \times x).$$

Composing with the realization functor and tensoring with $\mathbb{Q}$, gives

$$ff(F_{mot}(X, x)) : Fin^{op} \to \mathrm{Ch}(Vect(\mathbb{Q})).$$

Since $\mathrm{EHM}(k)$ is an abelian category satisfying the ascending chain condition, it follows that $\mathrm{Ind}(\mathrm{EHM}(k))$ is also an abelian category. Therefore, $\mathrm{Ch}(\mathrm{Ind}(\mathrm{EHM}(k)))$ is an abelian category and, in particular, $\mathbb{Q}$-Karoubian. Therefore, we may apply the constructions of section 3 to get differentially graded Lie algebras

$$\mathcal{P}^N_{F_{mot}(X,x)} \text{ and } \mathcal{P}^N_{ff(F_{mot}(X,x))}.$$

In the following, we shall drop the pointed variety (X, x) from our notation and write simply $\mathcal{P}^N_{F_{mot}}$ etc. One has an identification $ff(\mathcal{P}^N_{F_{mot}}) \otimes \mathbb{Q} = \mathcal{P}^N_{ff(F_{mot})}$. Furthermore, by the results of Gartz ([7], pg. 20), one has a natural isomorphism of Lie algebras:

$$\mathrm{H}_*(Tot(\mathcal{P}^N_{ff(F_{mot})})) \to \mathrm{H}_*(Tot(\mathcal{P}^N_S)).$$

Note that one has $ff(\mathrm{H}_*(Tot(\mathcal{P}^N_{F_{mot}}))) = \mathrm{H}_*(Tot(ff(\mathcal{P}^N_{F_{mot}})))$.

Remark 4.2. Note that, since $ff(\mathrm{H}_*(Tot(\mathcal{P}^N_{F_{mot}})))$ in finite dimensional, one has

$$\mathrm{H}_*(Tot((\mathcal{P}^N_{F_{mot}}))) \in \mathrm{EHM}(k).$$

Remark 4.3. Let $\mathrm{EHM}(k)_{\mathbb{Q}}$ denote the category of effective homological motives tensored with $\mathbb{Q}$. Then ff extends to a faithful exact functor $ff : \mathrm{EHM}(k)_{\mathbb{Q}} \to \mathrm{Ch}(Vect(\mathbb{Q}))$. If $M \in \mathrm{EHM}(k)$ then we let $M_{\mathbb{Q}}$ denote its image in $\mathrm{EHM}(k)_{\mathbb{Q}}$. Note, $ff(M_{\mathbb{Q}}) = ff(M) \otimes \mathbb{Q}$.

4.2. **Motivic structure on higher homotopy of the nilpotent spaces.** Let (X, x) be a pointed variety over $k \subset \mathbb{C}$. Associated to each such pointed variety, one has the differentially graded Lie algebras $\mathcal{P}^N_{F_{mot}}$ in $\mathrm{Ch}(\mathrm{Ind}(\mathrm{EHM}(k)))$. We begin by computing the homology of the Betti realization of this dgla. At the level of Betti realizations (tensored with $\mathbb{Q}$) we have an inverse system of dgla's

$$\cdots \to \mathcal{P}^N_S \to \mathcal{P}^{N-1}_S \to \cdots \to \mathcal{P}^1_S.$$

This gives a direct system of (graded) Lie coalgebras:

$$\cdots \leftarrow \mathrm{H}^*((Tot(\mathcal{P}^N_S))^\vee) \leftarrow \mathrm{H}^*((Tot(\mathcal{P}^{N-1}_S))^\vee) \leftarrow \cdots \to \mathrm{H}^*((Tot(\mathcal{P}^1_S))^\vee).$$

By Theorem 3.6 this direct system is isomorphic to the direct systems of Lie coalgebras:

$$\cdots \mathrm{H}^*(Q\mathfrak{B}^{-(N)}(A^{PL}(X))) \leftarrow \mathrm{H}^*(Q\mathfrak{B}^{-(N-1)}(A^{PL}(X))) \leftarrow \cdots \leftarrow \mathrm{H}^*(Q\mathfrak{B}^{-1}(A^{PL}(X))).$$

In particular, we have an isomorphism of graded Lie coalgebras

$$\varinjlim \mathrm{H}^*(Q\mathfrak{B}^{-N}(A^{PL}(X))) \to \varinjlim \mathrm{H}^*((Tot(\mathcal{P}^N_S))^\vee).$$

Since direct limits commute with cohomology, the left side can de identified with $\mathrm{H}^*(QB(A^{PL}(X)))$.

