# DE RHAM $\mathscr{E}$-FACTORS

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ABSTRACT. We construct a theory of de Rham epsilon factors. Given a smooth projective variety $X$ over a field of characteristic 0 and a $\mathscr{D}_X$-module $\mathscr{M}$, we may consider the determinant of de Rham cohomology $\det(R\Gamma_{dR}(X,\mathscr{M}))$ as a graded super line. Let $S$ be the singular support of $\mathscr{M}$, $U$ an open in $X$, $Y = X \setminus U$, and $\nu$ a 1-form on $U \subset X$ such that $\nu(U) \cap S = \emptyset$. Then we show the existence of a canonically defined graded super line $\varepsilon_{\nu,Y}(\mathscr{M})$ canonically isomorphic to $\det(R\Gamma_{dR}(X,\mathscr{M}))$. The key property is the local nature of the epsilon factor: it only depends on the restriciton of $\mathscr{M}$ and $\nu$ to any open neighborhood of $Y$. Restricting to the case of curves gives a theory of epsilon factors for curves previously defined by Beilinson, Bloch, Deligne, and Esnault.

## 1. INTRODUCTION

Let $X$ be a smooth variety over a field $k$ of characteristic zero. A complex $\mathscr{M}$ of holonomic $\mathscr{D}_X$-modules gives rise to a homotopy point $[\mathscr{M}]$ of the $K$-theory spectrum $K_h(\mathscr{D}_X)$ of holonomic $\mathscr{D}_X$-modules. If $X$ is projective, then we also get a homotopy point $[R\Gamma_{dR}(X,\mathscr{M})]$ of the $K$-theory spectrum $K(k)$. The structure morphism $X \to \mathrm{Spec}(k)$ induces a morphism of spectra $K_h(\mathscr{D}_X) \to K(k)$. The determinant construction induces a morphism on the corresponding fundamental groupoids $\Pi(K_h(\mathscr{D}_X)) \to \Pi(K(k))$. The latter groupoid is just the Picard groupoid of $\mathbb{Z}$-graded lines on $k$. Furthermore, after passing to determinants, the homotopy point $[R\Gamma_{dR}(X,\mathscr{M})]$ gives rise to the object $\det(R\Gamma_{dR}(X,\mathscr{M}))$ of $\Pi(K(k))$. The corresponding element of $K_0(k)$ is just the Euler characteristic of $\mathscr{M}$.

In this article we give a micro-local description of the homotopy point $[R\Gamma_{dR}(X,\mathscr{M})]$. Let $S = SS(\mathscr{M})$ denote the singular support of $\mathscr{M}$ contained in the cotangent bundle $T^*X$. Given a closed subvariety $Y \subset X$ and a 1-form $\nu$ on $X \setminus Y$ taking values in the complement of $S$, we construct a natural homotopy point $\mathscr{E}_{\nu,Y}(\mathscr{M})$ of $K(k)$. The homotopy point $\mathscr{E}_{\nu,Y}(\mathscr{M})$ has local nature and is determined by restrictions of $\mathscr{M}$ and $\nu$ on any open neighborhood of $Y$. There is also a natural identification of the homotopy points $\mathscr{E}_{\nu,Y}(\mathscr{M})$ and $[\det(R\Gamma_{dR}(X,\mathscr{M})]$. Passing to determinants gives a graded line $\varepsilon_{\nu,Y}(\mathscr{M}) := \det(\mathscr{E}_{\nu,Y}(\mathscr{M})$. If $Y$ is a disjoint union of finitely many $Y_\alpha$'s then we have $[R\Gamma_{dR}(X,\mathscr{M})] = \sum \mathscr{E}_{\nu,Y_\alpha}(\mathscr{M})$. Applying the determinant to this formula gives $\det(R\Gamma_{dR}(X,\mathscr{M})) = \otimes_\alpha \varepsilon_{\nu,Y_\alpha}(\mathscr{M})$.

The construction of $\mathscr{E}$-factors follows from a micro-local $K$-theoretic animation of the characteristic cycle $CC(\mathscr{M})$ of $\mathscr{M}$. It follows form the the work of Quillen ([Qui73]) that there is a morphism of spectra $K(\mathscr{D}_X) \to K(T^*X)$. We construct a lifting of this morphism to $K$-theory with supports in $S$. More precisely, let $K_S(\mathscr{D}_X)$ denote the $K$-theory spectrum of perfect complexes of $\mathscr{D}_X$-modules with support in $S$ and $K_S(T^*X)$ the $K$-theory of perfect complexes on $T^*X$

with support in $S$. Then we construct a natural homotopy morphism $gr_S : K_S(\mathscr{D}_X) \to K_S(T^*X)$. If $\mathscr{M}$ has singular support in $S$, then we get a natural homotopy point $gr_S([\mathscr{M}])$ of $K_S(T^*X)$. The induced morphism $gr : K_{0,S}(\mathscr{D}_X) \to K_{0,S}(T^*X)$ is given by sending $\mathscr{M}$ to the associated graded with respect to some good filtration. The image in the Chow groups is precisely $CC(\mathscr{M})$. Therefore, we may view the homotopy point $gr_S([\mathscr{M}])$ as a micro-local $K$-theoretic animation of $CC(\mathscr{M})$. On the other hand, intersecting $CC(\mathscr{M})$ with the zero section $T^*_X X$ gives the Euler characterisrtic of $\mathscr{M}$. One has the classical Dubson–Kashiwara formula:

$$\chi(X,\mathscr{M}) = \langle CC(\mathscr{M}), T^*_X X\rangle.$$

If we restrict the homotopy point $gr_S([\mathscr{M}])$ to the zero section and apply $R\Gamma$, then this identifies with the homotopy point $R\Gamma_{dR}(X,\mathscr{M})$. In particular, we get a $K$-theoretic animation of the Dubson–Kashiwara formula.

The idea that such an $\varepsilon$-factorization of $\det(R\Gamma)$ could exist comes from number theory. If $X$ is a smooth curve over a finite field of characteristic $p > 0$, and $\mathscr{F}$ an etale constructible sheaf, then a precise factorization format for the determinant of the Frobenius action on the cohomology of $\mathscr{F}$ was conjectured by Deligne ([Del73]) and proven by Laumon ([Lau87]).[1] On the other hand, it is still an open question if there is a geometric $\varepsilon$-factorization in the $l$-adic case which would give rise to the classical $\varepsilon$-factorization by passing to traces of Frobenius.[2] There does not exist, even in the classical sense, an $\varepsilon$-factorizaion format in the higher dimensional $l$-adic case.

In the case of curves, Deligne ([Del84]) gave a construction of the geometric $\varepsilon$-factorization in the de Rham setting. These were reinvented in [BBE02], where the $\varepsilon$-factors were further endowed with an $\varepsilon$-connection. Beilinson ([Bei09]) showed that these de Rham $\varepsilon$-factors have much richer structure. In particular, they form an *$\varepsilon$-factorization theory*. We refer the reader to ([Bei09]) for the definition. The key property is that, in addition to the global factorization fomat, the $\varepsilon$-factors come equipped with an additional *local factorization structure*. Beilinson shows that such local factorization structures are fairly rigid in a certain precise sense ([Bei09]). When restricted to curves, our $\varepsilon$-factors also give rise to an $\varepsilon$-factorization theory in the sense of Beilinson. When combined with the rigidity mentioned above, this should allow us to identify our de Rahm epsilon factors with the previous ones (the details will appear elsewhere). Therefore, we may view the constructions of this article as a generalization of the previous theory to the higher dimensional case.

A geometric factorization format in the Betti setting (where $X$ is a real analytic variety and $\mathscr{M}$ is a constructible complex) was constructed by Beilinson ([Bei07]). In fact, Beilinson constructs the $\varepsilon$-factorization format more generally at the level of homotopy points. The constructions of ([Bei07]) are the main motivation for the constructions of this paper. In particular, the idea that one could directly microlocalize on the K-theory spectrum and, consequently, obtain a

[1] Delinge, in fact, proved the conjecture for compatible systems. The general case is proved by Laumon[Lau87]

[2] In the $l$-adic situation, since the classical $\varepsilon$-factors depend on an additive character of the base field, the geometric theory will lie in an appropriate gerbe rather than be a super graded line. Furthermore, the Frobenius trace function should only give the classical counterpart when $\mathscr{F}$ has virtual rank zero. See [Bei09] for more details on this point.

factorization of the determinant line is due to Beilinson. The results of this paper are an attempt to obtain an (algebraic) de Rham version of this story. It is remarkable that such a factorization exists, given the highly transcendental nature of the Betti constructions.

In analogy to the $l$-adic situation, Deligne envisoned an $\varepsilon$-factorization format for the determinant of the period matrix on a curve. This format was recently constructed by Beilinson ([Bei09]). Given that one now has a geomeric theory of $\varepsilon$-factors in both the Betti and de Rham situations, a natural question is whether one can give a geometric $\varepsilon$-format for the determinant of the period matrix of higher dimensional varieties. We refer the reader to section 3.4 for a precise formulation.

The results of this article were obtained as a part of the author's thesis ([Pat08]).There we used the sheaf of microdifferential operators to microlocalize; here we have chosen to use filtered $\mathscr{D}_X$-modules instead. The use of filtered $\mathscr{D}_X$-modules allows us to work globally and therefore avoids delicate patching arguments with sheaves of $K$-theory spectra. We expect that the filtered $\mathscr{D}_X$-modules approach will allow us to compare the construction here with the Betti situation.

Now we give a brief outline of the paper. In Section 2, we recall various preliminaries on $K$-theory spectra and their homotopy points. The material in this section is standard. In Section 3, we give a construction of de Rham $\varepsilon$-factors and prove various functoriality properties. In Section 4, we specialize our results to the case of curves. The last section explains the passage between homotopy points of spectra and the corresponding determinant lines. The results of this section are well known to the experts , and are included here for the reader's convenience.

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## 2. PRELIMINARIES ON $K$-THEORY SPECTRA

In this section, we recall some basic facts about $K$-theory spectra. We begin by recalling standard facts about the category of simplicial spectra. Most of this material is standard, and is included here for the convenience of the reader. In particular, we discuss the notion of homotopy points and homotopy morphisms of spectra. In the last sub-section, we recall some basic constructions from $K$-theory and their relation to determinants.

2.1. **Simplicial Spectra and homotopy morphisms.** In this section, we review some basic facts and constructions about the category of simplicial spectra (as in [BF78]). We closely follow the presentation of Beilinson ([Bei07]). Recall that a *spectrum* is a sequence of pointed simplicial sets $(P_n)_{n\geq 0}$ together with structure maps $\sigma_n : S^1 \wedge P_n \to P_{n+1}$, where $S^1$ denotes the one sphere. We denote by $\mathscr{S}$ the category of spectra. The category $\mathscr{S}$ comes equipped with a simplicial structure. Given a simplicial set $K$, we can define a spectrum $K \wedge P$ whose $i$-th space is given by $K \wedge P_i = K_+ \wedge P_i$ with the obvious structure maps. Here $K_+$ is the simplicial set given by $K$ disjoint union a base point. This gives rise to the functor $K \wedge \cdot : \mathscr{S} \to \mathscr{S}$ which has a natural right adjoint given by $P \to P^K$. Then for two spectra $P, Q$ one can define a simplicial set $Map(P,Q)$ whose $n$-simplices are given by:

$$Map(P,Q)(n) = \mathrm{Hom}_{\mathscr{S}}(P, Q^{\Delta_n}) = \mathrm{Hom}_{\mathscr{S}}(\Delta_n \wedge P, Q).$$

Given a (pointed) simplicial set $K$, we denote by $|K|$ its geometric realization. Then the *homotopy groups* of a spectrum $P$ are defined by $\pi_i(P) = \varinjlim \pi_{i+n}(|P_n|)$ for all $i \in \mathbb{Z}$; here the limit is taken over the maps induced by the structure maps. A morphism of spectra is a *weak equivalence* if it induces an isomorphism on the corresponding homotopy groups. A morphism $f : P \to Q$ is a cofibration if the induced maps $P_0 \to Q_0$ and $P_n \cup_{S^1 \wedge P_{n-1}} (S^1 \wedge Q_{n-1}) \to Q_n$ are inclusions. The above notions of weak equivalence and cofibration give $\mathscr{S}$ the structure of a stable and proper simplicial model category (see [BF78], [Hir03]). The fibrations are then given by the morphisms satisfying the right lifting property with respect to acyclic cofibrations (i.e. morphisms which are both a weak equivalence and a cofibration). The category of spectra has an initial and terminal object. A spectrum is *fibrant* if the natural morphism to the terminal object is a fibration and it is *cofibrant* if the natural map from the initial object is a cofibration. Finally, the category of spectra has functorial fibrant-cofibrant replacements.

The homotopy category of $\mathscr{S}$ is denoted by $\mathrm{Ho}(\mathscr{S})$. By definition, this is the localization of $\mathscr{S}$ with respect to the weak equivalences. It follows from the general theory of model categories that for fibrant-cofibrant objects $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{S})}(P,Q) = \pi_0(Map(P,Q))$. A weak equivalence of spectra $P \to Q$ can be inverted as a morphism in the homotopy category. But, in general such a morphism cannot be inverted as a morphism of spectra. To remedy this situation, one must use the more general notion of a homotopy morphism of spectra. A *homotopy morphism* $P \to Q$ consists of a contractible simplicial set $K$ and a genuine morphism of spectra $f : K \wedge P \to Q$. We refer to $K$ as the base of the homotopy morphism, and by abuse of notation we shall denote the homotopy morphism by $f : P \to Q$. Given two homotopy morphisms $f, g$ with bases $K_f, K_g$, an *identification of* $f$ *and* $g$ is a homotopy morphism $h$ with base $K_h$ together with morphisms $K_f \to K_h \leftarrow K_g$ such that $f, g$ are the respective pullbacks of $h$. One can define the composition of two homotopy morphisms $f : P \to Q$ and $g : Q \to R$ as the composition $K_g \wedge K_f \wedge P \to K_g \wedge Q \to R$. A homotopy morphism from a sphere spectrum to a given spectrum $P$ will be referred to as a homotopy point of $P$. If $f$ and $g$ are identified, then they induce the same maps on homotopy groups.

We now recall the construction of the homotopy inverse of a morphism. Let $P, Q$ be fibrant-cofibrant spectra and $f : P \to Q$ a weak equivalence of spectra. Then, a right homotopy inverse to $f$ is a pair $(g_r, h_r)$, where $g_r$ is a morphism $Q \to P$ and $h_r$ is a homotopy $\Delta_1 \wedge Q \to Q$ between $fg_r$ and $\mathrm{Id}_Q$. Dually, one can define the notion of left homotopy inverses. One has analogs of these definitions for homotopy morphisms. Recall that in a model category a morphism between fibrant-cofibrant objects is a weak equivalence if and only if it is a homotopy equivalence. Furthermore, if it is a simplicial model category then the notion of left/right homotopy is the same as that of *simplicial homotopy*: Two morphisms of spectra $f, g : A \to B$ are homotopic if there is a $h : \Delta^1 \wedge P \to Q$ such that the composition $\Delta^0 \wedge P \xrightarrow{i_0 \wedge \mathrm{Id}} \Delta^1 \wedge P \xrightarrow{h} Q$ is $f$ and the composition $\Delta^0 \wedge P \xrightarrow{i_1 \wedge \mathrm{Id}} \Delta^1 \wedge P \xrightarrow{h} Q$ is $g$. Here $i_0$ and $i_1$ are the face maps corresponding to the vertices 0 and 1. In particular, the above notion of right homotopy inverse is consistent with the general model category definition. The following lemma shows that any weak equivalence between fibrant-cofibrant spectra can be canonically inverted as a homotopy morphism; an outline of the construction is given in [Bei07]. The proof given here works in the general setting of proper closed simplicial model categories. In this article, we shall be interested in constructing various micro-localization maps between K-theory spectra. During the course of these constructions (see section 3), we will often need to invert weak equivalences of K-theory spectra. The following lemma will allows us to do this in a canonical way.

**Lemma 2.1.1.** *Let $f : P \to Q$ be a weak equivalence of fibrant-cofibrant objects. Then there exists a canonical right homotopy inverse $g_r$ and left homotopy inverse $g_l$. Furthermore, there is a natural identification of $g_r$ with $g_l$.*

*Proof.* Since $P$, $Q$ are fibrant-cofibrant, it follows that $Map(P,Q)$ , $Map(\Delta^1 \wedge Q, Q)$, and $Map(Q,Q)$ are all fibrant. Now consider the pull-back square:

$$\begin{array}{ccc} Map(\Delta^1 \wedge Q, Q)^{\mathrm{Id}} & \longrightarrow & Map(\Delta^1 \wedge Q, Q) \\ \downarrow & & \downarrow \\ \mathrm{Id} & \longrightarrow & Map(Q,Q). \end{array}$$

Here the bottom horizontal consists of the inclusion of the vertex corresponding to the identity, and the right vertical is the map induced by the face map $\Delta^0 \xrightarrow{i_1} \Delta^1$ . Since the right vertical arrow is an acyclic fibration, it follows that the left vertical arrow is also an acyclic fibration; in particular, the pull-back is contractible. Consider also the pullback:

$$\begin{array}{ccc} K^r & \longrightarrow & Map(\Delta^1 \wedge Q, Q)^{Id} \\ \downarrow & & \downarrow \\ Map(Q,P) & \longrightarrow & Map(Q,Q). \end{array}$$

Here, the right vertical is induced by the face map $i_0$. The lower horizontal is induced by $f : P \to Q$. Since $P, Q$ are fibrant-cofibrant and $f$ is a weak equivalence, it follows that the bottom horizontal is a weak equivalence. Furthermore, the right vertical is a fibration. To see this, note that $Map(\Delta^1 \wedge Q, Q) = Map(\Delta^1, Map(Q,Q))$, and therefore the pull back is just the path space of $Map(Q,Q)$. Since the category of simplicial sets is a proper model category, it follows that the pull back of a weak equivalence along a fibration is a weak equivalence. In particular, the top horizontal is also a weak equivalence. Therefore, $K^r$ is a contractible simplicial set. Its vertices correspond to pairs $(g_r, h_r)$ where $g_r : Q \to P$ is a morphism and $h_r$ is a homotopy between $fg_r$ and the $\mathrm{Id} : Q \to Q$. Furthermore, the canonical map $Map(Q,P) \wedge Q \to P$ induces a canonical homotopy morphism $\tilde{g_r} : K^r \wedge Q \to P$. By construction, this is a right inverse to $f$. The composition $f\tilde{g_r} : K^r \wedge Q \to Q$ sends a vertex $(g_r, h_r, q)$ to $fg_r(q)$. This is homotopic to the identity via the map $\tilde{h_r} : \Delta^1 \wedge K^r \wedge Q \to Q$ which, on vertices, sends $(i, g_r, h_r, q)$ to $h_r(i, q)$. At the level of simplicial sets, $\tilde{h_r}$ is induced by the evaluation map $Map(\Delta^1 \wedge Q, Q) \wedge (\Delta^1 \wedge Q) \to Q$. We can also construct in a similar manner a left homotopy inverse $(g_l, K^l)$. Consider the following pullback diagram:

$$\begin{array}{ccc} Map(\Delta^1 \wedge P, P)^{\mathrm{Id}} & \longrightarrow & Map(\Delta^1 \wedge P, P) \\ \downarrow & & \downarrow \\ \mathrm{Id} & \longrightarrow & Map(P, P). \end{array}$$

Here, the right vertical is induced as before by the face map at the vertex corresponding to 1. Just as before, all the corners are fibrant and the left vertical is a weak equivalence. We can form the following pull-back as before:

$$\begin{array}{ccc} K^l & \longrightarrow & Map(\Delta^1 \wedge P, P)^{Id} \\ \downarrow & & \downarrow \\ Map(Q, P) & \longrightarrow & Map(P, P). \end{array}$$

The bottom horizontal is induced by pre-composing with $f$, and the right vertical is induced by the face map at the vertex corresponding to 0. Again, the corners are all fibrant, the right vertical is a fibration and the top horizontal is a weak equivalence. This gives a canonically defined left homotopy inverse $\tilde{g_l} : K^l \wedge Q \to P$. Finally, we identify these two homotopy morphisms. Consider, the following pull-back square:

$$\begin{array}{ccc} H_1 & \longrightarrow & Map(\Delta^1 \wedge Q, P) \\ \downarrow & & \downarrow \\ Map(\Delta^1 \wedge Q, P) & \longrightarrow & Map(Q, P). \end{array}$$

Here, the right vertical and bottom horizontal are induced by $i_0 : \Delta^0 \to \Delta^1$. In particular, both of these arrows are acyclic fibrations. Therefore, all the arrows in the diagram are acyclic fibrations. Note that the vertices of $H_1$ are pairs of homotopies $(h_1, h_2)$ which are the same at 0. Consider the face maps $i_{[0,1]}, i_{[0,2]} : \Delta^1 \to \Delta^2$ corresponding to the inclusion of the edges $[0,1]$ and $[0,2]$. These induce maps $i_{[0,1]}, i_{[0,2]} : Map(\Delta^2 \wedge Q, P) \to Map(\Delta^1 \wedge Q, P)$ which are acyclic fibrations. This follows from the fact that the inclusion of the faces are acyclic cofibrations. Since these maps agree when restricted to the 0-th vertex, we get a map $f : Map(\Delta^2 \wedge Q, P) \to H_1$. Furthermore, the resulting morphism is an acyclic fibration. Finally, consider the morphism $p : K^l \times K^r \to H_1$ which on vertices sends $(g_l, h_l, g_r, h_r)$ to $(h_l g_r, g_l h_r)$. We get the pull-back diagram:

$$\begin{array}{ccc} H & \longrightarrow & Map(\Delta^2 \wedge Q, P) \\ \downarrow & & \downarrow \\ K^l \times K^r & \longrightarrow & H_1. \end{array}$$

Since the right vertical is an acyclic fibration, it follows that so is the left vertical. In particular, $H$ is contractible. Note that $H$ is the simplicial set with vertices given by $(g_r, h_r, g_l, h_l, \tilde{h})$ where $(g_r, h_r, g_l, h_l)$ are left and right homotopy inverses as before and $\tilde{h} : \Delta^2 \wedge Q \to P$. The map $\tilde{h}$ is given by $g_l f g_r$ at the vertex 0, $g_r$ at the vertex 1, and $g_l$ at the vertex 2. Furthermore, it is given by $h_l g_r$ along the $[0,1]$ edge and $g_l h_r$ along the $[0,2]$ edge. We have a canonical homotopy morphism $\psi : \Delta^2 \wedge H \wedge Q \to P$ which on vertices sends $(i, g_r, h_r, g_l, h_l, \tilde{h}, q)$ to $\tilde{h}(i,q)$. We shall use $\psi$ to identify the homotopy morphisms $\tilde{g_r}$ and $\tilde{g_l}$. To do this, it is enough to identify each of these with $\psi$. First, consider the homotopy morphism $\eta : \Delta^0 \wedge H \wedge Q \to P$ which on vertices is given by sending $(i, g_r, h_r, g_l, h_l, \tilde{h}, q)$ to $g_r(q)$. The inclusion of $\Delta^0 \to \Delta^2$ given by sending 0 to 1 gives an identification of $\eta$ with $\psi$. On the other hand, the projection $H \to K^r$ gives and identification of $\eta$ and $\tilde{g_r}$. A similar argument allows one to identify $\tilde{g_l}$ and $\psi$. □

**Remark 2.1.2.** Note that the composition of $g_r$ or $g_l$ with $f$ induces the identity maps on the homotopy groups.

Suppose $P$ is a fibrant-cofibrant spectrum. Then $K \wedge P$ is also cofibrant. It follows from the lemma that any homotopy morphism between fibrant-cofibrant spectra which is a weak equivalence can also be inverted as a homotopy morphism.
One also has a notion of *homotopy sum* for spectra. Let $I$ be a finite set. Then one has a canonical morphism of spectra $e_I : \vee_I P \to P^I$ induced by the identity on the $(i,i)$-th component and trivial on other components. For $k \in I$, let $i_k : P \to \vee_I P$ denote the inclusion onto the $k$-th component.

