INTEGRAL MODELS OF $\mathbb{P}^1$ AND ANALYTIC DISTRIBUTION ALGEBRAS FOR GL_2

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Dedicated to Peter Schneider on the occasion of his sixtieth birthday.

Contents

1. Introduction

The purpose of this paper is to begin the study of connections between arithmetic differential operators on semistable integral and formal models of flag varieties on the one hand and locally analytic distribution algebras of p-adic reductive groups on the other hand. Here we only consider the case of the group GL_2 over $\mathbb{Z}_p$ and the corresponding flag variety is the projective line $\mathbb{P}^1_{\mathbb{Z}_p}$.

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These investigations are motivated by the wish to study locally analytic representations of p-adic groups geometrically. In [1] K. Ardakov and S. Wadsley work with 'crystalline' differential operators (of level zero) on the smooth model of the flag variety of a split reductive group. This is close in spirit to the classical localization theory of Beilinson-Bernstein [2] and Brylinski-Kashiwara [5]. In the paper [18] we have made a first step in merging the localization theory of Schneider-Stuhler for smooth representations [20] with that of [2]. A key ingredient is the embedding, first discovered by V. Berkovich, cf. [3], of the building in the non-archimedean analytic space $X^{\rm an}$ attached to the flag variety X (see also [19]). The connection between the building and $X^{\rm an}$ can also be seen in terms of formal models for the rigid analytic space $X^{\rm rig}$. Especially transparent is that relation for formal models of $\mathbb{P}^1$, cf. [12]. To better understand the significance of these models for representation theory, and its relation to distribution algebras, is the starting point for our work presented here.

Regarding distribution algebras, it turns out that the analytic distribution algebras as considered by M. Emerton in [8], are well suited to be compared to arithmetic differential operators. Not surprisingly, Emerton has introduced and studied these rings having Berthelot's theory of arithmetic differential operators in mind, cf. [8, sec. 5.2]. Arithmetic differential operators on integral smooth models and their completions have been studied by C. Noot-Huyghe in [10], [17], [11]. In particular, she proves that these smooth formal models are $\mathscr{D}^\dagger$-affine. Here we take up her work in [10], in the special (and easy) case of $\mathbb{P}^1$ and show that the ring of global sections of the arithmetic differential operators is isomorphic to the analytic distribution algebra $\mathcal{D}^{an}(\mathbb{G}(0)^\circ)$ of the 'wide open' rigid-analytic group $\mathbb{G}(0)^\circ$ whose $\mathbb{C}_p$-valued points are $\mathbb{G}(0)^\circ(\mathbb{C}_p) = 1 + {\rm M}_2(\mathfrak{m}_{\mathbb{C}_p})$. Let $\mathfrak{X}$ be the completion of $\mathbb{P}^1_{\mathbb{Z}_p}$ along its special fiber. Our first result is

Theorem 1. (Thm. 3.2.2) *There is a canonical isomorphism of (topological) $\mathbb{Q}_p$-algebras*

$$\mathcal{D}^{\rm an}(\mathbb{G}(0)^\circ)_{\theta_0} \simeq H^0(\mathfrak{X}, \mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}}) \ . \qquad \square$$

Here, the subscript θ_0 indicates a central reduction. The proof of this theorem consists of two parts. Firstly, we identify the analytic distribution algebra $\mathcal{D}^{\rm an}(\mathbb{G}(0)^\circ)$ with the inductive limit (over m) of completed 'restricted divided power enveloping algebras' $\widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}}$ (of level m) of $\mathfrak{g}_{\mathbb{Z}_p} = \mathfrak{gl}_2(\mathbb{Z}_p)$. Secondly, we relate the algebras $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}}$ to the global sections $H^0(\mathbb{P}^1_{\mathbb{Z}_p}, \mathcal{D}^{(m)})$ of the sheaf of integral differential operators $\mathcal{D}^{(m)}$ of level m. Much of what we do in this part of the proof (sec. 3) is already contained in [10]. We have chosen to redo most of the arguments here, in an entirely explicit manner, because the arguments and techniques will be used later in sections 4 and 5.