The following theorem relates $\mathrm{H}^*(QB(A^{PL}(X)))$ with the homotopy Lie algebra. In the following, for a pointed space (X,x), let $\mathfrak{g}_*(X)$ denote the corresponding homotopy Lie algebra. The proof of the following theorem is the same as ([10], Theorem 2.6.2).

Theorem 4.4. *Suppose (X,x) is a pointed algebraic variety, and $\mathbb{Q}X(\mathbb{C})$ is the $\mathbb{Q}$-completion of the topological space $X(\mathbb{C})$.*

(1) *There is a natural isomorphism of Lie coalgebras:*

$$\mathrm{Hom}_{\mathbb{Q}}(\varinjlim \mathrm{H}^*(Tot(\mathcal{P}^N_{ff(F_{mot})})^\vee, \mathbb{Q}) = \mathfrak{g}_*(\mathbb{Q}X(\mathbb{C})).$$

(2) *One has canonical isomorphisms:*

$$\mathrm{H}^0(Tot(\mathcal{R}^N_{F_S})^\vee) \to \mathrm{H}^0(Bar^{-N}(A^{PL}(X))) \to \mathrm{Hom}(\mathbb{Q}[\pi_1(X,x)]/I^{N+1}, \mathbb{Q})$$

Proof. First, it is enough to prove the statement with $\mathcal{P}^N_{ff(F_{mot})}$ replaced by $\mathcal{P}^N_S$. Therefore, by the remarks above, we may replace $\varinjlim \mathrm{H}^*(Tot(\mathcal{P}^N_{ff(F_{mot}})^\vee)$ by $\mathrm{H}^*(QB(A^{PL}(X))$. Since the natural morphism from a minimal model $M(X) \to A^{PL}(X)$ induces a quasi-isomorphism of Bar complexes, the corresponding map $\mathrm{H}^*(QB(M(X))) \to \mathrm{H}^*(QB(A(X)))$ is an isomorphism of Lie coalgebras. By a result of Hain ([9]), the natural map $Q\mathrm{H}^*(B(M(X)) \to \mathrm{H}^*(QB(M(X))$ is an isomorphism of Lie coalgebras. By a theorem of Chen, the former is isomorphic to $QM(X)[1]$ as a Lie coalgebra. By ([1], 12.8), the induced map

$$\mathfrak{g}_*(\mathbb{Q}X) \to \mathrm{Hom}(QM(X)[1], \mathbb{Q}))$$

is an isomorphism. For the second statement, by the proof of ([7], Lemma 5.4), the first arrow is an isomorphism. The second arrow is an isomorphism by the Chen-Sullivan de Rham theorem ([10], 2.4.3). □

In the following, $\mathfrak{g}^{\leq k}_*(X,x)$ will denote the truncated homotopy Lie algebra of X. In particular, it is the same as $\mathfrak{g}_*(X)$ in degree $\leq k$ and 0 otherwise. The following corollary was obtained by Gartz ([7]) in the case of simply connected spaces.

Corollary 4.5. *Let (X,x) be a pointed variety over $k \subset \mathbb{C}$ such that $X(\mathbb{C})$ is a nilpotent space. Then, for each $k > 1$, we can associate to (X,x) graded Lie algebras $\mathcal{M}^{k,*}_{(X,x)}$ in $\mathrm{EHM}(k)_{\mathbb{Q}}$ such that the betti realization of $\mathcal{M}^{k,*}_{(X,x)}$ is the Lie algebra $\mathfrak{g}^{\leq k}_*(\mathbb{Q}X(\mathbb{C}))$.*

Proof. Consider the direct system of graded Lie coalgebras

$$\mathrm{H}^{\leq k}((Tot(\mathcal{P}^N_{F_{mot}}))^\vee)_{\mathbb{Q}} := \oplus_{i=0}^{i=k} \mathrm{H}^i((Tot(\mathcal{P}^N_{F_{mot}}))^\vee)_{\mathbb{Q}}.$$

By Lemma 4.6, we can replace this inverse system by an inverse system of graded Lie coalgebras $M^{N,k,*}_{(X,x)}$ with injective transition maps such that the resulting direct system has the same direct limit. The realizations of the $M^{N,k,*}_{(X,x)}$ form a direct system with injective transition maps and the corresponding direct limit computes $(\mathfrak{g}^{\leq k}_*(\mathbb{Q}X(\mathbb{C})))^\vee$.