**Lemma 2.1.3.** *Suppose $P$ is a fibrant-cofibrant spectrum. Then one has a canonical homotopy morphism (the sum) $\Sigma_I : P^I \to P$ such that the composition $\Sigma_I e_I i_k : P \to P$ is given by $id_P$.*

*Proof.* Consider the morphism $f_I \in Map(\vee_I P, P)$ induced by identity from $P \to P$. Let $E(P)_I$ be the pull-back defined by the square:

$$\begin{array}{ccc} E(P)_I & \longrightarrow & Map(P^I, P) \\ \downarrow & & \downarrow \\ f_I & \longrightarrow & Map(\vee_I P, P). \end{array}$$

The right arrow is induced by the morphism $e_I$. Since $P$ is fibrant-cofibrant, so are $\vee_I P$ and $P^I$. Furthermore, $e_I$ is an acyclic cofibration. It follows from the theory of proper closed simplicial model categories that the right vertical map is an acyclic fibration of simplicial sets. In particular, the left vertical arrow is also a weak equivalence, and hence $E(P)_I$ is contractible. Furthermore, the canonical morphism $Map(P^I, P) \wedge P^I \to P$, induces a morphism $E(P)_I \wedge P^I \to P$. Note that by definition of the pull-back, the vertices of $E(P)_I$ are morphisms from $P^I \to P$ which are the identity when composed with each copy of $P$. The resulting homotopy morphism, denoted by $\Sigma_I : P^I \to P$, is the required construction.

□

We say that a square diagram of homotopy morphisms

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ R & \xrightarrow{f'} & S \end{array}$$

is *commutative* or *commutes* if the homotopy morphisms $g'f$ and $gf'$ are identified as homotopy morphisms.

Suppose one is given a diagram of fibrant-cofibrant spectra:

$$\begin{array}{ccccc} & & T & & \\ & & \downarrow{\scriptstyle h} & & \\ P & \xrightarrow{f} & Q & \xrightarrow{g} & R \end{array}$$

where $g \circ h$ is homotopic to zero and the bottom row is a homotopy fiber sequence. Then, given a choice of homotopy of $g \circ h$ to zero, one has a homotopy morphism $T \to P$ [1] such that the following diagram commutes (up to homotopy):

$$\begin{array}{ccccc} & & T & & \\ & \swarrow & \downarrow{\scriptstyle h} & & \\ P & \xrightarrow{f} & Q & \xrightarrow{g} & R \end{array} .$$

Here we can replace all the morphisms by homotopy morphisms.

**Remark 2.1.4.** We refer the reader to ([Hir03]) for details on homotopy fiber/cofiber sequences of spectra.

2.2. ***K*-theory spectra.** Let $\mathscr{E}$ be a small exact category. Then Quillen's $K$-theory construction gives a functor from the category of small exact categories to the category of spectra. Since $\mathscr{S}$ has functorial fibrant-cofibrant replacements, we assume from now on that the associated spectrum $K(\mathscr{E})$ is fibrant-cofibrant. More generally, if $\mathscr{E}$ is an essentially small exact category then we can associate to it a spectrum $K(\mathscr{E})$ by taking the $K$-theory spectrum of an associated small model. An equivalence of small exact categories gives a weak equivalence of the corresponding spectra; therefore, two different small models can be canonically identified in the

[1] In general, such a lifting is not unique. However, once a choice of homotopy is fixed, any two can be identified.

world of homotopy morphisms. In particular, we can associate to any essentially small exact category a $K$-theory spectrum, and any exact functor between essentially small exact categories gives rise to a homotopy morphism of the corresponding $K$-theory spectra. If $F_1 : \mathscr{E}_1 \to \mathscr{E}_2$ and $F_2 : \mathscr{E}_2 \to \mathscr{E}_3$ are exact functors, then the homotopy morphisms $K(F_2) \circ K(F_1)$ and $K(F_2 \circ F_1)$ are canonically identified. From now on we assume that all our categories are essentially small. More generally, Waldhausen associates to any category with cofibrations and weak equivalences a corresponding $K$-theory spectrum. Furthermore, an exact functor between Waldhausen categories induces a homotopy morphism between the corresponding spectra. In this article, we shall mostly be interested in complicial bi-Waldhausen categories and complicial exact functors; we refer the reader to ([TT90]) for details. If $\mathscr{E}$ is an exact category, then $C^b(\mathscr{E})$ is a complicial bi-Waldhausen category with weak equivalences. We will recall briefly this structure. Again, we refer the reader to ([TT90]) for details. Any exact category $\mathscr{E}$ can be embedded into an abelian category $\mathscr{A}$ such that a sequence in $\mathscr{E}$ is exact if and only if the corresponding sequence in $\mathscr{A}$ is exact. Then we have a fully faithful embedding $C^b(\mathscr{E}) \to C^b(\mathscr{A})$. This gives $C^b(\mathscr{E})$ the structure of a complicial bi-Waldhausen category. The cofibrations are given by the degree-wise strict monomorphisms. The weak equivalences are morphisms which are quasi-isomorphisms in $C^b(\mathscr{A})$. A fundamental result of Thomason–Trobaugh–Waldhausen–Gillet ([TT90]) shows that the inclusion of $\mathscr{E}$ into $C^b(\mathscr{E})$ as degree zero morphisms induces a canonical weak equivalence of spectra $K(\mathscr{E}) \to K(C^b(\mathscr{E}))$. Here the right side is the Waldhausen $K$-theory spectrum associated to $K(C^b(\mathscr{E}))$. This allows us to canonically identify various Quillen and Waldhausen $K$-theory spectra. Similar statements apply to the categories $C^-(\mathscr{E})$ and $C^+(\mathscr{E})$. In the following, we shall always assume all our spectra to be fibrant-cofibrant. In particular, the machinery from the previous section will allow us to invert various weak equivalences canonically as homotopy morphisms.

**Remark 2.2.1.** There is another structure of a bi-complicial Waldhausen category on $C^b(\mathscr{E})$ commonly used in the literature. The weak equivalences are the same as above, but the cofibrations are degree-wise split monomorphisms whose quotient lies in $C^b(\mathscr{E})$. Let $\mathscr{E}_1$ denote $C^b(\mathscr{E})$ with this structure of bi-complicial Waldhausen category and $\mathscr{E}_2$ denote $C^b(\mathscr{E})$ with the structure of bi-complicial Waldhausen category described in the previous paragraph. Then the inclusion $\mathscr{E}_1 \to \mathscr{E}_2$ is a complicial exact functor such that the induced morphism $K(\mathscr{E}_1) \to K(\mathscr{E}_2)$ is a weak equivalence (1.11.7, [TT90]). In this article, we shall always use the complicial structure given in the previous paragraph.

Given a Waldhausen category $\mathscr{A}$, we denote by $\mathscr{A}^{\mathrm{tri}}$ the associated homotopy category given by inverting the weak equivalences; note that this is a triangulated category. If $F : \mathscr{A} \to \mathscr{B}$ is a complicial exact functor between two complicial bi-Waldhausen categories such that the induced map on homotopy categories is an equivalence of categories, then the induced map on $K$-theory spectra is a weak equivalence. We will often consider derived functors which are *a priori* only defined on $\mathscr{A}^{tri}$. Usually, these can be lifted to functors on certain full complicial bi-Waldhausen subcategories $\mathscr{C} \subset \mathscr{A}$ such that the inclusion induces an equivalence on the associated triangulated categories. Using the formalism of homotopy morphisms, we can lift the derived functor to a morphism of $K$-theory spectra. A typical application is the following: Let

$X$ be a proper scheme over $k$, and let $K(X)$ be the $K$-theory spectrum of perfect complexes on $X$. Since $X$ is proper, we can define $R\Gamma : D^b_{\text{perf}}(X) \to D^b_{\text{perf}}(k)$. The above approach allows us to lift this to a homotopy morphism $R\Gamma : K(X) \to K(k)$, where $K(X)$ is the $K$-theory spectrum of category of perfect complexes on $X$ and similarly for $K(k)$. First, we may consider the (full) complicial bi-Waldhausen sub-category of flasque perfect complexes. On this subcategory, $R\Gamma$ is represented by $\Gamma$. Furthermore, the properness assumption implies that $\Gamma$ preserves perfectness. We refer to the article by Thomason–Trobaugh ([TT90]) for more details.

In the following, we will sometimes consider Waldhausen categories $\mathscr{W}$ with two different notions of weak equivalences given by subsets $v \subset \omega$ of morphisms in $\mathscr{W}$. We will use the notation $v\mathscr{W}$ and $\omega\mathscr{W}$ to distinguish between the two induced Waldhausen category structures.

2.3. **Determinants.** In this section, we briefly recall the determinant philosophy at the spectrum level; we refer the reader to section 5 for details. Given a Waldhausen category $\mathscr{A}$, any object $F$ in $\mathscr{A}$ gives rises to a homotopy point $[F]$ of the associated $K$-theory spectrum $K(\mathscr{A})$. In the situation of an exact category $\mathscr{E}$, this construction gives a canonical homotopy point $[F]$ of $K(C^b(\mathscr{E}))$ for all $F \in \text{Ob}(C^b(\mathscr{E}))$. Furthermore, to any [0,1]-connected $\Omega$-spectrum $K$ we can associate a canonical Picard groupoid denoted by $\Pi(K)$; any homotopy point of $K$ gives rise to an object of the associated Picard groupoid. For any spectrum $K$, we can functorially associate a $[0,1]$ connected $\Omega$-spectrum denoted $K^{[0,1]}$ with a morphism $K \to K^{[0,1]}$. In the case of $K(C^b(\mathscr{E}))$, we can apply the above to get an object $\text{Det}(x) \in \Pi(K(C^b(\mathscr{E}))^{[0,1]})$ for any homotopy point $x$ of $K(C^b(\mathscr{E}))$. Furthermore, the homotopy point construction induces a determinant functor $\text{Det} : (C^b(\mathscr{E}), w) \to \Pi(K(C^b(\mathscr{E}))^{[0,1]})$, which is a universal determinant functor in the sense of Knudsen ([Knu02]). This functor factors through the derived category $(D^b(\mathscr{E}), qis) \to \Pi(K(C^b(\mathscr{E}))^{[0,1]})$ as a tensor functor. Here $D^b(\mathscr{E})$ has tensor structure coming from the additive structure. Furthermore, an identification of homotopy points gives rise to an isomorphism of the corresponding determinants. If $x$ and $y$ are two homotopy points of $K(C^b(\mathscr{E}))$, then one has a canonical isomorphism $\cdot_{xy} : \text{Det}(x) \otimes \text{Det}(y) \to \text{Det}(x+y)$; here $x+y$ is the homotopy sum described in 2.1.

**Remark 2.3.1.** If $F$ and $G$ are objects in a Waldhausen category $\mathscr{A}$, then the homotopy point $[F \oplus G]$ of the direct sum of $F$ and $G$ in $\mathscr{A}$ is identified with the homotopy sum $[F] + [G]$. See 5.15 for details.

Let $Pic^{\mathbb{Z}}(k)$ denote the Picard groupoid of $\mathbb{Z}$-graded lines on $k$, whose objects are ordered pairs of one dimensional $k$-vector spaces and an integer $n$, the degree of the line. Then there is a canonical determinant functor $\det : C^b(k) \to Pic^{\mathbb{Z}}(k)$. If $V$ is a vector space in degree zero, this functor sends $V$ to the usual determinant line graded by the dimension of the vector space. In the particular case of a scheme $X$ proper over $k$, the determinant of $R\Gamma(X, F)$ is just the usual determinant of cohomology graded by the Euler characteristic. If $S$ is a scheme, then we can define the Picard groupoid $Pic^{\mathbb{Z}}(S)$ of $\mathbb{Z}$-graded lines on $S$. The grading will be a $\mathbb{Z}$-valued locally constant function on $S$. Then, as in the case of a field, one has a determinant functor $\det_S : C^b_{perf}(S) \to Pic^{\mathbb{Z}}(S)$. By universality, there is a canonical morphism of Picard groupoids

$\mathrm{Det}_S : \Pi(K(S)) \to Pic^{\mathbb{Z}}(S)$ such that the following diagram commutes:

$$\begin{array}{ccc} C^b_{perf}(S) & \xrightarrow{\mathrm{Det}} & \Pi(K(S)) \\ & \searrow^{\mathrm{det}_S} & \downarrow{\scriptstyle \mathrm{Det}_S} \\ & & Pic^{\mathbb{Z}}(S). \end{array}$$

In particular, if $A$ and $B$ are perfect complexes, then the image of $\mathrm{Det}_S(\cdot_{[A][B]})$ is just the usual isomorphism $\det_S(A) \otimes \det_S(B) \to \det_S(A \oplus B)$.
Suppose we have an exact functor of exact categories $\mathscr{E}_1 \to \mathscr{E}_2$. Then we get an induced map $F : K(\mathscr{E}_1) \to K(\mathscr{E}_2)$. If $A \in \mathrm{Ob}(\mathscr{E}_1)$, then there is a canonical identification of the homotopy points $F([A])$ and $[F(A)]$. Furthermore, an identification of homotopy points gives rise to an isomorphism of the corresponding determinants.

## 3. THE EPSILON FACTORIZATION

Let $X$ be a smooth variety over a field $k$ of characteristic zero. For a closed subset $S$ of $T^*X$, let $K_S(\mathscr{D}_X)$ denote the $K$-theory spectrum of $\mathscr{D}_X$-modules with singular support contained in $S$. Similarly, let $K_S(T^*X)$ denote the $K$-theory spectrum of perfect complexes on $T^*X$ with cohomology supported in $S$. The construction of de Rham epsilon factors proceeds in two steps: First, we construct a microlocalization morphism $K_S(\mathscr{D}_X) \to K_S(T^*X)$. Then, we further localize with respect to a non-vanishing 1-form $\nu$ to get the required epsilon factorizations. In the first subsection we construct the microlocalization morphism. The second section is devoted to the construction of epsilon factors. The third section is devoted to the study of various functoriality properties of epsilon factors. The last two subsections briefly discuss the connection with Betti epsilon factors and the Dubson–Kashiwara formula. For the Betti analogs of the results in this section see [Bei07].

3.1. **Preliminaries on filtered $\mathscr{D}_X$-modules.** We begin by reviewing some preliminaries on filtered $\mathscr{D}_X$-modules (see [Lau83]). Recall that the sheaf of differential operators $\mathscr{D}_X$ comes with an increasing filtration by $\mathscr{O}_X$ submodules:

$$\mathscr{O}_X = \mathscr{D}_0 \subset \mathscr{D}_1 \subset \cdots \subset \mathscr{D}_X,\ \mathscr{D}_i\mathscr{D}_j = \mathscr{D}_{i+j}, \bigcup_i \mathscr{D}_i = \mathscr{D}_X$$

where each $\mathscr{D}_i$ is a locally free $\mathscr{O}_X$-module. A *filtered $\mathscr{D}_X$-module* consists of a pair $(\mathscr{M}, \mathscr{F}.)$ where $\mathscr{M}$ is a $\mathscr{D}_X$-module and $\mathscr{F}.$ is an increasing $\mathbb{Z}$-filtration of $\mathscr{M}$ by $\mathscr{O}_X$-submodules such that $\mathscr{F}_i = 0$ for $i << 0$, $\bigcup_i \mathscr{F}_i = \mathscr{M}$, and $\mathscr{D}_i\mathscr{F}_j \subset \mathscr{F}_{i+j}$. A filtered $\mathscr{D}_X$-module is *quasi-coherent* if each $\mathscr{F}_i$ is a quasi-coherent $\mathscr{O}_X$-module. It follows that $\mathscr{M}$ is a quasi-coherent $\mathscr{D}_X$-module. Let $MF_{\mathrm{qcoh}}(\mathscr{D}_X)$ denote the category of quasi-coherent filtered $\mathscr{D}_X$-modules where the morphisms preserve the filtration. If $\mathscr{F}.$ is a filtration on $\mathscr{M}$, then we can define a new filtration $\mathscr{F}[k]$ where $\mathscr{F}[k]_i = \mathscr{F}_{i+k}$. We denote by $\mathscr{D}_X[j]$ the filtered $\mathscr{D}_X$-module $\mathscr{D}_X$ with the standard filtration shifted by $j$. In general, if $\mathscr{M}$ is a filtered $\mathscr{D}_X$-module, we denote by $\mathscr{M}[j]$ the filtered $\mathscr{D}_X$-module with same underlying module, but with filtration shifted by $j$.
Let $\mathrm{gr}.(\mathscr{D}_X)$ denote the associated graded sheaf and $MG_{\mathrm{qcoh}}(\mathrm{gr}.(\mathscr{D}_X))$ the corresponding category of quasi-coherent graded modules. We have a natural functor $\mathrm{gr}. : MF_{\mathrm{qcoh}}(\mathscr{D}_X) \to MG_{\mathrm{qcoh}}(\mathrm{gr}.(\mathscr{D}_X))$.

We denote by $\mathrm{gr}(\mathscr{D}_X)$ the same sheaf as before, but where we forget the grading. Let $M_{\mathrm{qcoh}}(\mathrm{gr}(\mathscr{D}_X))$ denote the corresponding category of quasi-coherent $\mathrm{gr}(\mathscr{D}_X)$-modules. Forgetting the grading gives a natural functor $\mathrm{gr} : MF_{\mathrm{qcoh}}(\mathscr{D}_X) \to M_{\mathrm{qcoh}}(\mathrm{gr}(\mathscr{D}_X))$. The categories $M_{\mathrm{qcoh}}(\mathrm{gr}(\mathscr{D}_X))$ and $MG_{\mathrm{qcoh}}(\mathrm{gr}_{\cdot}(\mathscr{D}_X))$ are abelian categories. The category $MF_{\mathrm{qcoh}}(\mathscr{D}_X)$ has all kernels, cokernels, images, and coimages. For example, the kernel of a morphism $f : (\mathscr{M}, \mathscr{F}_{\cdot}) \to (\mathscr{N}, \mathscr{G}_{\cdot})$ is the $\mathscr{D}_X$-module $\ker(f)$ with the filtration induced from $\mathscr{F}_{\cdot}$. We say that a sequence $0 \to (\mathscr{M}, \mathscr{F}_{\cdot}) \to (\mathscr{N}, \mathscr{G}_{\cdot}) \to (\mathscr{P}, \mathscr{H}_{\cdot}) \to 0$ in $MF_{\mathrm{qcoh}}(\mathscr{D}_X)$ is exact if , for all $i$, $0 \to F_i \to G_i \to H_i \to 0$ is an exact sequence of $\mathscr{O}_X$-modules. This is equivalent to requiring that $uv = 0$ and $0 \to \mathrm{gr}(\mathscr{M}) \to \mathrm{gr}(\mathscr{N}) \to \mathrm{gr}(\mathscr{P}) \to 0$ be exact as a sequence of $\mathrm{gr}(\mathscr{D}_X)$-modules. The above notion of exact sequences makes $MF_{\mathrm{qcoh}}(\mathscr{D}_X)$ into an exact category. We shall also consider the category $MFF_{\mathrm{qcoh}}(\mathscr{D}_X)$ of doubly filtered quasi-coherent $\mathscr{D}_X$-modules. The objects are triples $(\mathscr{M}, \mathscr{F}^1_{\mathscr{M},\cdot}, \mathscr{F}^2_{\mathscr{M},\cdot})$, where $\mathscr{M}$ is a $\mathscr{D}_X$-module and $\mathscr{F}^*_{\mathscr{M},\cdot}$ are filtrations by quasi-coherent $\mathscr{O}_X$-submodules as above. We will usually drop the subscript $\mathscr{M}$ and denote it simply by $(\mathscr{M}, \mathscr{F}^1_{\cdot}, \mathscr{F}^2_{\cdot})$ or $(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ if no confusion arises. The category $MFF_{\mathrm{qcoh}}(\mathscr{D}_X)$ is an exact category where a sequence is exact iff it is exact at the $i$-th filtration level for both filtrations. If $\mathscr{M}$ is a doubly filtered $\mathscr{D}_X$-module, then $\mathscr{M}[j,k]$ is the doubly filtered $\mathscr{D}$-module with the same underlying filtration, but with the first filtration shifted by $j$ and second filtration shifted by $k$.