After having obtained theorem 1 we have been informed by C. Noot-Huyghe that she has proved the general case of this theorem, for an arbitrary split reductive group and the corresponding smooth formal model of the flag variety, in an unpublished manuscript.

Furthermore, we give a description of the analytic distribution algebras $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$ of rigid-analytic wide open congruence subgroups $\mathbb{G}(n)^\circ$. Their $\mathbb{C}_p$-valued points are given by $\mathbb{G}(n)^\circ(\mathbb{C}_p) = 1 + p^n \mathrm{M}_2(\mathfrak{m}_{\mathbb{C}_p})$. The description of the distribution algebras is close to that in [8, sec. 5.2], but more suited to the material treated in the second part of this paper, i.e., sections 4 and 5.

In these sections we consider certain semistable integral models $\mathbb{X}_n$ of $\mathbb{P}^1_{\mathbb{Z}_p}$, and we study the sheaves $\mathcal{D}^{(m)}_{\mathbb{X}_n}$ of logarithmic differential operators of level m on these schemes. Denote by $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge$ the p-adic completion of $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})$, and put $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge_{\mathbb{Q}} = H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}$. Then we show

Theorem 2. (Thm. 5.2.1) *Given $n \geqslant 0$ let $n' = \lfloor n\frac{p-1}{p+1} \rfloor$ be the greatest integer less or equal to $n\frac{p-1}{p+1}$. Then we have natural inclusions*

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0} \hookrightarrow \varinjlim_m H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge_{\mathbb{Q}} \hookrightarrow \mathcal{D}^{\mathrm{an}}(\mathbb{G}(n')^\circ)_{\theta_0} .$$

□

Let $\mathfrak{X}_n$ be the formal completion of $\mathbb{X}_n$ along its special fiber. Then there is a canonical injection $\varinjlim_m H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge_{\mathbb{Q}} \hookrightarrow H^0(\mathfrak{X}_n, \mathscr{D}^\dagger_{\mathfrak{X}_n,\mathbb{Q}})$. We do not treat here the question whether this inclusion is in fact an isomorphism. This problem is related to the question whether the schemes $\mathbb{X}_n$ (resp. formal schemes $\mathfrak{X}_n$) are $\mathcal{D}$-affine, a topic we plan to discuss in a future paper.

Acknowledgements. The reader will have no difficulty in recognising the influence of Peter Schneider's work on the ideas contained in this paper. Over the many years we have spent together in Münster, we have greatly benefited from Peter's generosity in sharing his ideas with us and guiding us into many different mathematical worlds. We are grateful for this. It is a pleasure to dedicate this paper to him on the occasion of his sixtieth birthday.

Notation. If L is a field equipped with a non-archimedean absolute value we let $\mathfrak{o}_L$ be its valuation ring and $\mathfrak{m}_{\mathfrak{o}_L} \subset \mathfrak{o}_L$ the maximal ideal of its valuation ring. We let $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$ be the set of non-negative integers. If $\nu = (\nu_1, \ldots, \nu_d)$ is a tuple of integers, then we put $|\nu| = \nu_1 + \ldots + \nu_d$.

2. Distribution algebras of wide open congruence subgroups

2.1. **The group schemes $\mathbb{G}(n)$.** Let $n \geqslant 0$ always denote a non-negative integer. Put

$$\mathbb{G}(0) = \mathbb{G} = \mathrm{GL}_{2,\mathbb{Z}_p} = \mathrm{Spec}\left(\mathbb{Z}_p\left[a, b, c, d, \frac{1}{\Delta}\right]\right) ,$$

where $\Delta = ad - bc$, and the co-multiplication is the one given by the usual formulas. For $n \geqslant 1$ let a_n, b_n, c_n, and d_n denote indeterminates. Define an affine group scheme $\mathbb{G}(n)$ over $\mathbb{Z}_p$ by setting