Since the latter is finite dimensional, the induced map $ff(M^{N,k,*}_{(X,x)}) \to (\mathfrak{g}^{\leq k}_*(\mathbb{Q}X(\mathbb{C})))^\vee$ is an isomorphism for N large enough. Taking $\mathcal{M}^{k,*}_{X,x}$ to be dual of $M^{N,k,*}_{X,x}$ for large enough N gives the desired Lie algebra. □

We now describe an explicit recipe for replacing the inverse system $\mathrm{H}^{\leq k}((Tot(\mathcal{P}^N_{F_{mot}}))^\vee)_\mathbb{Q}$ by a direct system of graded Lie coalgebras with inejctive transition maps.

Lemma 4.6. *The direct system of graded Lie coalgebras* $\mathrm{H}^{\leq k}((Tot(\mathcal{P}^N_{F_{mot}}))^\vee)_\mathbb{Q}$ *can be replaced by an direct system of graded Lie coalgebras* $M^{N,k,*}_{(X,x)}$ *with injective transition maps such that the resulting direct system has the same direct limit.*

Proof. Recall, we have a surjective morphism of dg Lie algebras $\mathcal{P}^N_{F_{mot}} \to \mathcal{P}^{N-1}_{F_{mot}}$ in Ch(Ind(EHM(k))). Taking duals (and passing to ECM(k)) gives an injective morphism of dg Lie coalgebras $(\mathcal{P}^{N-1}_{F_{mot}})^\vee \to (\mathcal{P}^N_{F_{mot}})^\vee$. We can view this as an inclusion of double complexes. Let K^N denote the corresponding cokernel. Taking total complexes gives an exact sequence of complexes:

$$0 \to Tot((\mathcal{P}^{N-1}_{F_{mot}})^\vee) \to Tot((\mathcal{P}^N_{F_{mot}})^\vee) \to Tot(K^N) \to 0.$$

This gives a long exact sequence in $\mathrm{ECM(k)}_\mathbb{Q}$

$$\cdots \to \mathrm{H}^{i-1}(Tot(K^N))_\mathbb{Q} \to \mathrm{H}^i(Tot((\mathcal{P}^{N-1}_{F_{mot}})^\vee))_\mathbb{Q} \to \mathrm{H}^i(Tot((\mathcal{P}^N_{F_{mot}})^\vee))_\mathbb{Q} \to \cdots$$

Let $M^{N-1,i}_{(X,x)}$ denote the quotient of $\mathrm{H}^i(Tot((\mathcal{P}^{N-1}_{F_{mot}})^\vee))_\mathbb{Q}$ by the image of $\mathrm{H}^{i-1}(Tot(K^N))_\mathbb{Q}$. Then each $M^{N,*}_{(X,x)}$ is a graded lie coalgebra in $\mathrm{ECM}(k)_\mathbb{Q}$ and it's dual is a lie algebra in $\mathrm{EHM}(k)_\mathbb{Q}$. By construction, the natural maps

$$M^{N,*}_{(X,x)} \to M^{N+1,*}_{(X,x)}$$

are inclusions. Furthermore, there is a natural isomorphism of Lie coalgebras:

$$\varinjlim \mathrm{H}^*((Tot(\mathcal{P}^N_{F_{mot}})^\vee))_\mathbb{Q} \to \varinjlim M^{N,*}_{(X,x)}.$$

Since the realization functor is faithful exact, one can check these statements after taking realizations.