Given an exact category $\mathscr{E}$, let $C^*(\mathscr{E})$ (for $* = +, -, b$) denote the corresponding category of chain complexes and $D^*(\mathscr{E})$ the associated derived category. We will use the notation $CF^*(\mathscr{D}_X)$,$CFF^*(\mathscr{D}_X)$, and $CG^*(\mathrm{gr}_{\cdot}(\mathscr{D}_X))$ for the categories of chain complexes associated to the categories above; similarly, $DF^*(\mathscr{D}_X)$,$DFF^*(\mathscr{D}_X)$, and $DG^*(\mathrm{gr}_{\cdot}(\mathscr{D}_X))$ will denote the corresponding derived categories. Recall that $CF^*(\mathscr{D}_X)$ and $CFF^*(\mathscr{D}_X)$, are both complicial bi-Waldhausen categories (see section 2.2). Let $\omega_{CF}$ and $\omega_{CFF}$ denote the weak equivalences in $CF^*(\mathscr{D}_X)$ and $CFF^*(\mathscr{D}_X)$. If
$(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ is a doubly filtered complex, then $H^i(\mathscr{M})$, with filtration induced by $F^1$ and $F^2$, is a doubly filtered $\mathscr{D}_X$-module. We will use the notation $H^i(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ to denote this doubly filtered $\mathscr{D}_X$-module. Similar notation will be used for filtered $\mathscr{D}_X$-modules.

We will say that an object of $MF(\mathscr{D}_X)$ is *free of finite type* if it is isomorphic to an object of the form $\oplus_{\alpha=1}^r \mathscr{D}_X[-j_\alpha]$ for some $j_\alpha \in \mathbb{Z}$ and $r \geq 0$. An object of $MF(\mathscr{D}_X)$ is a *projective filtered $\mathscr{D}_X$-module* if it is locally a direct summand of a free module of finite rank. We have analogous definitions of free and projective doubly filtered modules where $\oplus_{\alpha=1}^r \mathscr{D}_X[-j_\alpha]$ is replaced by $\oplus_{\alpha=1}^r \mathscr{D}_X[-j_\alpha, -k_\alpha]$. One can now define various categories of perfect and coherent complexes ([Lau83], [[SG71]). In particular, we have the categories $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$, $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$, $DF^b_{\mathrm{perf}}(\mathscr{D}_X)$, and $DFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ of perfect complexes. An object in the respective category is a complex which is locally quasi-isomorphic to a bounded complex each of whose terms is projective (i.e. a strictly perfect complex). If $(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ is perfect, then each $H^n(gr_{\mathscr{F}^k}(\mathscr{M}))$ is a coherent $gr(\mathscr{D}_X)$-module (4.4.3, [Lau83]).

Any exact category $\mathscr{E}$ can be embedded in an abelian category $\mathscr{A}$ such that a sequence in $\mathscr{E}$ is exact if and only if the corresponding sequence in $\mathscr{A}$ is exact. If $\mathscr{E}$ is an exact category, then $C^b(\mathscr{E})$ comes equipped with a homotopy pushout, induced from the one in $C^b(\mathscr{A})$. Let $f : M^{\cdot} \to N^{\cdot}$ and $g : M^{\cdot} \to P^{\cdot}$ be morphisms in $C^b(\mathscr{E})$. Then set $(N^{\cdot} \cup^h_{M^{\cdot}} P^{\cdot})^n = N^n \oplus$

$M^{n+1} \oplus P^n$ with the differential given by $d(n,m,p) = (d_N(n) + f(m), -d_M(m), d_P(p) - g(m))$. If the induced morphism $M^\cdot \to N^\cdot \oplus P^\cdot$ is a strict monomorphism, then the canonical map from the homotopy pushout to the pushout is a quasi-isomorphism. This follows from the analogous statement for abelian categories (applied to $\mathscr{A}$). There is a dual notion of homotopy pullback. Given $f : M^\cdot \to P^\cdot$ and $g : N^\cdot \to P^\cdot$, let $(M^\cdot \times^h_{P^\cdot} N^\cdot)^k = M^k \oplus P^{k-1} \oplus N^k$ with $d(m,p,n) = (d_M(m), -d_P(p) + f(m) - g(n), d_N(n))$. If the induced map $M^\cdot \oplus N^\cdot \to P^\cdot$ is a strict epimorphism, then the canonical map from the puhout to the homotopy pushout is a quasi-isomorphism. In particular, both $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ and $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ come equipped with homotopy pushouts and pullbacks.

**Lemma 3.1.1.** *The category of perfect filtered complexes $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ (resp. $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$) is a full subcategory of $CF^b(\mathscr{D}_X)$ (resp. $CFF^b(\mathscr{D}_X)$) which is closed under extensions, fiber products along strict epimorphisms, and push-outs along strict monomorphisms. In particular, $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ and $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ are complicial bi-Walhausen categories.*

*Proof.* The statement for $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ follows from that for $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$. Recall that the complicial Waldhausen structure on $CF^b(\mathscr{D}_X)$ comes from an embedding into $C(\mathscr{A})$. Furthermore, $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ is a full exact subcategory of $CF^b(\mathscr{D}_X)$. The cofibrations and weak-equivalences in $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ are defined to be morphisms which are cofibrations or weak-equivalences in $CF^b(\mathscr{D}_X)$. This will give a bi-complicial Waldhausen structure on $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ if it is closed under extensions, fiber products along strict epimorphisms, and push-outs under strict monomorphisms (1.2.11, [TT90]). The closure under extensions is proved in ([Lau83]). We show closure under fiber products along strict epimorphisms. Consider a diagram of objects in $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$:

$$\begin{array}{ccc} & & (\mathscr{M},\mathscr{F}) \\ & & \downarrow \\ (\mathscr{N},\mathscr{F}) & \xrightarrow{g} & (\mathscr{P},\mathscr{F}) \end{array}$$

where $g$ is a strict epimorphism. Since $g$ is a strict epimorphism, the homotopy pullback is quasi-isomorphic to the pullback, so it is enough to prove the analogous statement for the homotopy pullback. As the statement is local, we may assume all our complexes are quasi-isomorphic to strictly perfect complexes; in fact, since X is smooth, we can even assume this globally. In any case, the homotopy pullback of the corresponding strictly perfect complexes will be quasi-isomorphic to the initial one. On the other hand, it is clear from the construction of homotopy pullback that the homotopy pullback of strictly perfect complexes is strictly perfect. A similar argument works for the homotopy pushout. ☐

Let $KFF(\mathscr{D}_X)$ and $KF(\mathscr{D}_X)$ denote the K-theory spectra of $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ and $CF^b_{\mathrm{perf}}(\mathscr{D}_X)$. We also have the $K$-theory spectrum $K(\mathscr{D}_X)$ of the category of perfect complexes of $\mathscr{D}_X$-modules. We have two functors $F^1, F^2 : CFF^b_{\mathrm{perf}}(\mathscr{D}_X) \to CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ where $F^i$ sends $(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ to

$(\mathscr{M}, \mathscr{F}^i)$. We also have the forgetful functor $\omega : CF^b_{\text{perf}}(\mathscr{D}_X) \to C^b_{\text{perf}}(\mathscr{D}_X)$. These are all complicial exact functors of complicial bi-Waldhausen categories. In particular, we have the induced homotopy morphisms of spectra $F^* : KFF(\mathscr{D}_X) \to KF(\mathscr{D}_X)$ and $\omega : KF(\mathscr{D}_X) \to K(\mathscr{D}_X)$.

**Theorem 3.1.2.** *The square of spectra*

$$\begin{array}{ccc} KFF(\mathscr{D}_X) & \xrightarrow{F^1} & KF(\mathscr{D}_X) \\ \downarrow{\scriptstyle F^2} & & \downarrow{\scriptstyle \omega} \\ KF(\mathscr{D}_X) & \xrightarrow{\omega} & K(\mathscr{D}_X) \end{array}$$

*is a homotopy pushout square.*

The theorem is proved in two parts. First, the fibers of the two horizontal rows are computed via Waldhausen's localization theorem (1.8.2, [TT90]). Then these fibers are canonically identified. We begin by setting up some notation. Let $\omega_1$ denote the morphisms $f : (\mathscr{M}^\cdot, \mathscr{F}^1, \mathscr{F}^2) \to (\mathscr{N}^\cdot, \mathscr{F}^1, \mathscr{F}^2)$ in $CFF^b_{\text{perf}}(\mathscr{D}_X)$ such that the induced morphism $gr_{\mathscr{F}^1}(f) : gr_{\mathscr{F}^1}(\mathscr{M}^\cdot) \to gr_{\mathscr{F}^1}(\mathscr{N}^\cdot)$ is a weak equivalence; we define $\omega_2$ analogously using $\mathscr{F}^2$ instead. Let $\nu$ denote the morphisms in $CF^b_{\text{perf}}(\mathscr{D}_X)$ which induce a weak equivalence on the underlying complex. Then $\nu CF^b_{\text{perf}}(\mathscr{D}_X)$ is also a Waldhausen category. Finally, one has the Waldhausen category $\omega_{CFF} CFF^b_{\text{perf}}(\mathscr{D}_X)^{\omega_1}$ consisting of the full subcategory of $\omega_1$ acyclic objects in $CFF^b_{\text{perf}}(\mathscr{D}_X)$ with weak equivalences given by $\omega_{CFF}$.

**Lemma 3.1.3.** *The Waldhausen category $\omega_1 CFF^b_{\text{perf}}(\mathscr{D}_X)$ satisfies the saturation and extension axioms.*

*Proof.* Let $f : (\mathscr{M}^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) \to (\mathscr{N}^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot)$ and $g : (\mathscr{N}^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) \to (P^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot)$ be morphisms in $\omega_1 CFF^b_{\text{perf}}(\mathscr{D}_X)$. The saturation axiom states that if any two of $f, g, g \circ f$ are in $\omega_1$ then so is the third. By definition, this amounts to checking that if two of $gr_{\mathscr{F}^1}(f), gr_{\mathscr{F}^1}(g)$, and $gr_{\mathscr{F}^1}(g \circ f)$ are quasi-isomorphisms of complexes of sheaves of $\text{gr}(\mathscr{D}_X)$-modules, then so is the third. Since $gr_{\mathscr{F}^1}(g \circ f) = gr_{\mathscr{F}^1}(g) \circ gr_{\mathscr{F}^1}(f)$, the result follows from the corresponding statement for complexes of sheaves of $\text{gr}(\mathscr{D}_X)$-modules.
For the extension axiom, we must show that for a diagram of cofibration sequences:

$$\begin{array}{ccccc} (\mathscr{M}^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) & \rightarrowtail & (\mathscr{N}^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) & \twoheadrightarrow & (\P^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ (\mathscr{M}'^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) & \rightarrowtail & (\mathscr{N}'^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot) & \twoheadrightarrow & (\P'^\cdot, \mathscr{F}^1_\cdot, \mathscr{F}^2_\cdot), \end{array}$$

if $a$ and $c$ are in $\omega_1$ then so is $b$. The image of the above diagram of cofibration sequences under $F^1$ is also a cofibration sequence. Furthermore, $f \in \omega_1$ if and only if $F^1(f) \in \omega_{CF}$. The result follows since $\omega_{CF} CF^b_{\text{perf}}(\mathscr{D}_X)$ is a complicial bi-Waldhausen category.

□

Before proceeding, we briefly recall the notion of a cylinder functor on a Waldhausen category $\mathscr{A}$ (see ([TT90], 1.3) for details). Let $Cat(1,\mathscr{A})$ denote the category of morphisms in $\mathscr{A}$. A cylinder functor on $\mathscr{A}$ is a functor $T : Cat(1,\mathscr{A}) \to \mathscr{A}$, together with three natural transformations satisfying the following properties. By definition, $T$ assigns an object $Tf \in \mathscr{A}$ to each morphism $f : A \to B$ and a morphism $T(a,b) : Tf \to Tf'$ to each commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array}$$

The natural transformations are maps $j_1 : A \to Tf$, $j_2 : B \to Tf$, and $p : Tf \to B$ such that $pj_1 = f$, $pj_2 = 1$ and such that the following diagram commutes:

$$\begin{array}{ccccc} A \cup B & \xrightarrow{j_1 \cup j_2} & Tf & \xrightarrow{p} & B \\ \downarrow{\scriptstyle a\cup b} & & \downarrow{\scriptstyle T(a,b)} & & \downarrow{\scriptstyle b} \\ A' \cup B' & \xrightarrow{j_1 \cup j_2} & Tf' & \xrightarrow{p} & B' \end{array}$$

The above data is also required to satisfy the following conditions:
(1) $j_1 \cup j_2 : A \cup B \to Tf$ is a cofibration.
(2) If $a$ and $b$ are weak equivalences, then $T(a,b)$ is a weak equivalence.
(3) If $a$ and $b$ are cofibrations, then $T(a,b)$ and $Tf \cup_{A\cup B} (A' \cup B') \to Tf'$ is a cofibration.
(4) $T(0 \to A) = A$ with $p = j_2 = Id$.
A cylinder functor satisfying the following additional axiom is said to satisfy the cylinder axiom:
(5) (Cylinder axiom) For all $f$, $p : Tf \to B$ is a weak equivalence.
Suppose $\mathscr{A}$ is an abelian category. We have seen that $C^b(\mathscr{A})$ is a complicial bi-Waldhausen category. For any $f : A \to B \in C^b(\mathscr{A})$, let $Tf = A \cup^h_A B$ denote the homotopy pushout of $f$ and $Id : A \to A$. Let $j_1$ and $j_2$ denote the canonical inclusions. Finally, let $p$ be given by $p(a,a',b) = fa + b$. It is shown in ([TT90], 1.3.5) that this defines a cylinder functor satisfying the cylinder axiom on $C^b(\mathscr{A})$. In particular, the category $C^b(\mathrm{gr}(\mathscr{D}_X))$ has a canonically defined cylinder functor. We shall denote this cylinder functor by $T^{gr}$. Since $C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$ is closed under homotopy pushouts in $C^b(\mathrm{gr}(\mathscr{D}_X))$, $T^{gr}$ induces a cylinder functor satisfying the cylinder axiom on $C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$.

**Lemma 3.1.4.** *$CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ has a cylinder functor which satisfies the cylinder axiom.*

*Proof.* Recall, we have already seen that homotopy pushouts exist in $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$. For a given morphism $f : A \to B$ in $CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ , we let $Tf = A \cup^h_A B$ denote the homotopy pushout of $f$ and $Id : A \to A$. The maps $j_1$ and $j_2$ are the canonical inclusions. The map $p$ is given by $p(a,a',b) = fa + b$. One can check directly that this construction satisfies the axioms above. For example, since cofibrations are degree-wise admissible monomorphisms, (1) follows from

the definition of the homotopy pushout. Alternatively, one can deduce properties (1)-(5) from analogous statements in $C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$.
One has $gr_{F^i}(Tf)$ is isomorphic to $T^{gr}(gr_{F^i}(f))$. Furthermore, $a$ and $b$ are weak-equivalences if and only if $gr_{F^i}(a)$ and $gr_{F^i}(b)$ are weak equivalences. Similarly, for cofibrations (i.e. admissible monomorphisms). Now (2), (4), and (5) follow directly from the analogous statements in $C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$. For (3) it is enough to show that $Tf \cup_{A\cup B} (A' \cup B') \to Tf'$ is a cofibration since clearly $Tf \to Tf \cup_{A\cup B} (A' \cup B')$ is a cofibration. Therefore if the first map is a cofibration, then so is the composition, which is precisely $T(a,b) : Tf \to Tf'$. On the other hand, $gr_{F^i}(M \cup_P N)$ is isomorphic to $gr_{F^i}(M) \cup_{gr_{F^i}(P)} gr_{F^i}(N)$ for all $N,M,P \in CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$. Therefore, the claim follows from the analogous claim in $C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$.

□

**Corollary 3.1.5.** *The following sequence is a homotopy cofiber sequence:*

$$K(\omega_{CFF} CFF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\omega^1_{CFF}}) \to KFF(\mathscr{D}_X) \to K(\omega^1 CFF^b_{\mathrm{perf}}(\mathscr{D}_X)).$$

*Proof.* Given the previous two lemmas, this is a direct consequence of the Localization Theorem (1.8.2, [TT90]). □

**Lemma 3.1.6.**
(1) *$\nu CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ satisfies the saturation and extension axioms.*
(2) *$CF^b_{\mathrm{perf}}(\mathscr{D}_X)$ has a cylinder functor given by the homotopy pushout and satisfies the cylinder axiom.*

*Proof.* The proof is similar to that of Lemmas 3.3 and 3.4. □

**Corollary 3.1.7.** *The following sequence is a homotopy cofiber sequence:*

$$K(\omega_{CF} CF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\nu}) \to KF(\mathscr{D}_X) \to K(\nu CF^b_{\mathrm{perf}}(\mathscr{D}_X)).$$

The forgetful functor $F^2$ induces a functor $F^2 : \omega_{CFF} CFF^b(\mathscr{D}_X)^{\omega^1} \to \omega_{CF} CF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\nu}$. To see this, note that if $(\mathscr{M}^{\cdot}, \mathscr{F}^1_{\cdot}, \mathscr{F}^2_{\cdot})$ is $\omega^1$-acyclic, then the underlying complex of $\mathscr{D}_X$-modules is also acyclic. Furthermore, $F$ preserves cofibrations, pushouts, and weak equivalences. It follows that $F^2$ is a complicial exact functor, and we have an induced homotopy morphism $F^2 : K(\omega_{CFF} CFF^b(\mathscr{D}_X)^{\omega^1}) \to K(\omega_{CF} CF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\nu})$.

**Proposition 3.1.8.** *$F^2$ is a weak equivalence of spectra.*

*Proof.* We apply the Approximation Theorem (1.9.1, [TT90]). First, note that both $\omega_{CFF} CFF^b(\mathscr{D}_X)^{\omega^1}$ and $\omega_{CF} CF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\nu}$ satisfy the saturation axioms. A morphism $f \in \omega_{CFF} CFF^b(\mathscr{D}_X)^{\omega^1}$ is a weak equivalence if and only if $gr_{\mathscr{F}^1}$ and $gr_{\mathscr{F}^2}$ are weak equivalences. Since objects of $\omega_{CFF} CFF^b(\mathscr{D}_X)^{\omega^1}$ are $gr_{\mathscr{F}^1}$-acyclic, $gr_{\mathscr{F}^1}$ is automatically a weak equivalence. It follows that $f$ is a weak equivalence iff $F^2(f)$ is a weak equivalence. Finally, to apply the Approximation Theorem we must verify the following statement: Given any $(\mathscr{M}^{\cdot}, \mathscr{F}^1_{\cdot}, \mathscr{F}^2_{\cdot}) \in \omega_{CFF} CFF^b(\mathscr{D}_X)^{\omega^1}$ and $x : (\mathscr{M}^{\cdot}, \mathscr{F}^2_{\cdot}) \to (\mathscr{N}^{\cdot}, \mathscr{F}^2_{\cdot}) \in \omega_{CF} CF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\nu}$, there exists $(\mathscr{M}'^{\cdot}, \mathscr{F}^1_{\cdot}, \mathscr{F}^2_{\cdot}) \in$

$\omega_{CFF}CFF^b(\mathscr{D}_X)^{\omega^1}$, a morphism $a:(\mathscr{M}^\cdot,\mathscr{F}^1_\cdot,\mathscr{F}^2_\cdot)\to(\mathscr{M}'^\cdot,\mathscr{F}^1_\cdot,\mathscr{F}^2_\cdot)$, and a weak equivalence $\tilde{x}:(\mathscr{M}'^\cdot,\mathscr{F}^2_\cdot)\to(\mathscr{N}^\cdot,\mathscr{F}^2_\cdot)$ in $\omega_{CF}CF^b_{\text{perf}}(\mathscr{D}_X)^\nu$ such that $x=\tilde{x}\circ F^2(a)$. Note that, for each $i$, there exists a $k_i$ such that $\mathscr{F}^1_{\mathscr{M}^i,j}=0$ for all $j\le k_i$. Furthermore, since $\mathscr{M}$ is a bounded complex, we can find a $k$ which works for all $i$. Fix such a $k$. Let $\mathscr{M}'=\mathscr{N}$ and $\mathscr{F}^2_{\mathscr{M}'}=\mathscr{F}^2_{\mathscr{N}}$. Let $\mathscr{F}^1_{\mathscr{N}}$ be the filtration such that $\mathscr{F}^1_{\mathscr{N},i}=0$ if $i\le k$ and $\mathscr{F}^1_{\mathscr{N},i}=\mathscr{N}^\cdot$ if $i>k$. Since $\mathscr{N}$ is acyclic, $gr_{\mathscr{F}_1}(\mathscr{N})$ is also acyclic. It follows that $(\mathscr{M}'^\cdot,\mathscr{F}^1_\cdot,\mathscr{F}^2_\cdot)$ with $\mathscr{F}^1_{\mathscr{M}'}=\mathscr{F}^1_{\mathscr{N}}$ is an object of $\omega_{CFF}CFF^b(\mathscr{D}_X)^{\omega^1}$. Furthermore, we have $x(\mathscr{F}^1_{\mathscr{M}})\subset\mathscr{F}^1_{\mathscr{M}'}$. This gives the required mapping $a$ and $\tilde{x}=\text{Id}$. □

Let $G:\omega^1CFF^b_{\text{perf}}(\mathscr{D}_X)\to\omega_{CF}CF^b_{\text{perf}}(\mathscr{D}_X)$ denote the functor which sends $(\mathscr{M},\mathscr{F}^1_\cdot,\mathscr{F}^2_\cdot)$ to $(\mathscr{M},\mathscr{F}^1_\cdot)$. Then $G$ is a complicial exact functor and induces a morphism of spectra: $G:K(\omega^1CFF^b_{\text{perf}}(\mathscr{D}_X))\to KF(\mathscr{D}_X)$.