$$\mathcal{O}(\mathbb{G}(n)) = \mathbb{Z}_p\left[a_n, b_n, c_n, d_n, \frac{1}{\Delta_n}\right] \ , \ \text{ where } \ \Delta_n = (1+p^n a_n)(1+p^n d_n) - p^{2n} b_n c_n \ ,$$

and let the co-multiplication

$$\mathcal{O}(\mathbb{G}(n)) \longrightarrow \mathcal{O}(\mathbb{G}(n)) \otimes_{\mathbb{Z}_p} \mathcal{O}(\mathbb{G}(n)) = \mathbb{Z}_p\left[a_n, b_n, c_n, d_n, a'_n, b'_n, c'_n, d'_n, \frac{1}{\Delta_n}, \frac{1}{\Delta'_n}\right]$$

be given by the formulas

$$\begin{array}{ccccccccc} a_n & \mapsto & a_n & + & a'_n & + & p^n a_n a'_n & + & p^n b_n c'_n \ , \\ b_n & \mapsto & b_n & + & b'_n & + & p^n a_n b'_n & + & p^n b_n d'_n \ , \\ c_n & \mapsto & c_n & + & c'_n & + & p^n c_n a'_n & + & p^n d_n c'_n \ , \\ d_n & \mapsto & d_n & + & d'_n & + & p^n d_n d'_n & + & p^n c_n b'_n \ . \end{array}$$

These group schemes are connected by homomorphisms $\mathbb{G}(n) \to \mathbb{G}(n-1)$ given on the level of algebras as follows:

$$a_{n-1} \mapsto pa_n \ , \quad b_{n-1} \mapsto pb_n \ , \quad c_{n-1} \mapsto pc_n \ , \quad d_{n-1} \mapsto pd_n \ ,$$

if $n > 1$. For $n = 1$ we put

$$a \mapsto 1 + pa_1 \ , \quad b \mapsto pb_1 \ , \quad c \mapsto pc_1 \ , \quad d \mapsto 1 + pd_1 \ .$$

For a flat $\mathbb{Z}_p$-algebra R the homomorphism $\mathbb{G}(n) \to \mathbb{G}(0) = \mathbb{G}$ induces an isomorphism of $\mathbb{G}(n)(R)$ with a subgroup of $\mathbb{G}(R)$, namely

$$\mathbb{G}(n)(R) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathbb{G}(R) \ \middle| \ a-1, b, c, d-1 \in p^n R \right\} \ .$$

Of course, the preceding formulas defining the group schemes are derived formally from this description by setting $a = 1 + p^n a_n$, $b = p^n b_n$, $c = p^n c_n$, and $d = 1 + p^n d_n$.

2.2. **The rigid-analytic groups $\mathbb{G}(n)^{\mathrm{rig}}$ and $\mathbb{G}(n)^{\circ}$.** Let $\widehat{\mathbb{G}}(n)$ be the completion of $\mathbb{G}(n)$ along its special fiber $\mathbb{G}(n)_{\mathbb{F}_p}$. This is a formal group scheme over $\mathrm{Spf}(\mathbb{Z}_p)$. Its generic fiber in the sense of rigid geometry is an affinoid rigid-analytic group over $\mathbb{Q}_p$ which we denote by $\mathbb{G}(n)^{\mathrm{rig}}$. We have for any completely valued field $L/\mathbb{Q}_p$ (whose valuation extends the p-adic valuation)

$$\mathbb{G}(n)^{\mathrm{rig}}(L) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{G}(\mathfrak{o}_L) \;\middle|\; a-1, b, c, d-1 \in p^n \mathfrak{o}_L \right\} .$$