□

Note that if the natural maps

$$\mathrm{H}^i(Tot((\mathcal{P}^{N-1}_{F_{mot}})^\vee)) \to \mathrm{H}^i(Tot((\mathcal{P}^N_{F_{mot}})^\vee))$$

were inclusions, then by simply taking N large enough we see that $\mathrm{H}^*(Tot((\mathcal{P}^N_{F_{mot}}))$ will give the desired Lie algebra. For example, if X is simply connected, then $\mathrm{H}^{i-1}(Tot(K^N)) = 0$ for N large relative to i. To see this, note that $\mathrm{H}^{i-1}(Tot(K^N))$ is a direct summand of $\mathrm{H}^{N+i-1}(X^N)$, where X^N denotes the smash product. An application of the Kunneth formula now forces this to be zero if N is larger than $i-1$. This follows simply because, in this case, there must be an H^1 in each factor of the Kunneth decomposition.

4.3. **The case of M-nilpotent spaces.** In the following, let R^N denote the motive in $\mathrm{EHM}(k)_{\mathbb{Q}}$ given by $\mathrm{H}^0(Tot(\mathcal{R}^N_{F_{mot}}))_{\mathbb{Q}}$. Recall, its realization is given by $\mathbb{Q}[\pi_1(X,x)]/I^{N+1}$. Our main goal in this section is to prove the following theorem.

Theorem 4.7. *Let (X,x) be a pointed variety such $X(\mathbb{C})$ is M-nilpotent.*

(1) *Then there is a Lie algebra in* EHM(k) *whose realization is the truncated homotopy lie algebra $\mathfrak{g}^{<M}(\mathbb{Q}X(\mathbb{C}))$.*
(2) *There is an inverse system of objects $\mathcal{M}^{M,q}_{(X,x)}$ in* EHM(k) *whose realization is $(\pi_M(X(\mathbb{C}))/I^q)\otimes\mathbb{Q}$. Furthermore, $\mathcal{M}^{M,q}_{(X,x)}$ is a module object over the ring object $R^{q-1}_{(X,x)}$.*

Proof. In degrees less than M, the homotopy Lie algebra $\mathfrak{g}^{<M}(\mathbb{Q}X(\mathbb{C}))$ is just the usual homotopy Lie algebra of X tensored with $\mathbb{Q}$ and these are all finite dimensional. Therefore, part one can be proved just as the case of nilpotent spaces above.
For the second part, first recall that the natural morphism $\mathcal{R}^N_F \to \mathcal{P}^N_F$ combined with the Lie algebra structure of $\mathcal{P}_F$ induces morphisms:

$$R^N \otimes \mathrm{H}_{-i}(Tot(\mathcal{P}^N_{F_{mot}})) \to \mathrm{H}_{-i}(Tot(\mathcal{P}^N_{F_{mot}})).$$

Furthermore, this action induces a morphism:

$$R^N \otimes \mathcal{M}^{i,N}_{(X,x)} \to \mathcal{M}^{i,N}_{(X,x)},$$

where $\mathcal{M}^{N,i}_{(X,x)}$ is the dual of $M^{N,i}$ (see proof of Lemma 4.6). If $q\geq 2$ is fixed, then for all $N\geq q$ we have surjections:

$$R^N \to R^{q-1}.$$

Let $R^{N,q-1}$ denote the kernel of this morphism. Then we have an exact sequence in $\mathrm{EHM}(k)$

$$0 \to R^{N,q-1} \to R^N \to R^{q-1} \to 0$$

whose realization is

$$0 \to I^q/I^{N+1} \to \mathbb{Q}[\pi_1(X,x)]/I^{N+1} \to \mathbb{Q}[\pi_1(X,x)]/I^q \to 0.$$

In the following we fix a $q\geq 2$. Then for each $N > q-1$, let $\mathcal{M}^{i,N,q}_{(X,x)}$ denote the quotient of $\mathcal{M}^{i,N}_{(X,x)}$ by the image of $R^{N,q-1}_{(X,x)}\times\mathcal{M}^{i,N}$. For $N<q$, let $\mathcal{M}^{i,N,q}_{(X,x)} = \mathcal{M}^{i,N}_{(X,x)}$. Then the $\mathcal{M}^{i,N,q}_{(X,x)}$ form an inverse system with surjective transition maps. Furthermore, by the previous remarks and Lemma 4.6, one has a canonical isomorphism at the level of realizations:

$$(\varinjlim(ff(\mathcal{M}^{M,N,q}_{(X,x)})^\vee)^\vee \to \pi_M(\mathbb{Q}X(\mathbb{C}))/I^q\pi_M(X(\mathbb{C})).$$