**Proposition 3.1.9.** *$G$ is a weak equivalence of spectra.*

*Proof.* According to Theorem 1.9.8 of ([TT90]) it's enough to show that the resulting map on the associated triangulated categories is an equivalence of categories. Let $\tilde{D}FF^b_{\text{perf}}(\mathscr{D}_X)$ and $DF^b_{\text{perf}}(\mathscr{D}_X)$ denote the corresponding triangulated categories. First, note that essential surjectivity is clear. Given perfect filtered $(\mathscr{M},\mathscr{F})$, we can lift it to $(\mathscr{M},\mathscr{F},\mathscr{F})$. Therefore, we must show that the resulting functor on homotopy categories is fully-faithful. To show that the functor is full, we must show that the following map is surjective:

$$\text{Hom}_{\tilde{D}FF^b_{\text{perf}}(\mathscr{D}_X)}((\mathscr{M},\mathscr{F}^1,\mathscr{F}^2),(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2))\to\text{Hom}_{DF^b_{\text{perf}}(\mathscr{D}_X)}((\mathscr{M},\mathscr{F}^1),(\mathscr{N},\mathscr{F}^1))$$

By the following lemma, any perfect filtered complex is quasi-isomorphic to a strictly perfect complex. In particular, we can assume $(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2)$ and $(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2)$ are strictly perfect. Now, by part (3) of the following lemma:

$$\text{Hom}_{DF^b_{\text{perf}}(\mathscr{D}_X)}((\mathscr{M},\mathscr{F}^1),(\mathscr{N},\mathscr{F}^1))=\text{Hom}_{KF^b_{\text{stp}}(\mathscr{D}_X)/KF^{b,\emptyset}_{\text{stp}}(\mathscr{D}_X)}((\mathscr{M},\mathscr{F}^1)),(\mathscr{N},\mathscr{F}^1))$$

Now, morphisms $\phi$ on the right can be represented by a diagram

$$(\mathscr{M},\mathscr{F}^1)\xleftarrow[f]{}(\mathscr{P},\mathscr{F}^1)\xrightarrow{g}(\mathscr{N},\mathscr{F}^1)$$

where $f$ is a quasi-isomorphism and $(\mathscr{P},\mathscr{F}^1)$ is strictly perfect. In particular, all the $\mathscr{F}^i$ are good filtrations. It follows that we can lift $f$ to a morphism $(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2[k])\leftarrow(\mathscr{P},\mathscr{F}^1,\mathscr{F}^1)$. To see this note that $f(\mathscr{F}^1_{\mathscr{P}})$ is a good filtration on the image and so is the restriction of $\mathscr{F}^2_{\mathscr{M}}$. In particular, for each $i$, there exists $k_i$, such that $f(\mathscr{F}^1_{\mathscr{P}^i,j})\subset\mathscr{F}^2_{\mathscr{M}^i,k_i+j}$. Since our complexes are bounded we can find one $k$ which works for all $i$. Then $f(\mathscr{F}^1_{\mathscr{P}})\subset\mathscr{F}^2_{\mathscr{M}}[k]$. A similar statement holds for $g$. Now if $k\ge 0$, then the identity induces a morphism $(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2)\to(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2[k])$. If $k<0$, the identity induces a morphism $(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2[k])\to(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2)$.

In either case, both morphisms are weak-equivalences in $\omega^1 CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ and, therefore, isomorphisms in $\tilde{D}FF^b_{\mathrm{perf}}(\mathscr{D}_X)$. In particular, we can lift $\phi$ to a morphism in $\tilde{D}FF^b_{\mathrm{perf}}(\mathscr{D}_X)$. To show that the functor is faithful, we must show that the following is injective:

$$\mathrm{Hom}_{\tilde{D}FF^b_{\mathrm{perf}}(\mathscr{D}_X)}((\mathscr{M},\mathscr{F}^1,\mathscr{F}^2),(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2)) \to \mathrm{Hom}_{DF^b_{\mathrm{perf}}(\mathscr{D}_X)}((\mathscr{M},\mathscr{F}^1),(\mathscr{N},\mathscr{F}^1))$$

Again, we may assume that both $(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2)$ and $(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2)$ are strictly perfect. It is enough to show that if $f:(\mathscr{M},\mathscr{F}^1,\mathscr{F}^2)\to(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2)$ is a morphism such that $G(f)$ is null homotopic then there is a weak equivalence $a:(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2)\to(\mathscr{N}',\mathscr{F}^1,\mathscr{F}^2)$ such that the composition $a\circ f$ is null homotopic. Let $h$ denote the homotopy of $G(f)$ to zero. Although $h$ is not a morphism of complexes, the above argument still applies to show that there is a $k$ such that $h(\mathscr{F}^2_{\mathscr{M}^n})\subset\mathscr{F}^2_{\mathscr{N}^{n-1}}[k]$ for all $n$. If $k\leq 0$, the $h$ lifts to a homotopy of $f$. On the other hand, if $k>0$ the identity gives a weak equivalence $a:(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2)\to(\mathscr{N},\mathscr{F}^1,\mathscr{F}^2[k])$. Then $h$ give a homotopy of $a\circ f$ with zero.

□

**Lemma 3.1.10.** (1) *Let $\mathscr{M}$ be a bounded perfect complex of $\mathscr{D}_X$-modules. Then there is a bounded strictly perfect complex of $\mathscr{D}_X$-modules $\mathscr{P}$ and a quasi-isomorphism of $\mathscr{P}\to\mathscr{M}$.*

(2) *Let $(\mathscr{M},\mathscr{F})$ be a bounded perfect complex of filtered $\mathscr{D}_X$-modules. Then there exists a bounded strictly perfect complex $(\mathscr{P},\mathscr{F})$ and a quasi-isomorphism $(\mathscr{P},\mathscr{F})\to(\mathscr{M},\mathscr{F})$.*

(3) *Let $KF^b_{stp}(\mathscr{D}_X)$ denote the homotopy category of bounded strictly perfect filtered complexes of $\mathscr{D}_X$-modules, and $KF^{b,\emptyset}_{stp}(\mathscr{D}_X)$ the full (thick) subcategory of acyclic complexes. Then the canonical functor $KF^b_{\mathrm{stp}}(\mathscr{D}_X)/KF^{b,}_{\mathrm{stp}}(\mathscr{D}_X)\to DF^b_{\mathrm{perf}}(\mathscr{D}_X)$ is an equivalence of categories.*

(4) *Similar statements hold for doubly filtered perfect complexes of $\mathscr{D}_X$-modules.*

*Proof.* (1) It is a standard result that any $\mathscr{D}_X$-module $\mathscr{M}$ has a finite resolution by locally projective $\mathscr{D}_X$-modules ([HTT08], 1.4.20). Furthermore, if $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module then we can find a resolution by finite rank locally projectives. Now one can extend this result to a perfect complex of $\mathscr{D}_X$-modules just as in the case of usual perfect complexes of $\mathscr{O}_X$-modules ([[SG71], Expose 2). For example, one can apply ([TT90], Lemma 1.9.5) where $\mathscr{A},\mathscr{D},\mathscr{C}$ in loc. cit. are the categories of $\mathscr{D}_X$-modules, locally projective $\mathscr{D}_X$-modules, and perfect complexes. The above statement guarantees that hypothesis (1.9.5.1) of ([TT90], Lemma 1.9.5) holds. The other hypotheses required to apply the lemma hold trivially since in our care $\mathscr{D}\to\mathscr{A}$ is fully faithful. In particular we can apply the conclusion of ([TT90], 1.9.5) exactly as in the proof of ([TT90], 2.3.1) where the analogous result for $\mathscr{O}_X$-modules is proved.
(2) Suppose every coherent filtered $\mathscr{D}_X$-module $(\mathscr{M},\mathscr{F})$ has a finite resolution by locally projective filtered $\mathscr{D}_X$-modules. Then we can again proceed as in (1) and apply ([TT90], 1.9.5), where $\mathscr{A}$ will denote the ambient abelian category of the exact category of quasi-coherent filtered $\mathscr{D}_X$-modules. The existence of filtered resolutions as above, and the explicit description

of $\mathscr{A}$ given in ([Lau83]) show that the hypothesis of loc. cit. is satisfied. So suppose $(\mathscr{M},\mathscr{F})$ is a coherent filtered $\mathscr{D}_X$-module. Then $\mathscr{F}$ is a good filtration. In particular, there exists an $i_0$ such that for all $j$ and $i \geq i_0$: $\mathscr{D}_j\mathscr{F}_i(\mathscr{M}) = \mathscr{F}_{i+j}(\mathscr{M})$. Since $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, one can find a coherent $\mathscr{O}_X$-submodule $\mathscr{M}_0$ which generates $\mathscr{M}$ as a $\mathscr{D}_X$-module ([HTT08], 1.4.17). Furthermore, since $\mathscr{F}_i(\mathscr{M})$ are all coherent $\mathscr{O}_X$-modules, we can assume (by making $\mathscr{M}_0$ larger) $\mathscr{F}_{i_0}(\mathscr{M}) \subset \mathscr{M}_0$. Consider the filtered $\mathscr{O}_X$-module $(\mathscr{M}_0,\mathscr{F})$. The filtration is the one induced from $\mathscr{M}$; in particular, it is finite since $\mathscr{M}_0$ is coherent. By construction, one has an epimorphism of filtered $\mathscr{D}_X$-modules: $\mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{M}_0,\mathscr{F}) \to (\mathscr{M},\mathscr{F})$. The filtration on the left term is the tensor product filtration ($\mathscr{D}_X$ with the canonical filtration). Now, we can also find a surjection $\mathscr{P} \to \mathscr{M}_0$ where $\mathscr{P}$ is a locally free $\mathscr{O}_X$ module. Pulling back the filtration from $\mathscr{M}_0$ gives a surjection $(\mathscr{P},\mathscr{F}) \to (\mathscr{M}_0,\mathscr{F})$ of filtered $\mathscr{O}_X$-modules. Furthermore, we may assume that the $\mathscr{P}$ is filtered by locally free $\mathscr{O}$-submodules. We prove this by induction on the length $n$ of the filtration on $\mathscr{P}$. By induction and after possibly re-indexing or shifting, we can assume we have a filtered $\mathscr{O}_X$-module

$$0 \subsetneq \mathscr{F}_{\mathscr{P},1} \subset \cdots \subset \mathscr{F}_{\mathscr{P},n+1} \subsetneq \mathscr{P}$$

where $\mathscr{F}_{\mathscr{P},i}$ is locally free for $1 \leq i \leq n$ and $\mathscr{P}$ is locally free. Let $\mathscr{F}' \to \mathscr{F}_{\mathscr{P},n+1}$ denote a surjection from a locally free $\mathscr{O}_X$-module. Pulling back the filtration on $\mathscr{F}_{\mathscr{P},n+1}$ gives a filtration on $\mathscr{F}'$. This is of length $n$ so that again we may assume by induction that these are locally free. Now let $\mathscr{P}' = \mathscr{F}' \oplus \mathscr{P}$. This has a natural filtration of length $n+1$ by locally free subsheaves, and there is a natural surjection of filtered $\mathscr{O}_X$-modules $(\mathscr{P}',\mathscr{F}) \to (\mathscr{P},\mathscr{F})$. So we assume now that we have a surjection $(\mathscr{P},\mathscr{F}) \to (\mathscr{M}_0,\mathscr{F})$, where the filtration on $\mathscr{P}$ is by locally free $\mathscr{O}_X$-modules. Then the resulting $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{P},\mathscr{F})$ is a locally free filtered $\mathscr{D}_X$-modules. Locally the filtration gives a flag (not necessarily complete) of $\mathscr{O}_X^r$ where $r$ is the rank. The corresponding filtration on the induced $\mathscr{D}_X$-module is locally just shifts of the canonical filtration. Finally, the composition $\mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{P},\mathscr{F}) \to \mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{M}_0,\mathscr{F}) \to (\mathscr{M},\mathscr{F})$ has the required property. Continuing in this way gives a resolution by filtered modules

$$0 \to (\mathscr{L}_{2d},\mathscr{F}) \to \cdots \to (\mathscr{L}_0,\mathscr{F}) \to (\mathscr{M},\mathscr{F}) \to 0$$

where $(\mathscr{L}_i,\mathscr{F})$ is a locally free filtered $\mathscr{D}_X$-module for $i < 2d$ and $d = dim(X)$. But, over an affine open $U$, the global dimension (as a filtered ring) of $\mathscr{D}_X(U)$ is less than or equal to $2d$. It follows that $(\mathscr{L}_{2d},\mathscr{F})$ is locally projective. This gives the desired resolution.
(3) This is a direct consequence of (2).
(4) The proof is the same as that for (2). If $\mathscr{F}^1$ and $\mathscr{F}^2$ are good filtrations on $\mathscr{M}$, $i_0^1$ and $i_0^2$ as in the proof of (2), then choose $\mathscr{M}_0$ such that it contains both $\mathscr{F}^1_{i_0^1}(\mathscr{M})$ and $\mathscr{F}^2_{i_0^2}(\mathscr{M})$ □

Let $\Phi : \nu CF^b_{\mathrm{perf}}(\mathscr{D}_X) \to \omega C^b_{\mathrm{perf}}(\mathscr{D}_X)$ be the forgetful functor. Since $\Phi$ is a complicial exact functor, we have an induced morphism $\Phi : K(\nu CF^b_{\mathrm{perf}}(\mathscr{D}_X)) \to K(\mathscr{D}_X)$ of spectra.

**Lemma 3.1.11.** *If $\mathscr{P}$ is a strictly perfect complex of $\mathscr{D}_X$-modules, then it can be lifted to a strictly perfect complex of filtered $\mathscr{D}_X$-modules.*

*Proof.* We may assume that the $\mathscr{P}^k = 0$ for $k \notin [0,n]$. We will proceed by induction on $n$. If $n = 0$, then the claim follows from the standard fact that any coherent $\mathscr{D}_X$-module admits a global good filtration. By induction, we can assume that the complex $[\cdots \to 0 \to \mathscr{P}^0 \to \cdots \mathscr{P}^{n-1} \to 0 \to \cdots]$ lifts to a perfect complex $[\cdots \to 0 \to (\mathscr{P}^0, \mathscr{F}_{\mathscr{P}^0}) \to \cdots (\mathscr{P}^{n-1}, \mathscr{F}_{\mathscr{P}^{n-1}}) \to 0 \to \cdots]$. Choose a good filtration $\mathscr{F}'_{\mathscr{P}^n}$ on $\mathscr{P}^n$, since $\mathscr{F}_{\mathscr{P}^{n-1}}$ and $\mathscr{F}'_{\mathscr{P}^n}$ are good filtrations, there exists an integer $l$ such that for all $r$, $d_{n-1}(\mathscr{F}_{\mathscr{P}^{n-1},r}) \subset \mathscr{F}'_{\mathscr{P}^n,r+l}$. Replacing $\mathscr{F}'_{\mathscr{P}^n}$ by $\mathscr{F}'_{\mathscr{P}^n}[l]$ gives the desired lift. □

**Proposition 3.1.12.** $\Phi$ *is a weak equivalence of spectra.*

*Proof.* According to Theorem 1.9.8 of ([TT90]) it's enough to show that the resulting map on the associated triangulated categories is an equivalence of categories; let $D_{\text{perf}}(\mathscr{D}_X)$ and $\tilde{D}F_{\text{perf}}(\mathscr{D}_X)$ denote these triangulated categories. First, a morphism $f$ is in $v$ if and only if $\Phi(f)$ is a weak equivalence. It follows from the previous lemma that if $\mathscr{P}$ is a strictly perfect complex of $\mathscr{D}$-modules then we can lift it to an object of $CF^b_{\text{perf}}(\mathscr{D}_X)$. Since any $\mathscr{M} \in C^b_{\text{perf}}(\mathscr{D}_X)$ is quasi-isomorphic to a strictly perfect complex, it follows that the functor is essentially surjective. We first show that the functor is full. Let $(\mathscr{M}, \mathscr{F})$ and $(\mathscr{N}, \mathscr{F})$ be two objects of $CF^b_{\text{perf}}(\mathscr{D}_X)$ and $x : \mathscr{M} \to \mathscr{N}$ a morphism in $D_{\text{perf}}(\mathscr{D}_X)$. Again, we can assume that $(\mathscr{M}, \mathscr{F})$ and $(\mathscr{N}, \mathscr{F})$ are strictly perfect. Now we can proceed as in the proof of Proposition 3.9. An element $x \in \text{Hom}_{D_{\text{perf}}(\mathscr{D}_X)}(\mathscr{M}, \mathscr{N})$ can be represented by a diagram:

$$\mathscr{M} \xleftarrow[f]{} \mathscr{P} \xrightarrow{g} \mathscr{N}$$

where $\mathscr{P}$ is strictly perfect and $f$ is a weak equivalence. Now lift $\mathscr{P}$ to a strictly perfect filtered complex $(\mathscr{P}, \mathscr{F})$. Then we can lift the above diagram to a diagram:

$$(\mathscr{M}, \mathscr{F}[k]) \xleftarrow[f]{} (\mathscr{P}, \mathscr{F}) \xrightarrow{g} (\mathscr{N}, \mathscr{F}[l])$$

for some $k$ and $l$. On the other hand, $(\mathscr{M}, \mathscr{F}[k])$ is isomorphic to $(\mathscr{M}, \mathscr{F})$ in $\tilde{D}F_{\text{perf}}(\mathscr{D}_X)$ and similarly for $(\mathscr{N}, \mathscr{F}[l])$. This shows that the functor on homotopy categories is full. To prove faithfulness, one can proceed as in the proof of 3.9. Again it is enough to show that if $f : (\mathscr{M}, \mathscr{F}) \to (\mathscr{N}, \mathscr{F})$ is a morphism of complexes such that $\Phi(f)$ is null homotopic then there is weak equivalence $a : (\mathscr{N}, \mathscr{F}) \to (\mathscr{N}', \mathscr{F})$ such that $a \circ f$ is null homotopic. As before, we may assume that everything is strictly perfect. If $h$ is a homotopy of $\Phi(f)$ with zero, then the is a $k$ such that $h(\mathscr{F}_{\mathscr{M}^n}) \subset \mathscr{F}_{\mathscr{N}^{n-1}}[k]$ for all n. If $k \leq 0$, then $h$ lifts to a null homotopy of $f$. If $k > 0$, we can compose $f$ with the weak equivalence induced by the identity $(\mathscr{N}, \mathscr{F}) \to (\mathscr{N}, \mathscr{F}[k])$. □

*Proof.* (Theorem 3.2) From Corollary 3.5 and Corollary 3.7 we have the following commutative diagram where the rows are homotopy cofiber sequences:

$$\begin{array}{ccccc} K(\omega_{CFF}CFF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\omega^1_{CFF}}) & \longrightarrow & KFF(\mathscr{D}_X) & \longrightarrow & K(\omega^1_{CFF}CFF^b_{\mathrm{perf}}(\mathscr{D}_X)) \\ \downarrow & & \downarrow & & \downarrow \\ K(\omega_{CF}CF^b_{\mathrm{perf}}(\mathscr{D}_X)^{\nu}) & \longrightarrow & KF(\mathscr{D}_X) & \longrightarrow & K(\nu CF^b_{\mathrm{perf}}(\mathscr{D}_X)). \end{array}$$

It follows that the square on the right is a homotopy pushout square. This gives the following commutative diagram where the left square is a homotopy pushout square:

$$\begin{array}{ccccc} KFF(\mathscr{D}_X) & \longrightarrow & K(\omega^1_{CFF}CFF^b_{\mathrm{perf}}(\mathscr{D}_X)) & \longrightarrow & KF(\mathscr{D}_X) \\ \downarrow & & \downarrow & & \downarrow \\ KF(\mathscr{D}_X) & \longrightarrow & K(\nu CF^b_{\mathrm{perf}}(\mathscr{D}_X)) & \longrightarrow & K(\mathscr{D}_X). \end{array}$$

Since the two horizontal arrows on the right are weak equivalences, the right square is also a homotopy pushout, and It follows that the outside square is also a homotopy pushout. On the other hand, the composition of the two top arrows is just the map $F^1$, and the bottom row is $\omega$. Since the left vertical is $F^2$, the result follows. □

We will now construct a homotopy commutative square:

$$\begin{array}{ccc} KFF(\mathscr{D}_X) & \longrightarrow & KF(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF(\mathscr{D}_X) & \longrightarrow & K(\mathrm{gr}(\mathscr{D}_X)). \end{array}$$

The universal property of the homotopy pushout will then give a morphism $K(\mathscr{D}_X) \to K(\mathrm{gr}(\mathscr{D}_X))$. Furthermore, there is a natural map from $K(\mathrm{gr}(\mathscr{D}_X))$ to $K(T^*X)$. Finally, the above methods will then give the desired microlocal version of this morphism: $K_S(\mathscr{D}_X) \to K_S(T^*X)$. We begin with some preliminaries on $K$-theory.

Taking complexes with coherent cohomology instead of perfect complexes, we can also define the categories $CFF^b_{\mathrm{coh}}(\mathscr{D}_X)$ and $CF^b_{\mathrm{coh}}(\mathscr{D}_X)$. We then have the corresponding $G$-theories, $GFF(\mathscr{D}_X)$ , $GF(\mathscr{D}_X)$ and $G(\mathscr{D}_X)$. We could also consider the categories of (locally) projective filtered $\mathscr{D}_X$-modules $PFF(\mathscr{D}_X)$ and $PF(\mathscr{D}_X)$, and similarly the categories of coherent filtered $\mathscr{D}_X$-modules $MFF_{\mathrm{coh}}(\mathscr{D}_X)$ and $MF_{\mathrm{coh}}(\mathscr{D}_X)$. Let $KFF'(\mathscr{D}_X) = K(PFF(\mathscr{D}_X))$ and similarly for $KF'(\mathscr{D}_X)$ and $K'(\mathscr{D}_X)$. Also, let $GFF'(\mathscr{D}_X) = K(MFF_{\mathrm{coh}}(\mathscr{D}_X))$ and similarly for $GF'(\mathscr{D}_X)$ and $G'(\mathscr{D}_X)$.