Furthermore, we let $\widehat{\mathbb{G}}(n)^\circ$ be the completion of $\mathbb{G}(n)$ in the closed point corresponding to the unit element in $\mathbb{G}(n)_{\mathbb{F}_p}$. This is a formal group scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ (not of topologically finite type). Its generic fiber in the sense of Berthelot, cf. [6, sec. 7.1], is a so-called 'wide open' rigid-analytic group over $\mathbb{Q}_p$ which we denote by $\mathbb{G}(n)^\circ$. We have for any completely valued field $L/\mathbb{Q}_p$ (whose valuation extends the p-adic valuation)

$$\mathbb{G}(n)^\circ(L) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{G}(\mathfrak{o}_L) \;\middle|\; a-1, b, c, d-1 \in p^n \mathfrak{m}_{\mathfrak{o}_L} \right\} .$$

The remainder of this section is inspired by M. Emerton's paper [8], especially sec. 5.

2.3. **The analytic distribution algebra of $\mathbb{G}(0)^\circ$.** Our goal in this subsection is to give a description of

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ) \underset{\mathrm{df}}{=} \mathcal{O}(\mathbb{G}(0)^\circ)'_b$$

in terms of 'divided power enveloping algebras' which is analogous to [8, 5.2.6]. However, the discussion in [8, sec. 5.2] does not apply here because the exponential function for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ does not map a lattice in

$$\mathfrak{g} \underset{\mathrm{df}}{=} \mathrm{Lie}(\mathrm{GL}_2(\mathbb{Q}_p))$$

bijectively onto $\mathrm{GL}_2(\mathbb{Z}_p)$. Nevertheless, it is possible to also treat $\mathbb{G}(0)^\circ$ by making use of the 'Kostant $\mathbb{Z}$-form' of the enveloping algebra $U(\mathfrak{g})$, cf. [15]. Set

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} , \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

and put

$$\mathfrak{g}_{\mathbb{Z}_p} \underset{\mathrm{df}}{=} \mathrm{M}_2(\mathbb{Z}_p) = \mathbb{Z}_p e \oplus \mathbb{Z}_p h_1 \oplus \mathbb{Z}_p h_2 \oplus \mathbb{Z}_p f .$$

For integers $m, n \in \mathbb{N}$ define

$$q_n^{(m)} \underset{\mathrm{df}}{=} \left\lfloor \frac{n}{p^m} \right\rfloor ,$$

that is, the greatest integer less or equal to $\frac{n}{p^m}$. For fixed m we then denote by $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ the $\mathbb{Z}_p$-submodule of $U(\mathfrak{g})$ generated by the elements

$$q^{(m)}_{\nu_1}!\frac{e^{\nu_1}}{\nu_1!}\cdot q^{(m)}_{\nu_2}!\binom{h_1}{\nu_2}\cdot q^{(m)}_{\nu_3}!\binom{h_2}{\nu_3}\cdot q^{(m)}_{\nu_4}!\frac{f^{\nu_4}}{\nu_4!}\ . \tag{2.3.1}$$

Lemma 2.3.2. *$U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ is a $\mathbb{Z}_p$-subalgebra of $U(\mathfrak{g})$.*

Proof. This is contained in [16, Prop. 2.3.1] and the remark before [16, Lemme 2.3.3], namely that $\mathcal{D}^{(m)}_{X,n}$ has a basis given by the operators $\partial_{<\underline{k}>}$, $|\underline{k}|\leqslant n$. Note also the description of $\partial_{<\underline{k}>}$ given in part (c) of that lemma. □

We now let $\widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ be the p-adic completion of $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$. Explicitly, its elements can be written as

$$\sum_{\nu=(\nu_1,\nu_2,\nu_3,\nu_4)\in\mathbb{N}^4}\gamma_\nu\cdot q^{(m)}_{\nu_1}!\frac{e^{\nu_1}}{\nu_1!}\cdot q^{(m)}_{\nu_2}!\binom{h_1}{\nu_2}\cdot q^{(m)}_{\nu_3}!\binom{h_2}{\nu_3}\cdot q^{(m)}_{\nu_4}!\frac{f^{\nu_4}}{\nu_4!}\ ,$$

where $\gamma_\nu\in\mathbb{Z}_p$ and $|\gamma_\nu|\to 0$ as $|\nu|\to\infty$. Furthermore, we put

$$\widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}} \underset{\mathrm{df}}{=} \widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\otimes_{\mathbb{Z}}\mathbb{Q}\ .$$