Since the right side is finite dimensional, we can argue as before to conclude that $ff(\mathcal{M}^{M,N,q}_{(X,x)}) \to \pi_M(\mathbb{Q}X(\mathbb{C}))/I^q\pi_M(X(\mathbb{C}))$ is an isomorphism for large enough N. □

4.4. **The case of hyperplane arrangements.** Let $Y = \mathbb{A}^M$ and $\mathcal{A}$ denote a set of hyperplanes in Y. Following Hattori ([10]), we say that $\mathcal{A}$ is generic if for all $\mathcal{B} \subset \mathcal{A}$ the intersection $\cap_{H\in\mathcal{B}} H$ has codimension $|\mathcal{B}|$ when $|\mathcal{B}| \leq M$ and is empty when $|\mathcal{B}| > M$. In this setting, the higher homotopy groups of $X(\mathcal{A}) = \mathbb{C}^M \setminus \mathcal{A}$ were computed by Hattori.

Theorem 4.8. *(Hattori) Let $\mathcal{A}$ denote a generic hyperplane arrangement in $\mathbb{C}^M$ such that $|\mathcal{A}| = n > M > 1$. Then*
(1) $\pi_1(X(\mathcal{A})) = \mathbb{Z}^n$
(2) $\pi_k(X(\mathcal{A})) = 0$ *for all* $1 < k < M$.
(3) $\pi_M(X(\mathcal{A}))$ *has a free* $\mathbb{Z}[\pi_1]$ *resolution of length* $n - M$.

Let $X = \mathbb{P}^M \setminus \{L_1, \ldots, L_{M+2}\}$ denote the complement of $M + 2$ hyperplanes in general position (generic in the above sense). Then we can apply Hattori's theorem to $X(\mathbb{C})$ and conclude that
(1) $\pi_1(X(\mathbb{C})) = \mathbb{Z}^{M+1}$
(2) $\pi_k(X(\mathbb{C})) = 0$ for all $1 < k < M$.
(3) $\pi_N(X(\mathbb{C}))$ is a free $\mathbb{Z}[\mathbb{Z}^M]$-module of rank 1.

In particular, $X(\mathbb{C})$ is an M-nilpotent space. Therefore, the results of the previous subsection give the following theorem.

Theorem 4.9. *Let $X = \mathbb{P}^M \setminus \{L_1, \cdots, L_{M+2}\}$ and $x \in X$ a k-rational point. Then there is an object $\mathcal{M}^{k,M}_{(X,x)}$ in Nori's category of motives whose Betti realization is $(\pi_M(X, x)/I^k\pi_M(X, x)) \otimes \mathbb{Q}$. Furthermore, $\mathcal{M}^{k,M}_{(X,x)}$ is a module object over $R^{k-1}_{(X,x)}$ in the category of mixed Tate motives. Recall, $R^{k-1}_{(X,x)}$ has betti realization given by $\mathbb{Q}[\pi_1(X, x)]/I^k$.*

One can also apply the results of the previous section to more general hyperplane arrangements. In particular, it follows from the results of Papadima and Suciu ([18]) that the class of hypersolvable (see loc. cit.) hyperplane arrangements also give rise to M-nilpotent spaces. Note that in general the first non-vanishing higher homotopy group may not be M (the dimension of the corresponding affine space) but will instead be given by the combinatorics of the intersection lattice.

5. Voevodsky's category

In this section, we render the results of the previous section to Voevodsky's category of motives. We begin by recalling some notation and facts regarding Voevodsky's category of motives. We refer to ([20]) for the details.

Let $DM_{gm}(k)$ denote Voevodsky's category of geometric motives and $DM^{eff}_{gm}(k)$ the corresponding category of effective motives. Let $DM_{gm}(k)_{\mathbb{Q}}$ denote the Karoubian completion of $DM_{gm}(k) \otimes \mathbb{Q}$. Since this is a $\mathbb{Q}$-linear Karoubi complete category, it is automatically $\mathbb{Q}$-Karoubian.