Since $X$ is smooth, the canonical morphisms $K'(\mathscr{D}_X) \to G'(\mathscr{D}_X)$ is a weak equivalence. This can be proven as in the case of usual K-theory of schemes ([TT90], 3.2.1). Any exact category $\mathscr{E}$ can be considered a complicial bi-Waldhausen category ([TT90], 1.2.12). One fixes an ambient abelian category $\mathscr{A}$ for $\mathscr{E}$ as before, and considers the complexes $C^\cdot$ in $C(\mathscr{A})$ such that $C^0 \in \mathscr{E}$

and $C^i = 0$ for all other $i$. The cofibrations are level-wise admissible monomorphisms and the weak equivalences are quasi-isomorphisms (for such complexes they are just isomorphisms in $\mathscr{E}$). Now we have already seen that $K(\mathscr{E})$ is the same as $K(C^b(\mathscr{E}))$ (section 2.2). Therefore, it is enough to show that $K(C^b(P(\mathscr{D}_X))) \to K(C^b(Coh(\mathscr{D}_X)))$ is a weak equivalence, where $P(\mathscr{D}_X)$ and $Coh(\mathscr{D}_X)$ are the categories of projective and coherent $\mathscr{D}_X$-modules respectively. This is an equivalence if the corresponding map on homotopy categories is an equivalence. Since $P(\mathscr{D}_X)$ is a full subcategory of $Coh(\mathscr{D}_X)$, by ([TT90], 1.9.7) it is enough to show that the functor on homotopy categories is essentially surjective. But, this can be proved as in Lemma 3.10. In particular, any bounded complex of coherent $\mathscr{D}_X$-modules is quasi-isomorphic to a bounded complex of projective $\mathscr{D}_X$-modules. A similar argument shows that $KFF^{'}(\mathscr{D}_X) \to GFF'(\mathscr{D}_X)$ and $KF'(\mathscr{D}_X) \to GF'(\mathscr{D}_X)$ are weak equivalences.
Furthermore, the canonical map $K'(\mathscr{D}_X) \to K(\mathscr{D}_X)$ is also a weak equivalence. Again, it is enough to show that the map on homotopy categories is a weak equivalence. It is clearly fully-faithful, and essential surjectivity follows from Lemma 3.10. A similar argument shows that the canonical maps $KFF'(\mathscr{D}_X) \to KFF(\mathscr{D}_X)$ and $KF'(\mathscr{D}_X) \to KF(\mathscr{D}_X)$ are weak equivalences. This result will be used in the proof of the Theorem 3.13 below. Note, it also true that $GFF'(\mathscr{D}_X) \to GFF(\mathscr{D}_X)$ is a weak equivalence and similarly for $GF'(D_X)$ and $G'(\mathscr{D}_X)$, but we will not use this below.
Recall that $\mathscr{D}_X$ comes with a canonical good filtration with associated graded denoted by $\mathrm{gr}(\mathscr{D}_X)$ . Let $C^b_{\mathrm{qcoh}}(\mathrm{gr}(\mathscr{D}_X))$ denote the category of bounded chain complexes of quasi-coherent $\mathrm{gr}(\mathscr{D}_X)$-modules (see section 2.2 for a description of the Waldhausen category structure). We will say that a complex in $C^b_{\mathrm{qcoh}}(\mathrm{gr}(\mathscr{D}_X))$ is perfect if it is locally quasi-isomorphic to a bounded complex all of whose components are projective of finite type. Let $C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$ denote the category of perfect complexes of $\mathrm{gr}(\mathscr{D}_X)$-modules and $K(\mathrm{gr}(\mathscr{D}_X)) = K(C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X)))$. Recall (section 2.2), we take the Waldhausen structure where the weak equivalences are quasi-isomorphisms and cofibrations are degree-wise admissible monomorphisms. We have induced complicial exact functors $gr : CF^b_{\mathrm{perf}}(\mathscr{D}_X) \to C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$ and $gr^1, gr^2 : CFF^b_{\mathrm{perf}}(\mathscr{D}_X) \to C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$ of complicial bi-Waldhausen categories. In particular, we have induced homotopy morphisms of the corresponding $K$-theory spectra. As in the previous paragraph, we have the $K$-theory spectrum $K'(\mathrm{gr}(\mathscr{D}_X))$ and a weak equivalence $K'(\mathrm{gr}(\mathscr{D}_X)) \to K(\mathrm{gr}(\mathscr{D}_X))$ ([TT90], 3.2.1).
Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Suppose $(\mathscr{F}^1, \mathscr{F}^2)$ are two good filtrations on $\mathscr{M}$. Then there exist integers $r$ and $s$ such that $\mathscr{F}^1_k \subset \mathscr{F}^2_{k+r} \subset \mathscr{F}^1_{k+s}$. We will say that $\mathscr{F}^1$ and $\mathscr{F}^2$ are $(r,s)$-*adjacent* if there exist $r$ and $s$ as above. If we can take $r = 0$ and $s = 1$, then we will say that $\mathscr{F}^1$ and $\mathscr{F}^2$ are *adjacent*. Let $PFF^{(r,s)}(\mathscr{D}_X)$ denote the category of doubly filtered projective modules such that the corresponding filtrations are $(r,s)$-adjacent. Since every exact sequence of doubly filtered projective modules splits locally (with filtration), this is a full exact subcategory of $PFF(\mathscr{D}_X)$. Denote the corresponding $K$-theory spectrum by $KFF^{(r,s)}(\mathscr{D}_X)$. Let $\mathscr{I}$ denote the set $\mathbb{Z} \times \mathbb{Z}$ with ordering $(r,s) \leq (r',s')$ iff $r \leq r'$, $s \leq s'$, and $r' - r \leq s' - s$. For each pair $(r,s) \leq (r',s')$ we have the embedding $PFF^{(r,s)}(\mathscr{D}_X) \subset PFF^{(r',s')}(\mathscr{D}_X)$. The colimit

over the directed set $I$ of the $PFF^{(r,s)}(\mathscr{D}_X)$ is $PFF(\mathscr{D}_X)$. It follows that the canonical morphism of spectra $\operatorname{colim}_{\mathscr{I}} KFF^{(r,s)}(\mathscr{D}_X) \to KFF'(\mathscr{D}_X)$ is a weak equivalence. [2]

**Theorem 3.1.13.** *One has a canonical homotopy commutative diagram:*

$$\begin{array}{ccc} KFF(\mathscr{D}_X) & \longrightarrow & KF(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF(\mathscr{D}_X) & \longrightarrow & K(\mathrm{gr}(\mathscr{D}_X)). \end{array}$$

*In particular, $gr^1$ and $gr^2$ are canonically identified as homotopy morphisms.*

*Proof.* It is enough to prove that the following diagram is commutative:

$$\begin{array}{ccc} KFF'(\mathscr{D}_X) & \longrightarrow & KF'(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF'(\mathscr{D}_X) & \longrightarrow & G'(\mathrm{gr}(\mathscr{D}_X)). \end{array}$$

Recall that $G'(\mathrm{gr}(\mathscr{D}_X))$ is the $K$-theory of the category of coherent $\mathrm{gr}(\mathscr{D}_X)$-modules; there is a canonical weak equivalence $G'(\mathrm{gr}(\mathscr{D}_X)) \to K(\mathrm{gr}(\mathscr{D}_X))$. We must construct a homotopy between $gr_{F^1} : KFF'(\mathscr{D}_X) \to G'(\mathrm{gr}(\mathscr{D}_X))$ and $gr_{F^2} : KFF'(\mathscr{D}_X) \to G'(\mathrm{gr}(\mathscr{D}_X))$. By the remarks of the previous paragraph, it is enough to construct homotopies between $gr_{F^1} : KFF^{(r,s)}(\mathscr{D}_X) \to G'(\mathrm{gr}(\mathscr{D}_X))$ and $gr_{F^2} : KFF^{(r,s)}(\mathscr{D}_X) \to G'(\mathrm{gr}(\mathscr{D}_X))$ which are compatible under the inclusion mappings for varying $(r,s)$. If $(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ is an object of $PFF^{(r,s)}(\mathscr{D}_X)$, let $H(n) = \mathscr{F}^1 + \mathscr{F}^2[n]$, which is a good filtration on $\mathscr{M}$. Let $gr_{H(n)} : KFF^{(r,s)}(\mathscr{D}_X) \to G'(\mathrm{gr}(\mathscr{D}_X))$ be the morphism induced by sending $(\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)$ to $gr_{H(n)}(\mathscr{M})$. If $n = r$, then $H(n) = F^2[n]$. It follows that $gr_{H(n)}$ is canonically homotopic to $gr_{F^2}[n]$. On the other hand, $gr_{F^2[n]} = gr_{F^2}$, so for $n = r$, $gr_{H(n)}$ is homotopic to $gr_{F^2}$. If $n = r - s$, then $H(n) = F^1$; therefore, it suffices to construct a homotopy between $gr_{H(n)}$ and $gr_{H(n+1)}$. Note that $H(n)$ and $H(n+1)$ are adjacent filtrations. It follows that we have exact sequences of coherent $\mathrm{gr}(\mathscr{D}_X)$-modules:

$$0 \to \oplus H(n+1)_k / H(n)_k \to gr_{H(n)}(\mathscr{M}) \to \oplus H(n)_k / H(n+1)_{k-1} \to 0$$
$$0 \to \oplus H(n)_k / H(n+1)_{k-1} \to gr_{H(n+1)}(\mathscr{M}) \to \oplus H(n+1)_k / H(n)_k \to 0.$$

If we let $f(n)((\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)) = H(n+1)_k / H(n)_k$ and $g(n)((\mathscr{M}, \mathscr{F}^1, \mathscr{F}^2)) = H(n)_k / H(n+1)_{k-1}$, then the above exact sequences give us with canonical homotopies $gr_{H(n)} \cong f(n) + g(n) \cong gr_{H(n+1)}$. Combining everything gives the required homotopy between $gr_{F^1}$ and $gr_{F^2}$ on $KFF^{(r,s)}(\mathscr{D}_X)$. It is clear that the construction is compatible for varying $(r,s)$, and the result follows.

□

[2] In the case at hand, the transition maps are all inclusions and therefore the colimit is just the union over all $(r,s)$ in $PFF(\mathscr{D}_X)$. In particular, it is just $PFF(\mathscr{D}_X)$. The weak equivalence of the resulting spectra follows as in ([TT90], see proof of 3.20.1).

If $Y$ is a scheme, we let $K(Y)$ denote the $K$-theory spectrum of perfect complexes on $X$. Let $\pi : T^*X \to X$ denote the canonical projection map; then we have a canonical isomorphism $\pi_*\mathscr{O}_{T^*X} \cong \mathrm{gr}(\mathscr{D}_X)$. Since $\pi$ is affine, we have an equivalence of categories: $\pi_* : C^b_{\mathrm{perf}}(\mathscr{O}_{T^*X}) \to C^b_{\mathrm{perf}}(\mathrm{gr}(\mathscr{D}_X))$. It follows that we have an equivalence of $K$-theory spectra: $\pi_* : K(T^*X) \to K(\mathrm{gr}(\mathscr{D}_X))$.

**Corollary 3.1.14.** *There is a canonical morphism of spectra* $gr : K(\mathscr{D}_X) \to K(T^*X)$.

*Proof.* By the universal property of homotopy pushout and Theorem 3.13, we have a canonical morphism $K(\mathscr{D}_X) \to K(\mathrm{gr}(\mathscr{D}_X))$. Composing this with $\pi_*^{-1} : K(\mathrm{gr}(\mathscr{D}_X)) \to K(T^*X)$ gives the required morphism. ☐

We will now construct a microlocal version of the morphism in the previous corollary. For a perfect complex $\mathscr{M} \in C^b_{\mathrm{perf}}(\mathscr{D}_X)$, let $SS(\mathscr{M}) \subset T^*X$ denote the singular support of $\mathscr{M}$ in the cotangent bundle. The singular support of a bounded complex of $\mathscr{D}_X$ modules is defined to be the union of the singular supports of the corresponding homology sheaves. Given $S \subset T^*X$, let $CFF^b_{\mathrm{perf},S}(\mathscr{D}_X) \subset CFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ denote the full subcategory of complexes such that the underlying complex has singular support contained in $S$. These are again complicial bi-Walhausen categories with the induced structure. Similar statements apply to $CF^b_{\mathrm{perf},S}(\mathscr{D}_X)$ and $C^b_{\mathrm{perf},S}(\mathscr{D}_X)$. Let $KFF_S(\mathscr{D}_X)$, $KF_S(\mathscr{D}_X)$ and $K_S(\mathscr{D}_X)$ denote the corresponding $K$-theory spectra; similarly, let $K_S(\mathrm{gr}(\mathscr{D}_X))$ denote the $K$-theory spectrum of perfect complexes such that the corresponding complex on $T^*X$ has support in $S$. The support of a perfect complex on $T^*X$ is defined to be the union of the supports of the corresponding homology sheaves.

**Theorem 3.1.15.** *(1): The following diagram is a homotopy pushout:*

$$\begin{array}{ccc} KFF_S(\mathscr{D}_X) & \longrightarrow & KF_S(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF_S(\mathscr{D}_X) & \longrightarrow & K_S(\mathscr{D}_X). \end{array}$$

*(2): There following diagram is commutative:*

$$\begin{array}{ccc} KFF_S(\mathscr{D}_X) & \longrightarrow & KF_S(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF_S(\mathscr{D}_X) & \longrightarrow & K_S(\mathrm{gr}(\mathscr{D}_X)). \end{array}$$

*Proof.* The proof proceeds exactly as the proofs of Theorems 3.2 and 3.13. One needs only to note that the singular support is independent of the choice of good filtrations and only depends on the underlying perfect complexes. ☐

**Corollary 3.1.16.** *There exists a canonical map* $gr_S : K_S(\mathscr{D}_X) \to K_S(T^*X)$.

The following is a naturality property for the microlocalization as $S$ varies.

**Proposition 3.1.17.** *If $S \subset S'$ then one has a homotopy commutative diagram:*

$$\begin{array}{ccc} K_S(\mathscr{D}_X) & \longrightarrow & K_S(T^*X) \\ \downarrow & & \downarrow \\ K_{S'}(\mathscr{D}_X) & \longrightarrow & K_{S'}(T^*X). \end{array}$$

*Proof.* Let $g_1$ denote the composition $K_S(\mathscr{D}_X) \to K_S(\mathrm{gr}(\mathscr{D}_X)) \to K_{S'}(\mathrm{gr}(\mathscr{D}_X))$ and $g_2$ denote the composition $K_S(\mathscr{D}_X) \to K_{S'}(\mathscr{D}_X)) \to K_{S'}(\mathrm{gr}(\mathscr{D}_X))$. Using the canonical equivalence $\pi_* : K_{S'}(T^*X) \to K_{S'}(\mathrm{gr}(\mathscr{D}_X))$, it is enough to show that $g_1$ and $g_2$ are canonically identified. Since $S \subset S'$, we have a canonical commutative diagram given by taking the associated graded:

$$\begin{array}{ccc} KFF_S(\mathscr{D}_X) & \longrightarrow & KF_S(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF_S(\mathscr{D}_X) & \longrightarrow & K_{S'}(\mathrm{gr}(\mathscr{D}_X)). \end{array}$$

The universal property of homotopy pushouts then gives a morphism $f : K_S(\mathscr{D}_X) \to K_{S'}(\mathrm{gr}(\mathscr{D}_X))$. The morphisms $KF_S(\mathscr{D}_X) \to K_{S'}(\mathrm{gr}(\mathscr{D}_X))$ in the above diagram factor as $KF_S(\mathscr{D}_X) \to K_S(\mathrm{gr}(\mathscr{D}_X)) \to K_{S'}(\mathrm{gr}(\mathscr{D}_X))$. It follows that the composition $g_1$ is also a solution to the homotopy pushout problem given by the above square. Another application of universal property of homotopy pushouts gives a homotopy equivalence between $f$ and $g_1$. A similar argument shows that $f$ is also canonically identified with $g_2$.

□

**Corollary 3.1.18.** *The following diagram commutes: If $S \subset T^*X$ then one has a homotopy commutative diagram:*

$$\begin{array}{ccc} K(\mathscr{D}_X) & \longrightarrow & K(T^*X) \\ \uparrow & & \uparrow \\ K_S(\mathscr{D}_X) & \longrightarrow & K_S(T^*X) \end{array}$$

*Proof.* Apply the previous proposition with $S' = T^*X$. □

It follows directly from the construction that the induced morphism $gr_S : K_{S,0}(\mathscr{D}_X) \to K_{S,0}(T^*X)$ on Grothendieck groups is given by sending a $\mathscr{D}_X$-module $\mathscr{M}$ to $gr_{\mathscr{F}}(\mathscr{M})$ where $\mathscr{F}$ is some good filtration on $\mathscr{M}$. This morphism was previously constructed by Laumon ([Lau83]). The above construction can be viewed as a lifting of this morphism to the whole $K$-theory spectrum.

3.2. **Construction of epsilon factors.** Any perfect complex of $\mathscr{D}_X$-modules $\mathscr{M}$ with singular support in $S$ gives rise to a homotopy point $[\mathscr{M}]$ of $K_S(\mathscr{D}_X)$. The results of the previous section then give a microlocalized homotopy point $gr_S([\mathscr{M}])$ of $K_S(T^*X)$. We shall now construct a morphism, depending on the choice of a 1-form $\nu$ on $U \subset X$ ($\mathscr{M}$ will be smooth off $Y = X \setminus U$

i.e. on $U$ it will be a vector bundle with connection), from $K_S(T^*X)$ to $K_Y(X)$. This will then give us the required epsilon factors. We begin with a lemma.

**Lemma 3.2.1.** *Let $\nu$ be a 1-form on $U = X \setminus Y$, with $Y \subset X$ closed such that $\nu(X \setminus Y) \cap S = \emptyset$. Let $V = T^*X \setminus S$. Then one has a commutative diagram:*

$$\begin{array}{ccc} K(V) & \xrightarrow{\nu^*} & K(U) \\ \uparrow & & \uparrow \\ K(T^*X) & \xrightarrow{(\pi^*)^{-1}} & K(X) \\ \uparrow & & \uparrow \\ K_S(T^*X) & \longrightarrow & K_Y(X) \end{array}$$

*Proof.* The columns are just the usual localization sequences. The homotopy morphism $\nu^*$ is just pullback by the given section. Furthermore, the middle horizontal is induced by the structure map $\pi : T^*X \to X$. Since $\pi \circ \nu = \mathrm{Id}$ the top square commutes. Since both vertical columns are homotopy cofiber sequences we get a map from $K_S(T^*X) \to K_Y(X)$. ☐

Note that the morphism $K_S(T^*X) \to K_Y(X)$ depends on $\nu$; we shall denote this morphism by $\mathscr{E}_\nu$. Denote the composition $gr_S \circ \mathscr{E}_\nu$ by $\mathscr{E}_{\nu,Y}$. Now given a complex $\mathscr{F} \in D^b_S(\mathscr{D}_X)$ we get a homotopy point $[\mathscr{F}]$ of $K_S(\mathscr{D}_X)$. Then, composing with $\mathscr{E}_{\nu,Y}$ gives a homotopy point $\mathscr{E}_{\nu,Y}(\mathscr{F})$ of $K_Y(X)$. This give us the required localization of the determinant of cohomology. We need to show that our epsilon factor satisfies a global product formula. We recall the statement of the global product formula.

Suppose $f : X \to Z$ is a proper morphism of smooth varieties. Then we have pushforward maps $Rf_* : D^b_{\mathrm{perf}}(\mathscr{D}_X) \to D^b_{\mathrm{perf}}(\mathscr{D}_Z)$. If $X$ is projective, then we get $R\Gamma_{dr} : D^b_{\mathrm{perf}}(\mathscr{D}_X) \to D^b_{\mathrm{perf}}(k)$. On the other hand, we also have the usual pushforward $R\Gamma : D^b_{\mathrm{perf}}(X) \to D^b_{\mathrm{perf}}(k)$. These induce morphisms of $K$-theory spectra $R\Gamma : K(X) \to K(k)$, $R\Gamma : K_Y(X) \to K(k)$, and $R\Gamma_{dr} : K(\mathscr{D}_X) \to K(k)$. The global product formula states that the homotopy points $[R\Gamma_{dr}(X, \mathscr{M})]$ and $[R\Gamma(\mathscr{E}_{\nu,Y}(\mathscr{M}))]$ of $K(k)$ are canonically identified.

In fact, the push forward functors exist at the filtered level ([Lau83]): Laumon's construction of filtered direct images also works for doubly filtered $\mathscr{D}_X$-modules. In particular, we have functors $Rf_* : DFF^b_{\mathrm{perf}}(\mathscr{D}_X) \to DFF^b_{\mathrm{perf}}(\mathscr{D}_X)$ and $Rf_* : DF^b_{\mathrm{perf}}(\mathscr{D}_X) \to DF^b_{\mathrm{perf}}(\mathscr{D}_X)$, which are furthermore compatible with the forgetful functors $DFF^b_{\mathrm{perf}}(\mathscr{D}_X)$
$\to DF^b_{\mathrm{perf}}(\mathscr{D}_X) \to D^b_{\mathrm{perf}}(\mathscr{D}_X)$.

We prove a few preliminary lemmas in preparation for the proof of the global product formula. First, we give another construction of the morphism $gr : K(\mathscr{D}_X) \to K(T^*X)$ due to Quillen. If $X$ is affine, then Quillen shows that the natural morphism $K'(X) \to K'(\mathscr{D}_X)$ given by sending an $\mathscr{O}_X$-module $\mathscr{M}$ to the left $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is a weak equivalence. These can be glued to get an analogous weak equivalence in the non-affine case ([Hod89]). On the other hand, we have a canonical weak equivalence $K'(X) \to K'(T^*X)$ given by pullback along the projection.

Then, inverting the above homotopy morphism gives us a morphism $gr^Q : K(\mathscr{D}_X) \to K(T^*X)$. Here we also use the identification of $K'(\mathscr{D}_X)$ and $K'(X)$ with $K(X)$ and $K(\mathscr{D}_X)$.

**Remark 3.2.2.** Note that the construction of Quillen does not give us a $gr_S$. This is the main reason for the complicated constructions of the previous section. On the other hand, Quillen's theorem, which is a statement about positively filtered rings, can be generalized to $\mathbb{Z}$-filtered rings. This then leads to a construction of a $gr_{\mathscr{E}} : K(\mathscr{E}_X) \to K(T^*X \setminus X)$, where $\mathscr{E}_X$ is the sheaf of micro-local differential operators. The microlocal $gr_{\mathscr{E}}$ allows the construction of a $gr_S$ as before. This was the approach taken in the author's thesis ([Pat08]). Unfortunately, with this method, one has to work with affine opens and then glue together the resulting $gr_{\mathscr{E}}$ using Mayer–Vietoris, and such gluings at the level of $K$-theory spectra require delicate arguments with sheaves of $K$-theory spectra. Furthermore, whereas these methods do not generalize to the analytic situation, the constructions using filtered $\mathscr{D}_X$-modules do generalize. We hope to report on the analytic situation (and the connection with Betti epsilon factors) elsewhere. The use of filtered $\mathscr{D}_X$-modules allows us to make arguments globally and avoid the use of sheaves of spectra as well as the theory of microdifferential operators. It also makes various functoriality properties (see subsections 3.4 and 3.5) easier to see.