We consider the unique $\mathbb{Q}_p$-algebra homomorphism $U(\mathfrak{g})\to\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)$ which sends $X\in\mathfrak{g}$ to the linear form

$$f\mapsto \frac{d}{dt}f(e^{tX})\Big|_{t=0}\ .$$

Here we follow the same convention as in [8, sec. 5] in that we consider the right regular action of a group on its ring of functions.

Proposition 2.3.3. *The map $U(\mathfrak{g})\to\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)$ just defined extends continuously to $\widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$. The family of these maps, for various m, induces a canonical isomorphism of topological $\mathbb{Q}_p$-algebras*

$$\varinjlim_m \widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}}\xrightarrow{\simeq}\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)\ .$$

Proof. The affine algebra of the formal group scheme $\widehat{\mathbb{G}}(0)^\circ$ is the completion of the ring $\mathbb{Z}_p\left[a,b,c,d,\frac{1}{\Delta}\right]$ with respect to the ideal $I=(p,a-1,b,c,d-1)$. (We write here a instead of a_0, b instead of b_0, etc.) Hence

$$\mathcal{O}\left(\widehat{\mathbb{G}}(0)^\circ\right)=\mathbb{Z}_p[[a-1,b,c,d-1]]\ .$$

For the ring of global functions of $\mathbb{G}(0)^\circ$ we then have, algebraically and topologically,

$$\mathcal{O}\left(\mathbb{G}(0)^\circ\right) = \varprojlim_{r<1} \mathcal{O}\left(\mathbb{G}(0)_r\right) \ ,$$

where

$$\mathcal{O}\left(\mathbb{G}(0)_r\right) = \left\{ \sum \xi_\mu (a-1)^{\mu_1} b^{\mu_2} c^{\mu_3} (d-1)^{\mu_4} \ \middle| \ |\xi_\mu| r^{|\mu|} \to 0 \text{ as } |\mu| \to \infty \right\} \ .$$

It is easily checked that

$$\left[\frac{e^{\nu_1}}{\nu_1!}\binom{h_1}{\nu_2}\binom{h_2}{\nu_3}\frac{f^{\nu_4}}{\nu_4!}\right] . \left[(a-1)^{\mu_1} b^{\mu_2} c^{\mu_3} (d-1)^{\mu_4}\right] = \left\{ \begin{array}{ccc} 1 & , & \nu = \mu \\ 0 & , & \nu \neq \mu \end{array} \right. \ .$$

We thus find that $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)$ consists of sums

$$\sum_{\nu=(\nu_1,\nu_2,\nu_3,\nu_4)\in\mathbb{N}^4} \gamma_\nu \frac{e^{\nu_1}}{\nu_1!}\binom{h_1}{\nu_2}\binom{h_2}{\nu_3}\frac{f^{\nu_4}}{\nu_4!} \ ,$$

which have the property that there is $R > 1$ for which $|\gamma_\nu| R^{|\nu|} \to 0$ as $|\nu| \to \infty$. The rest of the proof is as in [8, 5.2.6]. Because

$$v_p\left(q^{(m)}_{\nu_1}! q^{(m)}_{\nu_2}! q^{(m)}_{\nu_3}! q^{(m)}_{\nu_4}!\right)$$

is asymptotic to

$$\frac{\nu_1+\nu_2+\nu_3+\nu_4}{(p-1)p^m} \quad \text{as} \quad |\nu| \to \infty \ ,$$

it follows that $\widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}}$ embeds into $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)$. Furthermore, the inductive limit of the spaces $\mathcal{O}(\mathbb{G}(0)_r)'_b$, for $r \uparrow 1$, is equal to the the inductive limit of the rings $\widehat{U}(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}}$, as $m \to \infty$. □