The category $DM_{gm}(k)$ comes equipped with a betti realization functor $R_B : DM_{gm}(k) \to D$, where D denotes the bounded derived category of $\mathbb{Q}$-vector spaces ([13], [11]). There is also a natural functor from $Var_k \to DM_{gm}(k)$ which extends to a functor $C^b(Var_k) \to DM_{gm}(k)$. We assume here that an imbedding $k \to \mathbb{C}$ has been fixed.

We briefly recall the strategy of Huber's ([13], [11]) construction of R_B. First, note that one has a natural functor, $C^b(Var_k) \to CoCh(Vect(\mathbb{Q}))$, which sends a variety X to the dual of the corresponding singular chain complex of $X(\mathbb{C})$ tensored with $\mathbb{Q}$ and a complex of varieties to the corresponding total complex. Note that R_B is a contravariant functor. Huber ([13], [11]) shows that this functor gives rise to a commutative diagram:

$$\begin{array}{ccc} C^b(\mathbb{Z}[Var_k]) & \longrightarrow & CoCh(Vect(\mathbb{Q})) \\ \downarrow & & \downarrow \\ DM_{gm}(k) & \longrightarrow & D(Vect(\mathbb{Q})) \end{array}$$

Note that the right hand column is $\mathbb{Q}$-Karoubian, and so one has an induced diagram:

$$\begin{array}{ccc} C^b(\mathbb{Q}[Var_k]^\kappa) & \longrightarrow & CoCh(Vect(\mathbb{Q})) \\ \downarrow & & \downarrow \\ DM_{gm}(k)_{\mathbb{Q}} & \longrightarrow & D(Vect(\mathbb{Q})) \end{array}$$

Now there is a natural map $\mathbb{Q}[PVar_k]^\kappa \to C^b(\mathbb{Q}[Var_k]^\kappa)$, it sends a pair (X, Y) to the complex $[Y \to X]$. Given any pointed variety (X, x), we have the functor $F(X, x) : Fin^{op} \to \mathbb{Q}[PVar_k]^\kappa$ which, after composing with the previous functor gives rise to a functor $F'(X, x) : Fin^{op} \to C^b(\mathbb{Q}[Var_k]^\kappa)$. This gives rise to a differentially graded Lie algebra $\mathcal{P}^N_{F'}$ in $C^b(\mathbb{Q}[Var_k]^\kappa)$. Taking the associated total complex gives rise to a differentially graded Lie algebra $\mathcal{P}^N_{mot}$ in $\mathbb{Q}[Var_k]^\kappa$. We can this of this object as complex, and, in particular, as an element of $C^b(\mathbb{Q}[Var_k]^\kappa)$. Let $\mathcal{P}^n_v$ denote its image $DM_{gm}(k)_{\mathbb{Q}}$.

Theorem 5.1. *Let (X, x) be a pointed algebraic variety. Then one has an isomorphism:*

$$\mathrm{Hom}_{\mathbb{Q}}(\varinjlim \mathrm{H}^*(R_B(\mathcal{P}^N_v)), \mathbb{Q}) \cong \mathfrak{g}_*(\mathbb{Q}X(\mathbb{C})).$$

Proof. This follows from the fact that $R_B(\mathcal{P}^n_v)$ is quasi-isomorphic to the complex $(Tot(\mathcal{P}^N_S))^\vee$ and Theorem 4.4. $\square$

It is not known if Voevodsky's category has a good t-structure. In particular, one does not know how to take homology of the total complex given by $\mathcal{P}^n_v$. On the other hand, if k is a number field (which we shall assume from now on), then it is known that triangulated category of mixed Tate motives, $DMT(k)_{\mathbb{Q}} \subset DM_{gm}(k)_{\mathbb{Q}}$ has a good t-structure ([6]). In particular, one has the corresponding abelian category of mixed Tate motives $MT(k)_{\mathbb{Q}}$. Furthermore, if the motive of X is mixed Tate (i.e. lies in $DMT(k)_{\mathbb{Q}}$), then so will $\mathcal{P}^N_v$. We assume from now on that the motive of X is mixed Tate. Note that the realization functor restricts to a realization functor on $DMT(k)_{\mathbb{Q}}$. Furthermore, this functor is t-exact ([21], pg. 20). Therefore, by t-exactness and contravariance of R_B,