**Lemma 3.2.3.** *The homotopy morphisms $gr^Q$ and $gr$ are canonically identified.*

*Proof.* Note that we have a commutative diagram:

$$\begin{array}{ccc} KFF(\mathscr{D}_X) & \longrightarrow & KF(\mathscr{D}_X) \\ \downarrow & & \downarrow \\ KF(\mathscr{D}_X) & \longrightarrow & K(T^*X). \end{array}$$

The arrows $KF(\mathscr{D}_X) \to K(T^*X)$ come from the composition $KF(\mathscr{D}_X) \to K(\mathscr{D}_X) \to^{gr^Q} K(T^*X)$. It follows from the universal property of the homotopy pushout that $gr$ and $gr^Q$ canonically identified. ☐

**Lemma 3.2.4.** *Let $X$ be a smooth projective variety over $k$. Then the composition*

$$K(\mathscr{D}_X) \xrightarrow{gr} K(T^*X) \xrightarrow{(\pi^*)^{-1}} K(X) \xrightarrow{R\Gamma} K(k)$$

*is homotopic to $R\Gamma_{dr} : K(\mathscr{D}_X) \to K(k)$.*

*Proof.* By the previous lemma we may replace $gr$ by $gr^Q$. Now $gr^Q : K(\mathscr{D}_X)$ $\to K(T^*X)$ was defined to be the composition

$$K(\mathscr{D}_X) \longrightarrow K(X) \xrightarrow{\pi^*} K(T^*(X).$$

Therefore it is enough to see that the following diagram commutes:

$$\begin{array}{ccc} K(\mathscr{D}_X) & \longrightarrow & K(X) \\ & \overset{R\Gamma_{dr}}{\searrow} & \downarrow{\scriptstyle R\Gamma} \\ & & K(k) \end{array}$$

In fact, it is enough to show that the following diagram commutes:

$$\begin{array}{ccc} K(X) & \longrightarrow & K(\mathscr{D}_X) \\ & \overset{R\Gamma}{\searrow} & \downarrow{\scriptstyle R\Gamma_{dr}} \\ & & K(k) \end{array}$$

here the top arrow sends $\mathscr{M}$ to $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}$. The result is now a standard fact from the theory of $\mathscr{D}_X$ modules at the level of derived functors and derived categories. ☐

**Corollary 3.2.5.** *(Global Product Formula) With the notation as before, the homotopy points $[R\Gamma_{dr}(X,\mathscr{M})]$ and $[R\Gamma(\mathscr{E}_{\nu,Y}(\mathscr{M}))]$ are canonically identified.*

*Proof.* By Lemma 3.18 and Corollary 3.17 we have a commutative diagram:

$$\begin{array}{ccccc} K(\mathscr{D}_X) & \longrightarrow & K(T^*X) & \longrightarrow & K(X) \\ \uparrow & & \uparrow & & \uparrow \\ K_S(\mathscr{D}_X) & \longrightarrow & K_S(T^*X) & \longrightarrow & K_Y(X) \end{array}$$

Our $\mathscr{M}$ gives a homotopy point of $[\mathscr{M}]$ of $K_S(\mathscr{D}_X)$. By the previous lemma, composing with the top row followed by $R\Gamma : K(X) \to K(k)$ gives the homotopy point $[R\Gamma_{dr}(X,\mathscr{M})]$. On the other hand, the following diagram commutes:

$$\begin{array}{ccc} K(X) & \xrightarrow{R\Gamma} & K(k) \\ \uparrow & \overset{R\Gamma}{\nearrow} & \\ K_Y(X) & & \end{array} .$$

Therefore, composition with the bottom row followed by $R\Gamma : K_Y(X) \to K(k)$, which is $[R\Gamma(\mathscr{E}_{\nu,Y}(\mathscr{M}))]$, is canonically identified with $[R\Gamma_{dr}(X,\mathscr{M})]$. ☐

Let $\varepsilon_{\nu,Y}(\mathscr{M}) = \det([R\Gamma(\mathscr{E}_{\nu,Y}(\mathscr{M}))])$ be the corresponding element of $Pic^{\mathbb{Z}}(k)$. Passing to determinants gives a canonical isomorphism $\eta_{dr,\nu} : \det(R\Gamma(X,\mathscr{M})) \to \varepsilon_{\nu,Y}(\mathscr{M})$; the former is by definition the determinant of de Rham cohomology. Note that the epsilon factor has a local nature in the sense that it only depends on the values of the form and $\mathscr{M}$ on an open neighborhood of $Y$.

**Lemma 3.2.6.** *Let $\mathscr{G},\mathscr{F} \in D^b_S(\mathscr{D}_X)$, and $\nu$ be as above. Suppose $\mathscr{F}|_{U'} = \mathscr{G}|_{U'}$ for some open $U' \subset X$ such that $Y \subset U'$. Then $\mathscr{E}_{\nu,Y}(\mathscr{F})$ and $\mathscr{E}_{\nu,Y}(\mathscr{G})$ are canonically identified.*

*Proof.* First we can restrict everything to $U'$. This gives rise to homotopy points $\mathscr{E}_{\nu,Y}(\mathscr{G}|U')$ and $\mathscr{E}_{\nu,Y}(\mathscr{F}|U')$ of $K(Y)$. Furthermore, by the given hypothesis these two are canonically identified. Therefore, it is enough to show that $\mathscr{E}_{\nu,Y}(\mathscr{F})$ is canonically identified with $\mathscr{E}_{\nu,Y}(\mathscr{F}|U')$. This follows from the following commutative diagram:

$$\begin{array}{ccccc} K_S(\mathscr{D}_X) & \longrightarrow & K_S(T^*X) & \longrightarrow & K_Y(X) \\ \downarrow & & \downarrow & & \downarrow \\ K_{S_{U'}}(\mathscr{D}_{U'}) & \longrightarrow & K_{S_{U'}}(T^*U') & \longrightarrow & K_Y(X). \end{array}$$

Here $S_U = T^*U \cap S$; the left two vertical arrows are the natural restriction maps and the right is the identity map. □

**Lemma 3.2.7.** *Suppose $\nu = \mu$ on an open neighborhood $U'$ of $Y$. Then $\mathscr{E}_{\nu,Y}(\mathscr{F})$ and $\mathscr{E}_{\mu,Y}(\mathscr{F})$ are canonically identified.*

*Proof.* The proof is similar to that of the previous lemma. □

**Remark 3.2.8.** (1) Since $X$ is smooth, one can identify $K(X)$ with $G(X)$, where $G(X)$ is the K-theory of the abelian category of coherent sheaves on $X$. It follows from comparing with the localization sequence for K-theory, that the fiber $K_Y(X)$ of $K(X) \to K(U)$ can be canonically identified with $G(Y)$.
(2) It follows from the previous comment that if $Y$ is the disjoint union of $Y_i$ then $K_Y(X) = \prod K_{Y_i}(X)$. In particular, one has homotopy points $\mathscr{E}_{\nu,Y_i}(\mathscr{F})$ of $K(Y_i)$ and a canonical identification $\mathscr{E}_{\nu,Y}(\mathscr{F}) = \sum_i \mathscr{E}_{\nu,Y_i}(\mathscr{F})$.
(3) If $k'$ is a finite extension of $k$, then $\varepsilon_{\nu,Y}(\mathscr{F}) \otimes_k k' = \varepsilon_{\nu_{k'},Y_{k'}}(\mathscr{F}_{k'})$.

3.3. **Compatibility properties of epsilon factors.** In this section we discuss the compatibility of epsilon factors under pushforward and pullback.
Let $f : X \to Y$ be a smooth morphism of smooth varieties over $k$. Then one has pullback functors $Lf^* : D^b_{\text{perf}}(\mathscr{D}_Y) \to D^b_{\text{perf}}(\mathscr{D}_X)$. These pullback functors can be lifted to the categories of filtered $\mathscr{D}_X$-modules which are compatible with the forgetful functors. One has a commutative diagram:

$$\begin{array}{ccccc} T^*X & \xleftarrow[\rho_f]{} & X \times_Y T^*Y & \xrightarrow{pr_X} & X \\ & & \downarrow{\scriptstyle pr_{T^*Y}} & & \downarrow{\scriptstyle f} \\ & & T^*Y & \xrightarrow{\pi_Y} & Y. \end{array}$$

Here the arrow on the left is the natural one induced by $f$ and the right commutative square is cartesian. It is well known that for a complex of perfect $\mathscr{D}_Y$ modules $\mathscr{M}$, $SS(Lf^*(\mathscr{M})) \subset \rho_f pr^{-1}_{T^*Y}(SS(\mathscr{M}))$; in fact, since $f$ is smooth, the inclusion is an equality. For $Y'$ closed in $Y$, let $\nu$ be a non-vanishing 1-form on $Y \setminus Y'$ such that $\nu(Y \setminus Y') \cap SS(\mathscr{M}) = \emptyset$. Suppose $f$ is etale. Then $f^*\nu$ is a 1-form on $X \setminus f^{-1}(Y')$ such that $f^*\nu(X \setminus f^{-1}(Y')) \cap SS(Lf^*(\mathscr{M})) = \emptyset$.

**Proposition 3.3.1.** *Let $X,Y,f$, and $\nu$ be as above. Furthermore, denote $f^{-1}(Y')$ by $X'$. One has a commutative diagram of spectra:*

$$\begin{array}{ccc} K_S(\mathscr{D}_Y) & \xrightarrow{\varepsilon_{\nu,Y'}} & K_{Y'}(Y) \\ \downarrow Lf^* & & \downarrow f^* \\ K_{\rho_f pr^{-1}_{T^*Y}(S)}(\mathscr{D}_X) & \xrightarrow{\varepsilon_{\nu,X'}} & K_{X'}(X) \end{array}$$

*In particular, the two homotopy points $[f^*(\varepsilon_{\nu,Y'}(\mathscr{M}))]$ and $[\varepsilon_{\nu,X'}(Lf^*\mathscr{M}))]$ are canonically identified.*

*Proof.* In fact, in the situation of the proposition $f^*$ is exact. Furthermore, the pullback of the cotangent bundle of $Y$ is isomorphic to the cotangent bundle of $X$. The diagram in question factors as:

$$\begin{array}{ccccc} K_S(\mathscr{D}_Y) & \longrightarrow & K_S(T^*Y) & \longrightarrow & K_{Y'}(Y) \\ \downarrow Lf^* & & \downarrow & & \downarrow f^* \\ K_{\rho_f pr^{-1}_{T^*Y}(S)}(\mathscr{D}_X) & \longrightarrow & K_{S'}(T^*X) & \longrightarrow & K_{X'}(X). \end{array}$$

The arrow in the middle is given by pull-back to $T^*Y \times_Y X$ followed by pushforward along the natural isomorphism $T^*Y \times_Y X \to T^*X$ . It is clear that the right square commutes. For the left square, it is enough to show that the following square commutes:

$$\begin{array}{ccc} KF_S(\mathscr{D}_Y) & \longrightarrow & K_S(gr(\mathscr{D}_Y)) \\ \downarrow & & \downarrow \\ KF_{S'}(\mathscr{D}_X) & \longrightarrow & K_{S'}(\mathrm{gr}(\mathscr{D}_X)). \end{array}$$

The right vertical is given by pullback along $f$ and pushing forward along the natural isomorphism $f^*(gr(\mathscr{D}_Y)) \to \mathrm{gr}(\mathscr{D}_X)$. The diagram commutes since, by construction of the filtered pull back, $gr(f^*(\mathscr{M},\mathscr{F})) \cong f^*(gr_{\mathscr{F}}(\mathscr{M}))$. □

Now we consider the behavior of epsilon factors under pushforward. In [Lau83], Laumon has defined push forward maps $\int_f : DF^b_{\mathrm{perf}}(\mathscr{D}_X) \to DF^b_{\mathrm{perf}}(\mathscr{D}_Y)$ for proper morphisms $f : X \to Y$. The idea is to factor the morphism into a closed immersion followed by a smooth projection.

**Lemma 3.3.2.** *One has a commutative diagram of spectra:*

$$\begin{array}{ccc} KF(\mathscr{D}_X) & \xrightarrow{\int_f} & KF(\mathscr{D}_Y) \\ \downarrow gr & & \downarrow gr \\ K(T^*X) & \xrightarrow{G} & K(T^*Y), \end{array}$$

*where $G$ is induced by sending $M$ to $Rpr_{T^*Y,*}\rho^!_f M[d_f]$. Here $d_f$ is the relative dimension of $f$.*

*Proof.* This is just a restatement of Laumon (5.6.1, [Lau83]). There it is shown that the diagram commutes at the level of derived categories. In fact, Laumon constructs a canonical morphism at the level of complexes. The idea is to factor the given morphism into a closed immersion followed by a smooth projection. For closed immersions, the pushforward is exact, while for smooth morphisms, the pushforward can be defined canonically at the level of complexes using the relative de Rham complex. In particular, we can define the pushforward functors at the level of complexes. The same is true for $G$ as long as we work with appropriate models for the $K$-theory spectra. For smooth morphisms, $\rho_f^!$ is given by the usual pullback twisted by the sheaf of relative differentials. So if we work with perfect complexes of flat modules, then the two morphisms $gr \circ \int_f$ and $G \circ gr$ can be defined at the level of complexes. Furthermore, there is a natural morphism of functors $G \circ gr \to gr \circ \int_f$, which is an isomorphism on the derived categories. There is also a canonical choice for such a factoring using the graph of $f$. Therefore, we get a canonical identification of the corresponding homotopy morphisms. □

**Remark 3.3.3.** We could make the homotopy constructed in the previous lemma even "more" canonical. The homotopy constructed in the proof depended on a certain factorization of $f$ into a closed morphism followed by a smooth morphism. On the other hand, it follows from ([Lau83]) that the homotopies constructed from two different factorizations are canonically identified, and we could take the colimit over all such homotopies. Then we would even get naturality for composition of proper maps $f : X \to Y$ and $g : Y \to Z$. Another way to do this is to compare the corresponding graphs and relate the resulting relative de Rham complexes; this gives rise to a canonical morphism $\int_{g \circ f} \to \int_g \circ \int_f$ on the level of complexes ([Lau83]).

The homotopy constructed in Lemma 3.28 is essentially a Riemann-Roch theorem for higher $K$-theory of $\mathscr{D}_X$-modules; it is a lifting of Laumon's construction from $K_0$ to the whole $K$-theory spectrum. We can now microlocalize to get a microlocal Riemann–Roch. Let $S = SS(\mathscr{M}) \subset T^*X$. Then $SS(\int_f(\mathscr{M})) \subset pr_{T^*Y}(\rho_f^{-1}(S)) = S'$. Therefore, the above lemma gives a commutative diagram:

$$\begin{array}{ccc} KF_S(\mathscr{D}_X) & \xrightarrow{\int_f} & KF_{S'}(\mathscr{D}_Y) \\ \downarrow{\scriptstyle gr} & & \downarrow{\scriptstyle gr} \\ K_S(T^*X) & \xrightarrow{G} & K_{S'}(T^*Y). \end{array}$$

We need only check that if $\mathscr{F}$ is a perfect complex on $T^*X$ with support in $S$ then $\operatorname{Supp}(G(\mathscr{F})) \subset S'$. More generally, let $f : X \to Y$ be a morphism of schemes. Let $\mathscr{F}$ be a quasi-coherent sheaf on $X$ with support in $S \subset X$. If $f$ is proper, then the support of $f_*(\mathscr{F})$ is contained in $f(S)$. If $\mathscr{F}$ is a quasi-coherent sheaf on $Y$ with support $S$, then the support of $f^!(\mathscr{F})$ is contained in $f^{-1}(S)$. Let $f : X \to Y$ be a proper morphism with $S$ and $S'$ as above. Let $\nu$ be a non-vanishing 1-form on $Y \setminus Y'$ such that $\nu(Y \setminus Y') \cap S' = \emptyset$ and $f$ is etale. Then $f^*\nu(X \setminus X') \cap S = \emptyset$.

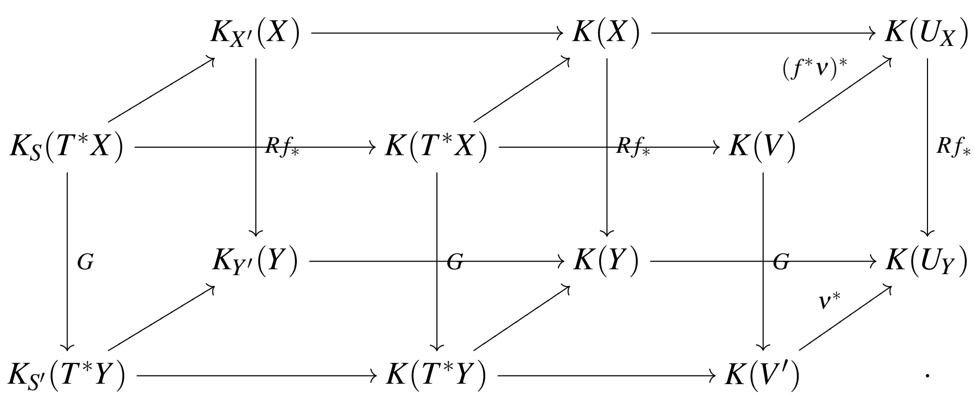
**Corollary 3.3.4.** *Let $X,Y,X',Y'$ and $\nu$ be as before. Suppose that $f$ is etale. One has a canonical commutative diagram:*

$$\begin{CD} K_S(\mathscr{D}_X) @>>> K_{S'}(\mathscr{D}_Y) \\ @V{\varepsilon_{f^*\nu,X'}}VV @VV{\varepsilon_{\nu,Y'}}V \\ K_{X'}(X) @>{Rf_*}>> K_{Y'}(Y). \end{CD}$$

*Proof.* It is enough to check the commutativity of the following diagram:

$$\begin{CD} K_S(T^*X) @>{G}>> K_{S'}(T^*Y) \\ @VVV @VVV \\ K_{X'}(X) @>{Rf_*}>> K_{Y'}(Y). \end{CD}$$

Consider the following diagram:




Here $U_X = X \setminus X'$, $V = T^*X \setminus S$ and similarly for $U_Y$ and $V'$. Since $f$ is etale, $T^*Y \times_Y X \cong T^*X$. It follows that the morphism $G : K(T^*X) \to K(T^*Y)$ is just the usual push-forward by an etale morphism. This also gives a morphism $G : K_S(T^*X) \to K_{S'}(T^*Y)$ simply by restriction. On the other hand, the morphism $G : K(V) \to K(V')$ is not defined in general. However, for the purposes of proving our commutativity we may replace $S$ by $f^{-1}(S')$. To see this, note that the following diagram commutes:

$$\begin{array}{ccc} K_S(T^*X) & & \\ \downarrow & \searrow & \\ K_{f^{-1}(S')}(X) & \longrightarrow & K(X') \end{array}$$

The arrow on the bottom exists since $f^*\nu(X \setminus X') \cap f^{-1}(S') = \emptyset$. In particular, we may assume that $V$ gets mapped to $V'$ under the natural etale morphism $T^*X \to T^*Y$. Therefore, we can again define $G : K(V) \to K(V')$ by the usual push-forward by an etale morphism. Now we have already seen that all the squares commute except possibly the ones with $G$ in them. The two squares in the front clearly commute. We would like to show that the three vertical slits commute. The left-most slit is the one in the theorem. The horizontal rows are homotopy cofiber sequences; furthermore, the horizontal arrows in the left-most slit (i.e. the $\mathscr{E}_\nu$) are defined by the universal property of these cofiber sequences. So we need only show that the left two vertical slits commute. The right-most vertical slit is commutative by base change ([TT90], 3.18). For the middle slit, we have to show that the following commutes:

$$\begin{array}{ccc} K(T^*X) & \xleftarrow[\pi_X^*]{} & K(X) \\ \downarrow G & & \downarrow Rf_* \\ K(T^*Y) & \xleftarrow[\pi_X^*]{} & K(Y). \end{array}$$

Recall that $G$ sends $\mathscr{M}$ to $Rpr_{T^*Y,*}\rho_f^!\mathscr{M}[d_f]$. But, since $f$ is etale, $\rho_{f^!} = \rho_{f^*}$ and $d_f = 0$. Then we have $Rpr_{T^*Y,*}\rho_f^!(\pi_X^*(\mathscr{M}))[d_f] = Rpr_{T^*Y,*}\rho_f^*(\pi_X^*(\mathscr{M})) \to Rpr_{T^*Y,*} \circ pr_X^*(\mathscr{M}) \to \pi_Y^* \circ Rf_*(\mathscr{F})$, where the arrows are quasi-isomorphisms. The last one follows from the cartesian square:

$$\begin{array}{ccc} X \times_Y T^*Y & \xrightarrow{pr_X} & X \\ \downarrow pr_{T^*Y} & & \downarrow f \\ T^*Y & \xrightarrow{\pi_Y} & Y. \end{array}$$

□

3.4. **Comparison with topological epsilon factors.** In this section we explain a conjectural relation between de Rham and Betti epsilon factors. We begin by rendering the story of the last two sections to the holonomic setting. Let $C^b_{hol}(\mathscr{D}_X) \subset C^b_{\mathrm{perf}}(\mathscr{D}_X)$ denote the full subcategory of complexes with holonomic cohomology. We can also consider the corresponding filtered categories $CF^b_{hol}(\mathscr{D}_X)$ and $CFF^b_{hol}(\mathscr{D}_X)$. We will use similar notation for the corresponding derived categories $D^b_{hol}(\mathscr{D}_X)$, *etc.* We denote the corresponding $K$-theory spectra by $K_{hol}(\mathscr{D}_X)$, *etc.* The results of the previous sections give us canonical morphisms of spectra $gr_S : K_{hol,S}(\mathscr{D}_X) \to K_S(T^*X)$.

**Remark 3.4.1.** We could have started with complexes of holonomic $\mathscr{D}_X$-modules, as the resulting derived categories are equivalent to the ones above. For the non-filtered versions this is a well known result of Beilinson. One can prove a similar result in the non-filtered case. However, this result will not be used in the following.

Let $X$ be smooth and proper over $\mathbb{C}$. The de Rham complex construction gives a functor $DR : D^b_{hol}(\mathscr{D}_X) \to D^b_c(X^{an})$, where $D^b_c(X^{an})$ denotes the derived category of constructible sheaves in the classical topology on $X$. If $\mathscr{M} \in D^b_{hol}(\mathscr{D}_X)$ and $\nu$ is a 1-form $X \setminus X'$ such that $SS(\mathscr{M}) \cap \nu(X \setminus X') = \phi$, then we have constructed a de Rham epsilon factor $\varepsilon^{dR}_{\nu,X'}(\mathscr{M})$. This factor comes equipped with a global period isomorphism: $\eta_{dR} : \det(R\Gamma(X,\mathscr{M})) \to \varepsilon^{dR}_{\nu,X'}(\mathscr{M})$. On the other hand, Beilinson ([Bei07]) has constructed a Betti epsilon factor $\varepsilon^{\mathrm{B}}_{\nu,X'}(DR(\mathscr{M}))$. The Betti epsilon factor also comes equipped with a global period isomorphism: $\eta_B : \det(R\Gamma(X^{an}, DR(\mathscr{M}))) \to$

$\varepsilon^B_{\nu,X'}(DR(\mathscr{M}))$. Finally, we have the global de Rham isomorphism:

$$DR : \det(R\Gamma(X,\mathscr{M}) \to \det(R\Gamma(X^{an}, DR(\mathscr{M})).$$

Then one expects canonical local (i.e only depending on $X'$ and $\nu$ around $X'$) isomorphisms $DR_{\nu,X'} : \varepsilon^{dR}_{\nu,X'}(\mathscr{M}) \to \varepsilon^B_{\nu,X'}(DR(\mathscr{M}))$ such that the following diagram commutes:

$$\begin{array}{ccc} \det(R\Gamma(X,\mathscr{M})) & \xrightarrow{DR} & \det(R\Gamma(X^{an}, DR(\mathscr{M}))) \\ \downarrow{\scriptstyle \eta_{dR}} & & \downarrow{\scriptstyle \eta_B} \\ \varepsilon^{dR}_{\nu,X'}(\mathscr{M}) & \xrightarrow{DR_{\nu,X'}} & \varepsilon^B_{\nu,X'}(DR(\mathscr{M})). \end{array}$$

It follows from the work of Beilinson ([Bei09]) that the conjecture is true for curves. More generally, one expects an identification at the level of homotopy points of spectra. The above picture would result after taking determinants.