Remark 2.3.4. The Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ is nothing else than the distribution algebra $\mathrm{Dist}(\mathrm{GL}_{2,\mathbb{Z}_p})$ of the group scheme $\mathrm{GL}_{2,\mathbb{Z}_p}$ as defined in [13, I.7], cf. [13, II.1.12] for the explicit relation between the Kostant $\mathbb{Z}$-form and the distribution algebra. One can then use the very definition of the distribution algebra in [13, I.7] to give an intrinsic proof of 2.3.3 which should generalize to any split reductive group scheme over $\mathbb{Z}_p$. □

2.4. **The analytic distribution algebra of $\mathbb{G}(n)^\circ$ for $n \geqslant 1$.** In this subsection we derive a description of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ) \underset{\mathrm{df}}{=} \mathcal{O}(\mathbb{G}(n)^\circ)'_b$, for $n \geqslant 1$, from the decription in 2.3.3. The open embedding of rigid spaces $\mathbb{G}(n)^\circ \hookrightarrow \mathbb{G}(0)^\circ$ induces a restriction map on spaces of functions $\mathcal{O}(\mathbb{G}(0)^\circ) \to \mathcal{O}(\mathbb{G}(n)^\circ)$ which has dense image. Taking the continuous dual spaces gives hence an injection

$$\mathcal{D}^{\rm an}(\mathbb{G}(n)^\circ) \hookrightarrow \mathcal{D}^{\rm an}(\mathbb{G}(0)^\circ) \;.$$

We will describe the left hand side as a subalgebra of the right hand side. To this end, let $U(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ be the $\mathbb{Z}_p$-submodule of $U(\mathfrak{g})$ generated by the elements

$$q^{(m)}_{\nu_1}!\frac{(p^ne)^{\nu_1}}{\nu_1!} \cdot q^{(m)}_{\nu_2}!p^{n\nu_2}\binom{h_1}{\nu_2} \cdot q^{(m)}_{\nu_3}!p^{n\nu_3}\binom{h_2}{\nu_3} \cdot q^{(m)}_{\nu_4}!\frac{(p^nf)^{\nu_4}}{\nu_4!} \;. \tag{2.4.1}$$

As before, we find that $U(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ is a $\mathbb{Z}_p$-subalgebra of $U(\mathfrak{g})$, and we let $\widehat{U}(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ denote its p-adic completion.

Remark 2.4.2. We caution the reader that $U(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ and $\widehat{U}(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ are merely notations. That is, these rings are not what one would get by formally replacing (the basis of) $\mathfrak{g}_{\mathbb{Z}_p}$ by (the basis of) $p^n\mathfrak{g}_{\mathbb{Z}_p}$ in the definition of $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$. The reason is that, obviously,

$$\binom{p^nh_i}{\nu} \neq p^{n\nu}\binom{h_i}{\nu} \;,$$

if $\nu > 1$. It is the term on right which one has to work with here, not the term on the left. □

The algebra homomorphism $U(\mathfrak{g}) \rightarrow \mathcal{D}^{\rm an}(\mathbb{G}(0)^\circ)$ defined right before 2.3.3 obviously factors as $U(\mathfrak{g}) \rightarrow \mathcal{D}^{\rm an}(\mathbb{G}(n)^\circ) \rightarrow \mathcal{D}^{\rm an}(\mathbb{G}(0)^\circ)$.