$$R_B(\mathrm{H}^{-*}(\mathcal{P}^N)) \cong \mathrm{H}^*(Tot(\mathcal{P}_S^N)),$$

where $\mathrm{H}^*(\mathcal{P}^N)$ is cohomology with respect to the t-structure. The previous discussion can also be applied to $\mathcal{R}$, and therefore one has an object $\mathrm{H}^0(\mathcal{R}_v^N)$ in $MT(k)_{\mathbb{Q}}$. Furthermore, one has a natural action map as before:

$$\mathrm{H}^0(\mathcal{R}_v^N) \otimes \mathrm{H}^i(\mathcal{P}_v^N) \to \mathrm{H}^i(\mathcal{P}_v^N).$$

Theorem 5.2. *Let (X, x) be a pointed algebraic variety such that $X(\mathbb{C})$ is M-nilpotent and the motive of X is mixed Tate. Then there are are objects $M^{(M,k)} \in MT(k)_{\mathbb{Q}}$ for all $i \leq M$ such that the Betti realization of $M^{M,k}$ is $\pi_M(X)/I^k\pi_M(X) \otimes \mathbb{Q}$.*

Proof. Given previous remarks, and the fact that $MT(k)_{\mathbb{Q}}$ is a Tannakian category, we can repeat the proof of Theorem 4.7. □

The motives arising from complements of hyperplane arrangements are all mixed Tate. Therefore, generic hyperplane arrangements, in the sense of Hattori, give rise to varieties satisfying the hypotheses of the theorem.

Remark 5.3. By the results of ([6]), the motives from Theorem 5.2 will lift to the category of mixed Tated motives $MT(\mathcal{O}_{k,S})_{\mathbb{Q}}$ over the the ring of S-integers in k whenever the motive of (X, x) has good reduction over $\mathcal{O}_{k,S}$.

References

[1] A. K. Bousfield and V. K. A. M. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 8(179):ix+94, 1976.

[2] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[3] Francis Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012.

[4] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* **Q** *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[5] Pierre Deligne. Le groupe fondamental unipotent motivique de $\mathbf{G}_m - \mu_N$, pour $N = 2, 3, 4, 6$ ou 8. *Publ. Math. Inst. Hautes Études Sci.*, (112):101–141, 2010.

[6] Pierre Deligne and Alexander B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[7] Kaj Michael Gartz. *A construction of a differential graded Lie algebra in the category of effective homological motives.* ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–The University of Chicago.

[8] Richard M. Hain. The de Rham homotopy theory of complex algebraic varieties. I. *K-Theory*, 1(3):271–324, 1987.

[9] Richard M. Hain. The de Rham homotopy theory of complex algebraic varieties. II. *K-Theory*, 1(5):481–497, 1987.

[10] Akio Hattori. Topology of C^n minus a finite number of affine hyperplanes in general position. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 22(2):205–219, 1975.

[11] A. Huber. Corrigendum to: "Realization of Voevodsky's motives" [J. Algebraic Geom. **9** (2000), no. 4, 755–799; mr1775312]. *J. Algebraic Geom.*, 13(1):195–207, 2004.

[12] A. Huber and S. Muller-Stach. On the relation between nori motives and kontsevich periods. arXiv:1105.0865v4 [math.AG].
[13] Annette Huber. Realization of Voevodsky's motives. *J. Algebraic Geom.*, 9(4):755–799, 2000.
[14] Marc Levine. Mixed motives. In *Handbook of K-theory. Vol. 1, 2*, pages 429–521. Springer, Berlin, 2005.
[15] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.
[16] John W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.
[17] M. Nori. An abelian category of motives. Personal Communitcation.
[18] Stefan Papadima and Alexander I. Suciu. Higher homotopy groups of complements of complex hyperplane arrangements. *Adv. Math.*, 165(1):71–100, 2002.
[19] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.
[20] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.
[21] J. Wildeshaus. Notes on artin-tate motives. arXiv:0811.4551v2 [math.AG].

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, U.S.A.
E-mail address: patel471@purdue.edu