3.5. **Animation of the Dubson–Kashiwara Formula.** Let $X/k$ be be a smooth variety of dimension $d$. Given $\mathscr{M} \in D^b_{\text{perf}}(\mathscr{D}_X)$,we can associate to $\mathscr{M}$ its characteristic cycle $CC(\mathscr{M}) \in CH^d(T^*X)$. Then the classical Dubson–Kashiwara formula states that the Euler characteristic of $\mathscr{M}$ is equal to the degree of the zero cycle obtained by intersecting $CC(\mathscr{M})$ with the zero section $T^*_XX$: $\chi(X,\mathscr{M}) = (T^*_XX, CC(\mathscr{M}))$. This formula has a natural animation at the level of $K$-theory spectra. A similar animation was given by Beilinson in the case of constructible sheaves on real analytic $X$ ([Bei07]).
We have constructed a morphism $gr_S : K_S(\mathscr{D}_X) \to K_S(T^*X)$ such that the following diagram commutes:

$$\begin{array}{ccc} KF_{S,0}(\mathscr{D}_X) & & \\ \downarrow{\scriptstyle \omega} & \searrow^{gr_S} & \\ K_{S,0}(\mathscr{D}_X) & \xrightarrow{gr_S} & K_{S,0}(T^*X). \end{array}$$

Let $\tau_X : K_0(X) \to CH^*(X)$ denote the usual Riemann–Roch morphism. The image of the homotopy point $\text{gr}(\mathscr{M})$ in $K_0(\mathscr{D}_X)$, denoted also by $\text{gr}(\mathscr{M})$, is given by $gr(\mathscr{M},\mathscr{F})$ for some $(\mathscr{M},\mathscr{F}) \in DF^b_{\text{perf}}(\mathscr{D}_X)$ lifting $\mathscr{M}$. Furthermore, if $\mathscr{M} \in D^b_{hol}(\mathscr{D}_X)$, then it follows from Laumon ([Lau83], 6.6.1) and the above commutative diagram (with $S = T^*X$) that the image of $\text{gr}(\mathscr{M}) \in CH^*(T^*X)$ is given by $CC(\mathscr{M}) \in CH^d(T^*X)$, where $d = \dim(X)$. It follows that the image of $\text{gr}(\mathscr{M})$ in $CH^0(X)$ is given by the intersection of $CC(\mathscr{M})$ with the zero section. The image in $CH^0(k) = \mathbb{Z}$ is just $(T^*_XX, CC(\mathscr{M}))$. On the other hand, the image of $gr(\mathscr{M})$ in $K_0(k)$ is given by $R\Gamma(X,\mathscr{M})$. Finally, its image in $CH^0(k) = \mathbb{Z}$ is just the Euler characteristic. In fact, we even get a microlocal animation of $CC(\mathscr{M})$ given by $gr_S(\mathscr{M})$.

## 4. The case of curves

In this section, we render our story to the case of curves. In particular, we will see that our theory of epsilon factors gives rise to a classical theory of epsilon factors. Let us recall the definition

of such a theory in the de Rham case (see ([BBE02]). Let $k$ be a field of characteristic zero and $F = k'((x))$ for some finite extension $k'$ of $k$. Let $\mathscr{D}_F$ be the ring of differential operators on $F$ and $M_{hol}(\mathscr{D}_F)$ the category of $\mathscr{D}_F$-modules such that the underlying $F$-vector space is finite dimensional. If $X$ is a curve and $x \in X$ is a closed point, then we let $K_x$ denote the function field of $O_x := \hat{\mathscr{O}}_{X,x}$ and $k(x)$ denote the residue field at $x$. Any holonomic $\mathscr{D}_X$ module $\mathscr{M}$ can be pulled back to an object $\mathscr{M}_x \in M_{hol}(\mathscr{D}_{K_x})$. Similarly, a non-vanishing meromorphic 1-form $\nu$ on $X$ gives rise to a 1-form $\nu_x \in \omega(K_x)$. If $t_x$ is a uniformizer at $x$, then we have $O_x = k(x)[[t_x]]$ and $K_x = k(x)((t_x))$.
A classical theory of epsilon factors is rule which associates to a datum $(F, \mathscr{M})$, where $\mathscr{M} \in M_{hol}(\mathscr{D}_F)$, a function $\varepsilon(F, \mathscr{M}) : \omega(F)^\times \to \mathrm{Ob}(Pic^{\mathbb{Z}}(k))$ such that the following properties hold:
(1): For a short exact sequence $0 \to \mathscr{M}_1 \to \mathscr{M} \to \mathscr{M}_2 \to 0$ one has

$$\varepsilon(F, \mathscr{M}, \nu) = \varepsilon(F, \mathscr{M}_1, \nu) \otimes \varepsilon(F, \mathscr{M}_2, \nu).$$

(2): Let $F'/F$ be a finite separable extension, $\mathscr{M}' \in M_{hol}(\mathscr{D}_{F'})$ and $\nu \in \omega(F)^\times \subset \omega(F')^\times$. Then one has

$$\varepsilon(F', \mathscr{M}', \nu) = \varepsilon(F, \mathscr{M}', \nu) \otimes_k k'.$$

(3): Most importantly, one has a product formula: Given a smooth projective curve $X$ over $k$, a divisor $D \subset X$, $U = X \setminus D$, $\nu$ a non-vanishing 1-form on $U$, and a holonomic $\mathscr{D}_X$ module $\mathscr{M}$ smooth off $D$ such that $j_* j^* \mathscr{M}$ where $j : U \to X$, one has a global product isomorphism:

$$\varepsilon_{dR} : \det(R\Gamma(X, \mathscr{M})) \to \otimes_x \varepsilon(K_x, \mathscr{M}_x, \nu_x).$$

Let $M_{hol}(\mathscr{D}_{O_x})$ denote the category of finitely generated $\mathscr{D}_{O_x}$-modules $\mathscr{M}$ such that $\mathscr{M}_{K_x} := \mathscr{M} \otimes_{O_x} K_x$ is a finite dimensional $K_x$ vector space. We have the associated $K$-theory spectra $K_{hol}(\mathscr{D}_{O_x})$ and $K_{hol}(\mathscr{D}_{K_x})$. The rings of differential operators $\mathscr{D}_{O_x}$ and $\mathscr{D}_{K_x}$ come equipped with filtration by order of differential operator. We denote by $T^*_x X$ the pull-back $T^*X \times_X Spec(O_x)$. Just as before, we have the categories of $\mathscr{D}$-modules with good filtrations $MF_{hol}(\mathscr{D}_{O_x})$ and $MF_{hol}(\mathscr{D}_{K_x})$. Recall that a filtration $\mathscr{F}$ on $\mathscr{M}$ is good if and only if the corresponding $gr_{\mathscr{F}}(\mathscr{M})$ is finitely generated; in particular, we have the usual notion of the singular support. There is a natural pushforward functor: $j_* : M_{hol}(\mathscr{D}_{K_x}) \to M_{hol}(\mathscr{D}_{O_x})$. This is an embedding with essential image consisting of $\mathscr{M}$ with invertible $t_x$-action. In particular, it consists of $\mathscr{D}_{O_x}$-modules with singular support contained in $t_x z_x = 0$, where $z_x$ is the image of $\partial_{t_x}$ in the associated graded. Denote this set by $S_x \subset T^*_x X$, and $V_x$ be its complement.

**Proposition 4.0.1.** *Let $\nu_x \in \omega(K_x)^{\times}$. We have a commutative diagram of spectra:*

$$\begin{array}{ccccc}
 & & K(V_x) & \xrightarrow{\nu_x} & K(K_x) \\
 & & \uparrow & & \uparrow \\
K_{hol}(\mathscr{D}_{O_x}) & \xrightarrow{gr_x} & K(T^*_x X) & \xrightarrow{(\pi_*)^{-1}} & K(O_x) \\
\uparrow & & \uparrow & & \uparrow \\
K_{hol,S_x}(\mathscr{D}_{O_x}) & \xrightarrow{gr_{S,x}} & K_{S_x}(T^*_x X) & \xrightarrow{\mathscr{E}_{\nu_x}} & K(k(x))
\end{array}$$

*Proof.* The existence of $gr_x$ and $gr_{S,x}$ are proved exactly as for the global case from section 3. The two vertical columns on the right are the usual localization sequences. Finally, $\mathscr{E}_{\nu_x}$ is constructed exactly as in Lemma 3.19. □

The proposition gives rise to a classical theory of epsilon factors: Apply the proposition to $F$ and $O_F$. For every $F$ and $\nu \in \omega(F)^{\times}$ as above, we have a sequence $K_{hol}(\mathscr{D}_F) \to K_S(gr(\mathscr{D}_F)) \to K(k')$. We denote the composition by $\varepsilon(F,\nu)$. Any holonomic $\mathscr{D}_F$-module $\mathscr{M}$ gives a homotopy point of $K_{hol}(\mathscr{D}_F)$. Then $\varepsilon(F,\nu)(\mathscr{M})$ is a homotopy point of $K(k')$, and passing to determinants gives $\varepsilon(F,\mathscr{M},\nu) \in Pic^{\mathbb{Z}}(k')$. This gives rise to a classical theory of epsilon factors satisfying (1) and (2) above.
The inculsion $j_x : Spec(O_x) \to X$ gives rise to a commutative diagram:

$$\begin{array}{ccc}
T^*X \times_X Spec(O_x) & \longrightarrow & Spec(O_x) \\
\downarrow & & \downarrow \\
T^*X & \longrightarrow & X.
\end{array}$$

**Proposition 4.0.2.** *The following diagram commutes.*

$$\begin{array}{ccccc}
K_{hol,S}(\mathscr{D}_X) & \xrightarrow{gr_S} & K_S(T^*X) & \xrightarrow{\varepsilon_\nu} & K(Y) \\
\downarrow & & \downarrow & & \downarrow \\
K_{hol,S_x}(\mathscr{D}_{O_x}) & \xrightarrow{gr_{S,x}} & K_{S_x}(T^*_x X) & \xrightarrow{\varepsilon_{\nu_x}} & K(k(x))
\end{array}$$

*Proof.* First note that for any open neighborhood $U$ of $x$, the corresponding diagram for the open immersions $U \to X$ commutes. In particular, we may assume that $X$ is affine. We can further assume that $Y$ consists of a single point by passing to $U$ small enough. We will first prove that the following diagram commutes:

$$\begin{array}{ccccc}
K_{hol,S}(\mathscr{D}_X) & \xrightarrow{gr_S} & K_S(T^*X) & \xrightarrow{\varepsilon_\nu} & K(Y) \\
\downarrow & & \downarrow & & \downarrow \\
K_{hol,S_x}(\mathscr{D}_{O_{X,x}}) & \xrightarrow{gr_{S,x}} & K_{S_x}(T^*X \times_X Spec(O_{X,x})) & \xrightarrow{\varepsilon_{\nu_x}} & K(k(x))
\end{array}$$

Here, $S_x$ is defined as before in $T^*X \times Spec(O_{X,x})$. Let $V_x$ denote the complement. Also, we can pull back $\nu$ to get a section $\nu_x : Spec(F(x)) \to V_x$, where $F(x)$ is the fraction field of $Spec(O_{X,x})$. To show that the right square commutes it is enough to show that the following diagram given by base change commutes:

$$\begin{array}{ccccccc}
 & & K(T^*X \times_X Spec(O_{X,x})) & \longrightarrow & & & K(V_x) \\
 & \nearrow & & & & \nearrow & \\
K(T^*X) & & \longrightarrow & & K(V) & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & K(Spec(O_{X,x})) & \longrightarrow & & & K(F(x)) \\
 & \nearrow & & & & \nearrow & \\
K(X) & & \longrightarrow & & K(U) & &
\end{array}$$

For this it is enough to check that both vertical slits commute. But this is a consequence of flat base change ([TT90], 3.18). Now to show that the left square commutes, it is enough to show that the following commutes:

$$\begin{array}{ccc}
KF(\mathscr{D}_X) & \longrightarrow & K(T^*X) \\
\downarrow & & \downarrow \\
KF(\mathscr{D}_{O_{X,x}}) & \longrightarrow & K(T^*X \times_X Spec(O_{X,x}))
\end{array}$$

Here $KF(\mathscr{D}_X)$ is K-theory of $\mathscr{D}_X$-modules with filtrations and similarly for $Spec(O_{X,x})$. A filtration on a $\mathscr{D}_X$-module can be pulled back to give one on the resulting $\mathscr{D}_{O_{X,x}}$-module. The result follows from the compatibility of taking associated graded and passing to stalks. This proves the result before passing to the completion. On the other hand, since completion is faithfully flat, a similar argument allows to pass to the completion. □

If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module on $X$ such that $j_*j^*(\mathscr{M}) = \mathscr{M}$, then $\varepsilon_{\nu,Y}(\mathscr{M}) = \sum_{x\in Y} \varepsilon_{\nu,x}(\mathscr{M})$. It follows from the previous proposition that the homotopy points $\varepsilon_{\nu_x}(\mathscr{M}_x)$ and $\varepsilon_{\nu,x}(\mathscr{M})$ are canonically identified as homotopy points of $K(k(x))$. Therefore, the resulting $\varepsilon(F,\mathscr{M},\nu)$ satisfy a global product formula. In particular, the $\varepsilon(F,\mathscr{M},\nu)$ give rise to a classical theory of epsilon factors.

In the case of curves, a classical theory of epsilon factors constructed was constructed by Deligne and the rediscovered in ([BBE02]). In ([BBE02]), it is shown that the $\varepsilon(F,\mathscr{M},\nu)$ glue together for varying $\nu$ into a local system $\varepsilon(F,\mathscr{M})$ on $\omega(F)^{\times}$. Furthermore, the epsilon factors are expected to have an even more precise local nature. In particular, they should form a de Rham factorization line (see [Bei09]). Our theory of epsilon factors also gives rise to de Rahm factorization lines. For example, the connection on the epsilon line results from the

fact that $K(S_{red})$ is equivalent to $K(S)$ for any scheme $S$. Furthermore, this line is canonically identified with that of ([Bei09]). The details will appear elsewhere.
Above we constructed a theory of epsilon factors at the level of homotopy points of spectra. One gets the result for determinants by truncating the $K$-theory spectrum at $K_1$. If we truncate at $K_0$, we get a local description of the Euler characteristic. On the other hand, the constructions of ([BBE02]) are already at the level determinants. The above construction can be thought of as a lifting of this construction to the whole $K$-theory spectrum.

## 5. Appendix

Let $\mathscr{E}$ be an exact category. Then $K_0(\mathscr{E})$ can be thought of as the universal Euler characteristic. More precisely, any function on isomorphism classes of objects of $\mathscr{E}$ into an abelian group, which is additive on short exact sequences, factors through $K_0(\mathscr{E})$. One can ask for a similar universal description for the $K$-theory spectrum $K(\mathscr{E})$. In this section, we show that the $[0,1]$-connected cover of $K(\mathscr{E})$, denoted by $K(\mathscr{E})^{[0,1]}$, can be thought of as the universal determinant functor on $\mathscr{E}$. The description of $K(\mathscr{E})^{[0,1]}$ as a universal determinant is certainly known to the experts; in particular, Deligne ([Del87]) constructs a universal Picard groupoid $\mathscr{P}(\mathscr{E})$ such that $\pi_0(\mathscr{P}(\mathscr{E})) = K_0(\mathscr{E})$ and $\pi_1(\mathscr{P}(\mathscr{E})) = K_1(\mathscr{E})$. Here we show how to associate canonically a Picard groupoid to a $K$-theory spectrum. Furthermore, we show that this Picard groupoid satisfies the properties of a universal determinant for $\mathscr{E}$. We begin by showing that the category of Picard groupoids and that of $[0,1]$-connected spectra are homotopically equivalent. Next, we recall the notion of universal determinant functors on exact categories and the result of Deligne mentioned above. Finally, we show that the homotopy point construction allows us to construct a universal determinant functor on $\mathscr{E}$ taking values in the Picard groupoid associated to the spectrum $K(\mathscr{E})^{[0,1]}$. In this section, we shall assume all our categories are small.

5.1. **Picard groupoids and spectra.** In this section, we prove an equivalence between the homotopy category of $[0,1]$-connected spectra and that of Picard groupoids. This equivalence is well known to the experts ([Bei07]). On the other hand, we could not find a reference in the literature. We include a proof here for the sake of completeness. Recall that the category of spectra, denoted by $\mathscr{S}$, has two natural closed simplicial model category structures (strict and stable). We shall denote by $Ho(\mathscr{S})^{str}$ and $Ho(\mathscr{S})^{stb}$ the corresponding homotopy categories. A spectrum $P$ is $[0,1]$-connected if all its homotopy groups vanish except in degrees 0 and 1. We shall denote by $Ho(\mathscr{S}^{[0,1]})^{stb}$ the full subcategory corresponding to the $[0,1]$-connected spectra. We refer the reader to ([BF78]) for details on these model structures and their relation to $\Gamma$-spaces.

**Definition 5.1.1.** A Picard groupoid is a category $\mathscr{P}$ with the following additional data:

(1) A bifunctor $\otimes : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$.
(2) For $X,Y,Z \in Ob(\mathscr{P})$, associativity constraints $\phi_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ natural in $X,Y$, and $Z$.
(3) For $X$ and $Y$, commutativity constraints $\psi_{X,Y} : X \otimes Y \to Y \otimes X$.

We require these data to satisfy the following:

A: Every morphism in $\mathscr{P}$ is an isomorphism.
B: For all $W$, the functor $X \to X \otimes W$ is an autoequivalence.
C: $\psi_{X,Y} \circ \psi_{Y,X} = \mathrm{Id}_{X\otimes Y}$.
D: The following commutes (Pentagonal axiom):

$$\begin{array}{ccccc}
 & & (X\otimes Y)\otimes(Z\otimes T) & & \\
 & \swarrow{\scriptstyle \phi_{X,Y,Z\otimes T}} & & \searrow{\scriptstyle \phi^{-1}_{X\otimes Y,Z,T}} & \\
X\otimes(Y\otimes(Z\otimes T)) & & & & ((X\otimes Y)\otimes Z)\otimes T \\
\downarrow{\scriptstyle \mathrm{Id}_X\otimes\phi^{-1}_{Y,Z,T}} & & & & \downarrow{\scriptstyle \phi_{X,Y,Z}\otimes\mathrm{Id}_T} \\
X\otimes((Y\otimes Z)\otimes T) & & \xrightarrow{\phi_{X,Y\otimes Z,T}} & & (X\otimes(Y\otimes Z))\otimes T.
\end{array}$$

E: The following commutes (Hexagonal axiom):

$$\begin{array}{ccccc}
 & & X\otimes(Y\otimes Z) & & \\
 & \swarrow{\scriptstyle \phi^{-1}_{X,Y,Z}} & & \searrow{\scriptstyle \mathrm{Id}_X\otimes\psi_{Y,Z}} & \\
(X\otimes Y)\otimes Z & & & & X\otimes(Z\otimes Y) \\
\downarrow{\scriptstyle \psi_{X\otimes Y,Z}} & & & & \downarrow{\scriptstyle \phi^{-1}_{X,Z,Y}} \\
Z\otimes(X\otimes Y) & & & & (X\otimes Z)\otimes Y \\
 & \searrow{\scriptstyle \phi^{-1}_{Z,X,Y}} & & \swarrow{\scriptstyle \psi_{X,Z}\otimes\mathrm{Id}_Y} & \\
 & & (Z\otimes X)\otimes Y & & .
\end{array}$$

**Definition 5.1.2.** A *unit* in a Picard groupoid consists of $(\mathbb{I},\lambda,\rho)$ where $\mathbb{I}\in Ob(\mathscr{P})$, $\lambda_X : X \to \mathbb{I}\otimes X$ is a natural isomorphism and similarly for $\rho_X : X \to X\otimes\mathbb{I}$.

Every Picard groupoid has a unit which is unique up to unique isomorphism; we shall always assume our Picard groupoids already come equipped with a chosen unit. Let $\pi_0(\mathscr{P})$ be the abelian group of isomorphism classes of objects of $\mathscr{P}$ with the mulitplication given by $\otimes$ and $\pi_1(\mathscr{P}) = \mathrm{Aut}_{\mathscr{P}}(\mathbb{I})$. Let *Pic* denote the category of Picard groupoids. A morphism of Picard groupoids will be a functor preserving the monoidal structure. Such a morphism induces a map of the $\pi_0$ and $\pi_1$; we say that two Picard groupoids are *homotopic* if there is a morphism which induces an isomorphism on $\pi_0$ and $\pi_1$. Let $Ho(\mathscr{P})$ denote the corresponding homotopy category (i.e,. localized at the homotopy equivalences).

**Theorem 5.1.3.** *There is an equivalence of categories $F : Ho(\mathscr{S}^{[0,1]})^{stb} \to Ho(Pic)$. Furthermore, $F$ preserves $\pi_i$.*

To prove the theorem we first construct functors in both directions. These constructions are due to May and Segal, and are achieved via the Segal machine of $\Gamma$-spaces. Accordingly, we first recall some basic aspects of the theory (see [Seg74]). Let $\Gamma^o$ be the category of pointed finite sets and pointed maps. For $n \geq 0$, let $n_+ = \{0,1,\ldots,n\}$ with base point 0. Let $\underline{\mathrm{C}}$ be a pointed category with initial and terminal object $*$. A $\Gamma$-*object over* $\underline{C}$ is a functor $\Gamma^o \to \underline{C}$ such that $0_+$ maps to $*$. A $\Gamma$-*space* is a $\Gamma$-object over $\mathbf{Sset}_*$ (the category of pointed simplicial sets). A $\Gamma$-object is determined by its values on $n_+$; therefore, in practice we shall only specify its values on such sets. We'll denote by $\Gamma_{spc}$ the category of $\Gamma$-spaces.

Now we recall how to construct functors between $\Gamma_{spc}$ and $\mathscr{S}$, following Segal ([Seg74]). Let $A$ be a $\Gamma$-space. Then we can extend $A$ to a functor $\mathbf{Sset}_* \to \mathbf{Sset}_*$, also denoted $A$. First let $A(W) = \mathrm{colim}_{V \subset W} A(V)$, where $W$ is a pointed set and the limit is over all $V \in \Gamma^0$. If $K$ is a simplicial set, then $K[n]$ denotes the set of $n$-simplices. Now for $K \in \mathbf{Sset}_*$ define $A(K)[n] = A(K[n])[n]$; this gives a simplicial set $A(K)$. Given $K, L \in \mathbf{Sset}_*$ one has a natural morphism $L \wedge A(K) \to A(L \wedge K)$ where $x \wedge y \in L[n] \wedge A(K)[n]$ goes to the image of $y$ under the natural map $A(x \wedge -) : A(K[n])[n] \to A(L[n] \wedge K[n])$. Given a spectrum $P$ define a spectrum $A(P)$ by setting $A(P)_n = A(P_n)$ with structure maps $S^1 \wedge A(P_n) \to A(S^1 \wedge P_n) \to A(P_{n+1})$. Given spectra $P$ and $P'$ let $\Phi(P,P')$ be the $\Gamma$-space defined by $\Phi(P,P')(V) = \mathrm{Hom}_{\mathscr{S}}(P^V, P')$ for $V \in \Gamma^o$; thus, given a spectrum $P$ we can associate to it the $\Gamma$-space $\Phi(\mathbb{S}, P)$. We shall denote this $\Gamma$-space associated to a spectrum $P$ by $\Gamma(P)$. Conversely, given a $\Gamma$-space $A$ we can associate to it the spectrum $A(\mathbb{S})$. These constructions give us well defined functors and in fact induce equivalences of various homotopy categories.

A $\Gamma$-space $A$ is called *special* if for all $n \geq 1$ the maps $A(p_1) \times \cdots \times A(p_n) : A(n_+) \to A(1_+) \times \cdots \times A(1_+)$ are weak equivalences, where $p_i : n_+ \to 1_+$ is defined by $p_i(i) = 1$ and $p_i(j) = 0$ for all $j \neq i$. If $A$ is a special $\Gamma$-space then $\pi_0(A(1_+))$ is an abelian monoid with multiplication given by $\pi_0(A(1_+) \times \pi_0(A(1_+)) \leftarrow \pi_0(A(2_+)) \to \pi_0(A(1_+))$. The first arrow is induced by $A(p_1) \times A(p_2)$ and the second arrow is induced by $\mu : 2_+ \to 1_+$, given by sending 0 to 0, 1 to 1 and 2 to 1. A $\Gamma$-space $A$ is *very special* if it is special and $\pi_0(A(1_+))$ is an abelian group. It follows from [BF78] that $\Gamma(P)$ is very special if $P$ is an $\Omega$-spectrum. Furthermore, if $A$ is very special then $A(\mathbb{S})$ is an $\Omega$-spectrum.

Let $\mathscr{P}$ be a Picard groupoid. Given $n_+$ or more generally a finite pointed set $(S,*)$, let $\Gamma(\mathscr{P})(S)$ denote the category whose objects are collections $X_U$ for each $U \subset S \setminus \{*\}$, where $X_U$ is an object of $\mathscr{P}$, with isomorphisms $X_U \otimes X_V \to X_{U \cup V}$ for all disjoint $U$ and $V$ such that the following diagram commutes:

$$\begin{array}{ccc} X_U \otimes X_V & \longrightarrow & X_{U \cup V} \\ \downarrow & \nearrow & \\ X_V \otimes X_U & & \end{array}$$

Here the vertical arrow is given by the commutativity constraint. We also require that the given system of isomorphisms are compatible with pairwise disjoint triples, $X_\phi = \mathbb{I}$ and $X_\phi \otimes X_U \to X_U$ is the unit in $\mathscr{P}$. Then $\Gamma(\mathscr{P})$ is a $\Gamma$-category (i.e. a $\Gamma$-object over the category of categories). Let $K_\Gamma(\mathscr{P})$ denote the associated $\Gamma$-space obtained by taking nerves. We shall denote the associated spectrum by $K(\mathscr{P})$. See ([May74]) for details of this construction in the more general setting of permutative categories.

**Lemma 5.1.4.** *Let $\mathscr{P}$ be a Picard groupoid. Then the associated $\Gamma$-space $K_\Gamma(\mathscr{P})$ is very special.*

*Proof.* This follows directly from the definitions. The map $\Gamma(\mathscr{P})(n_+) \to \Gamma(\mathscr{P})(1_+) \times \cdots \times \Gamma(\mathscr{P})(1_+)$ is an equivalence of categories. It follows that the resulting map on nerves is a weak equivalence. Furthermore, since $\Gamma(\mathscr{P})(1_+) = \mathscr{P}$, $\pi_0(K_\Gamma(\mathscr{P})(1_+)) = \pi_0(\mathscr{P})$. The latter is a group by definition. □

**Lemma 5.1.5.** *We have $\pi_i(K(\mathscr{P})) = 0$ for all $i \neq 0,1$.*

*Proof.* By definition the spectrum $K(\mathscr{P})$ is given by $\Gamma(\mathscr{P})(\mathbb{S})$; i.e., the Gamma space associated to $\mathscr{P}$ evaluated at the sphere spectrum. Since this is an infinite loop space, $\pi_i(K(\mathscr{P})) = \pi_i(K(\mathscr{P})_0)$. The zeroth space $K(\mathscr{P})_0$ is equal to $N(\Gamma(\mathscr{P})(\mathbb{S})_0) = N(\Gamma(\mathscr{P})(S^0))$. The right side is the simplicial set associated to $S^0$ which is just $1_+$ (i.e., the pointed two point set). Since $\Gamma(\mathscr{P})(1_+) = \mathscr{P}$, $\pi_i(N(\mathscr{P})) = \pi_i(\mathscr{P})$ for $i = 0,1$, and $\pi_i(N(\mathscr{P})) = 0$ for all other $i \geq 2$, the result follows. □

Now suppose $A$ is a very special $\Gamma$-space. Then we can consider the Poincaré groupoid associated to the space $|A(1_+)|$. Recall that this is the groupoid whose objects are points in $|A(1_+)|$ and whose morphisms are homotopy classes of paths. If $P$ is an $\Omega$-spectrum, let $\Pi(P)$ denote the Poincaré groupoid of the associated very special $\Gamma$-space. In general, given a topological space $X$ we denote by $\Pi(X)$ the associated Poincaré groupoid. Note that, given an $\Omega$-spectrum $P$, we have described two ways to associate a groupoid to $P$. The first is to take the Poincaré groupoid associated to the very special $\Gamma$-space $\Gamma(P)$, and the other is to take the Poincaré groupoid of the geometric realization of the zeroth space $P_0$. Both of these procedures give the same groupoid (up to equivalence).

**Proposition 5.1.6.** *If $P$ is an $\Omega$-spectrum, then $\Pi(P)$ is a Picard groupoid.*

*Proof.* Let $P_0$ be the zeroth space. Then $P_0$ is an infinite loop space, and the structure maps give a homotopy equivalence $P_0 \to \Omega P_1$. Therefore, the induced map on Poincaré groupoids $\Pi(P_0) \to \Pi(\Omega P_1)$ is an equivalence of categories. A choice of homotopy inverse to $P_0 \to \Omega P_1$ gives rise to a quasi-inverse to $\Pi(P_0) \to \Pi(\Omega P_1)$. Therefore, it is enough to show that $\Pi(\Omega P_1)$ has the structure of a Picard groupoid. On the other hand, the $H$-space structure on $\Omega P_1$ induces a symmetric monoidal structure on the corresponding groupoid. Furthermore, it also has canonically defined homotopy inverse. In particular, all the properties of Definition 5.1 are satisfied. □

Note that the Picard groupoid structure constructed in Proposition 5.6 depends on the choice of a homotopy inverse to $P_0 \to \Omega P_1$. However, there is a canonical Picard groupoid structure on $\Omega P_1$. By abuse of notation, we shall continue to use the notation $\Pi(P)$ to denote the Picard groupoid $\Pi(\Omega P_1)$. Let $\mathscr{S}^{fc}$ denote the full sub-cateogry of fibrant-cofibrant spectra. Note that the fibrant objects are precisely the $\Omega$-spectra. Recall that we have fixed a fibrant-cofibrant replacement functor $\mathscr{S} \to \mathscr{S}^{fc}$. In particular, we have a functor $\Pi : \mathscr{S} \to Pic$. On the other hand, we have already constructed a functor $K : Pic \to \mathscr{S}$. It follows from Lemmas 5.5 and 5.6 that we have induced functors $\Pi : \mathscr{S}^{[0,1]} \leftrightarrow Pic : K$. To show that these descend to the homotopy categories, we must show that both $\Pi$ and $K$ preserve weak equivalences. This is an easy consequence of the following folklore result about Picard groupoids ([Bre10]).

**Lemma 5.1.7.** *([Bre10]) Let $\mathscr{P}$ and $\mathscr{P}'$ be Picard groupoids and $A = (M, c) : \mathscr{P} \to \mathscr{P}'$ a moniodal functor. Then the following are equivalent:*

(a): *$M$ is an equivalence of categories.*
(b): *$M$ is an equivalence of Picard groupoids.*
(c): *$M$ induces an isomorphism on $\pi_0$ and $\pi_1$.*

**Corollary 5.1.8.** *If $f : P \to Q$ is a homotopy equivalence of $[0,1]$-connected spectra, then $\Pi f : \Pi(P) \to \Pi(Q)$ is an equivalence of Picard groupoids. If $g : P \to Q$ is an equivalence of Picard groupoids then the resulting map $K(P) \to K(Q)$ is a weak equivalence. In particular, we have induced funtors $\Pi : Ho(\mathscr{S}^{[0,1]})^{stb} \leftrightarrow Ho(Pic) : K$.*

*Proof.* It is clear that both functors preserve homotopy groups. The second statement follows. Now, a morphism of $[0,1]$-connected spectra induces a monoidal functor of the associated groupoids. If the original morphism is a weak equivalence then the induced morphism on the homotopy groups of the corresponding Picard groupoids are isomorphisms. Lemma 5.7 now shows that the resulting morphism of Picard groupoids is an equivalence. This proves the first statement. □

*Proof.* (Theorem 5.3) Let $\mathscr{P}$ be a Picard groupoid. Then we have canonical equivalences of Picard groupoids $\Pi(K(\mathscr{P})_0) \to \mathscr{P}$ and $\Pi(K(\mathscr{P})_0) \to \Pi(\Omega K(\mathscr{P})_1)$; it follows that $\Pi$ is essentially surjective. If $L$ is a fibrant-cofibrant spectrum in $\mathscr{S}^{[0,1]}$, then we have a canonical isomorphism $L \to K(\Pi(L))$ in the homotopy category. It follows that $\Pi$ is full and faithful on the homotopy category. □

5.2. **Determinant Functors.** In this section we recall the notion of determinant functors on exact categories; the results are due to Deligne ([Del87]). A version of the theory for triangulated categories is due to Breuning ([Bre10]).

**Definition 5.2.1.** Let $\mathscr{E}$ be an exact category. Let $w$ be a class of morphisms in $\mathscr{E}$ closed under composition and containing the isomorphisms, and let $\mathscr{E}_w$ be the subcategory with morphisms restricted to $w$. Then a determinant functor for $(\mathscr{E}, w)$ is a pair $(F, \mathscr{P})$ where $\mathscr{P}$ is a Picard groupoid and $F = (F_1, F_2)$, where $F_1 : \mathscr{E}_w \to \mathscr{P}$ is a functor and $F_2$ is a rule which associates to every short exact sequence $\delta : 0 \to A \to B \to C \to 0$ an isomorphism $F_2(\delta) : F_1(B) \to F_1(A) \otimes F_2(C)$ such that:

(1) For every morphism of short exact sequences

$$\begin{array}{ccccccccc} \delta : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ \delta' : 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

where the vertical maps are in $w$, one has a commutative diagram

$$\begin{array}{ccc} F_1(B) & \longrightarrow & F_1(A)\otimes F_1(C) \\ \big\downarrow{\scriptstyle F_1(b)} & & \big\downarrow{\scriptstyle F_1(a)\otimes F_1(c)} \\ F_1(B') & \longrightarrow & F_1(A')\otimes F_1(C') \end{array}$$

where the rows are given by $F_2(\delta)$ and $F_2(\delta')$.

(2) For every commutative diagram of short exact sequences:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C' \\ \big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow & & \big\downarrow \\ A & \longrightarrow & C & \longrightarrow & B' \\ & & \big\downarrow & & \big\downarrow \\ & & A' & \xrightarrow{\mathrm{Id}} & A' \end{array}$$

one has a commutative diagram:

$$\begin{array}{ccc} F_1(C) & \xrightarrow{F_2} & F_1(A)\otimes F_1(B') \\ & & \big\downarrow{\scriptstyle \mathrm{Id}\otimes F_2} \\ \big\downarrow{\scriptstyle F_2} & & F_1(A)\otimes(F_1(C')\otimes F_2(A')) \\ & & \big\downarrow \\ F_1(B)\otimes F_1(A') & \longrightarrow & (F_1(A)\otimes F_1(C'))\otimes F_1(A') \end{array}$$

where the bottom row is given by $F_2\otimes\mathrm{Id}$.

(3) For every pair of short exact sequences $\delta : A \to A\oplus B \to C$ and $\delta' : B \to A\oplus B \to A$ the following diagram is commutative:

$$\begin{array}{ccccc} & & F_1(A\oplus B) & & \\ & \swarrow{\scriptstyle F_2(\delta)} & & \searrow{\scriptstyle F_2(\delta')} & \\ F_1(A)\otimes F_1(B) & & \longrightarrow & & F_1(B)\otimes F_1(A). \end{array}$$

A morphism of determinant functors $F = (F_1, F_2)$ and $G = (G_1, G_2)$ from $\mathscr{E}_w \to \mathscr{P}$ is a natural transformation $h : F_1 \to G_1$ such that $(h(A)\otimes h(B))\circ F_2(\delta) = G_2(\delta)\circ h(B) : F_1(B) \to G_1(A)\otimes G_1(C)$ for all short exact sequences $\delta$. The category of determinant functors from $(\mathscr{E}, w)$ to $\mathscr{P}$ is denoted by $det(\mathscr{E}_w, \mathscr{P})$. The objects of this category are determinant functors and morphisms are morphisms of determinant functors.

**Definition 5.2.2.** A universal determinant functor for $(\mathscr{E}, w)$ is a determinant functor $\lambda$ with values in a Picard groupoid $\mathscr{V}$ such that for every Picard groupoid $\mathscr{P}$ the functor $Hom_{\mathrm{Pic}}(\mathscr{V}, \mathscr{P}) \to det(\mathscr{E}_w, \mathscr{P})$ induced by composition with $\lambda$ is an equivalence of categories.

**Theorem 5.2.3.** *(Deligne) A universal determinant functor exists for $(\mathscr{E},w)$, where $w$ is the class of isomorphisms and $\mathscr{E}$ is a small exact category.*

**Remark 5.2.4.** 1: One can also define a notion of determinant functor on any triangulated category ([Bre10]). Furthermore, there is an analog of the above theorem in this setting as well.
2: One can associate to any exact category $\mathscr{E}$ the bounded derived category $D^b(\mathscr{E})$. One can show that any determinant functor on $(\mathscr{E},w)$ extends canonically to a determinant functor on $D^b(\mathscr{E})$ ([Knu02]).
3: Let $\pi_i(\mathscr{E}) = K_i(\mathscr{E})$. Then the universal determinant functor induces an isomorphism on $\pi_i$ for $i = 0,1$.

5.3. **Homotopy points and universal determinants.** In this section, we will use homotopy points to construct a universal determinant functor with values in $\Pi(K(\mathscr{E}))$. As already mentioned, the existence of universal determinants on exact categories has already been shown by Deligne. The point here is that, by using the fundamental groupoid associated to the $K$-theory spectrum, we get something which is canonically associated to the $K$-theory spectrum. In particular, various decompositions at the level of homotopy points of spectra will descend to the corresponding Picard groupoids. As a result, we will be able to relate decompositions of homotopy points to factorizations of determinants. The result in the language of DG-categories appear in the work of Beilinson ([Bei07]). The results here are essentially a rendering of those results in the language of Walhausen categories (applicable to the constructions in section 3 and 4).
Given an object $A$ in a Waldhausen category $W$ we can associate to it a homotopy point of $K(W)$. First, we recall some facts about the Waldhausen $K$-theory construction. Recall that to each category with cofibrations and weak equivalences $\mathscr{C}$ (i.e. each Waldhausen category), Waldhausen associates a simplicial category $S.\mathscr{C}$. Each $S_n\mathscr{C}$ is also a Waldhausen category, and the classifying space of the associated simplicial category of weak equivalences gives the $K$-theory spectrum. The $0^{th}$-space is $\Omega|wS.\mathscr{C}|$. Waldhausen shows that this is an infinite loop space (by iterating the $S.$ construction). Now $wS_1\mathscr{C} = w\mathscr{C}$ and $wS_0\mathscr{C}$ is the category with one object and one morphism. As observed by Waldhausen, it follows that the 1-skeleton is naturally isomorphic to $S^1 \wedge |w\mathscr{C}|$, which results in an inclusion $S^1 \wedge |w\mathscr{C}| \to |wS.\mathscr{C}|$. Therefore, by adjunction, one has a map $|w\mathscr{C}| \to \Omega|wS.\mathscr{C}|$. Now $|w\mathscr{C}|$ is itself an infinite loop space ($w\mathscr{C}$ is a symmetric monoidal category) and the above map is a morphism of loop spaces. An object of $\mathscr{C}$ gives rise to a point in $|w\mathscr{C}|$ and therefore in $\Omega|wS.\mathscr{C}|$. Now a point in an infinite loop space gives a homotopy point of the associated spectrum. Thus, we have associated to an object of a Waldhausen category a homotopy point of the associated $K$-theory spectrum. Finally, a weak equivalence in $\mathscr{C}$ gives rise to a path in the corresponding infinite loop space. This gives an identification of the corresponding homotopy points.

**Lemma 5.3.1.** *A homotopy point of an $\Omega$-spectrum $P$ gives rise to an object of the associated (unique up to unique isomorphism) Picard groupoid. Furthermore, an identification of homotopy points gives rise to a canonical isomorphism in the Picard groupoid.*

*Proof.* Let $K \wedge \mathbb{S} \to P$ be a homotopy point of $P$. Then we have an induced map on fundamental groupoids $\Pi(K \wedge S^0) \to \Pi(P)$. Since $K$ is contractible, the former groupoid is equivalent $\Pi(S^0)$. Therefore, we get a map from $\Pi(S^0) \to \Pi(P)$. Then we can consider the image of the unique non-identity object in $\Pi(S^0)$. Note, a different choice of contracting homotopy of $K$ will give another object of $\Pi(P)$ canonically isomorphic to the original one. A similar argument shows that an identification of homotopy points gives rise to a unique isomorphism between the corresponding objects. ☐

As a result of the lemma, we can define a functor $\mathrm{Det} : (C^b(\mathscr{E}), qis) \to \Pi(K(C^b(\mathscr{E})))$. The resulting functor is a universal determinant functor.

**Theorem 5.3.2.** *The functor* $\mathrm{Det} : (C^b(\mathscr{E}), qis) \to \Pi(K(C^b(\mathscr{E})))$ *is a determinant functor in the sense of definition 5.9.*

*Proof.* Let $\delta : 0 \to A \to B \to C \to 0$ be an exact sequence. This is a cofibration sequence in the corresponding Waldhausen category. Let $[A], [B]$, and $[C]$ denote the correspoding points in the infinite loop space $\Omega|wS.C^b(\mathscr{E})|$, and let $[A] + [C]$ denote the composition of the corresponding loops. Then the image in the corresponding Picard groupoid is $\mathrm{Det}(A) \otimes \mathrm{Det}(B)$. The cofibration sequence $\delta$ gives rise to a path from $[B]$ to $[A] + [C]$. This gives us an isomorphism $\mathrm{Det}(B) \to \mathrm{Det}(A) \otimes \mathrm{Det}(C)$. If $\delta'$ is another cofibration sequence and $H$ a morphism of cofibration sequences, then it follows from the definition of $\Omega|wS.C^b(\mathscr{E})|$ that we get a homotopy from the path given by $\delta$ to the one given by $\delta'$. In particular, the diagram in (1) of Definition 5.9 commutes. The remaining properties follow in a similar manner. ☐

**Remark 5.3.3.** Suppose $\mathscr{E}$ is an exact category and $A, B \in C^b(\mathscr{E})$. It follows from the proof above that the exact sequence $0 \to A \to A \oplus B \to B \to 0$ gives an identification of the two homotopy points $[A \oplus B]$ and the homotopy sum (Lemma 2.3) of the homotopy points $[A]$ and $[B]$ since this homotopy point corresponds to the composition $[A] + [B]$ in the corresponding infinite loop space.

**Corollary 5.3.4.** *The functor* $\mathrm{Det} : (C^b(\mathscr{E}), qis) \to \Pi(K(C^b(\mathscr{E})))$ *gives a universal determinant functor.*

*Proof.* Let $(\mathscr{P}^{uni}, det)$ be the universal determinant (which is known to exist) functor for $C^b(\mathscr{E})$. One has a commutatve diagram (by universality):

$$\begin{array}{ccc} C^b(\mathscr{E}) & \xrightarrow{det} & \mathscr{P}^{uni} \\ & \searrow^{det} & \downarrow \\ & & \Pi(K(C^b(\mathscr{E}))). \end{array}$$

Now, by construction, both the top and bottom rows induce an isomorphism on $\pi_0$ and $\pi_1$. Therefore the vertical arrow is a morphism of Picard groupoids which induces an isomorphism on homotopy, and hence, it is an equivalence of Picard groupoids. ☐

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