Proposition 2.4.3. *The map $U(\mathfrak{g}) \rightarrow \mathcal{D}^{\rm an}(\mathbb{G}(n)^\circ)$ extends continuously to $\widehat{U}(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ and there is a canonical isomorphism of topological $\mathbb{Q}_p$-algebras*

$$\varinjlim_m \widehat{U}(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}} \stackrel{\simeq}{\longrightarrow} \mathcal{D}^{\rm an}(\mathbb{G}(n)^\circ) \;.$$

Proof. We proceed here as in the proof of 2.3.3. The affine algebra of the formal group scheme $\widehat{\mathbb{G}}(n)^\circ$ is $\mathbb{Z}_p[[a_n,b_n,c_n,d_n]]$ and the coordinates a_n,b_n,c_n,d_n on $\mathbb{G}(n)^\circ$ are related to the coordinates a,b,c,d on $\mathbb{G}(0)^\circ$ by

$$a_n = \frac{1}{p^n}(a-1) \;, \quad b_n = \frac{1}{p^n}b \;, \quad c_n = \frac{1}{p^n}c \;, \quad d_n = \frac{1}{p^n}(d-1) \;.$$

¿From the proof of 2.3.3 we get

$$\left[\tfrac{(p^ne)^{\nu_1}}{\nu_1!}p^{n\nu_2}\tbinom{h_1}{\nu_2}p^{n\nu_3}\tbinom{h_2}{\nu_3}\tfrac{(p^nf)^{\nu_4}}{\nu_4!}\right] . \left[\left(\tfrac{a-1}{p^n}\right)^{\mu_1}\left(\tfrac{b}{p^n}\right)^{\mu_2}\left(\tfrac{c}{p^n}\right)^{\mu_3}\left(\tfrac{d-1}{p^n}\right)^{\mu_4}\right] = \begin{cases} 1 & , \quad \nu = \mu \\ 0 & , \quad \nu \neq \mu \end{cases} \;.$$

And the remainder of the proof is along the same lines as in 2.3.3. □

Remark 2.4.4. For $n \geqslant 1$ ($n \geqslant 2$ if $p = 2$) the group $\mathbb{G}(n)(\mathbb{Z}_p) = 1 + p^n \mathrm{M}_2(\mathbb{Z}_p)$ is uniform pro-p and its integral Lie algebra in the sense of [7, sec. 9] is $p^n \mathfrak{g}_{\mathbb{Z}_p}$ when considered as a $\mathbb{Z}_p$-submodule of $\mathfrak{g}$. We can thus apply [8, sec. 5.2] to get a description of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$ in terms of divided power enveloping algebras. The relation between the two descriptions is as follows.

In [8], $\mathbb{G}(n)^\circ$ is identified with the rigid-analytic four-dimensional wide open polydisc $(\mathbb{B}^\circ)^4$ via the 'coordinates of the second kind'

$$(t_1, t_2, t_3, t_4) \mapsto \exp(t_1 p^n e) \exp(t_2 p^n h_1) \exp(t_3 p^n h_2) \exp(t_4 p^n f) \ .$$

Functions $\mathcal{O}(\mathbb{G}(n)^\circ)$ are then considered as functions on $(\mathbb{B}^\circ)^4$ via pull-back. Using this identification, we consider elements in $U(\mathfrak{g})$ as differential operators on $\mathcal{O}\left((\mathbb{B}^\circ)^4\right)$. [8, 5.2.6] then tells us that $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$ is the inductive limit of rings

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)^{(m)}$$

$$\underset{\mathrm{df}}{=} \left\{ \sum_\nu \gamma_\nu \frac{q_{\nu_1}^{(m)}! q_{\nu_2}^{(m)}! q_{\nu_3}^{(m)}! q_{\nu_4}^{(m)}!}{\nu_1! \nu_2! \nu_3! \nu_4!} (p^n e)^{\nu_1} (p^n h_1)^{\nu_2} (p^n h_2)^{\nu_3} (p^n f)^{\nu_4} \,\middle|\, |\gamma_\nu| \to 0 \text{ as } |\nu| \to 0 \right\} \ .$$

The relation of these rings to the rings $\widehat{U}(p^n \mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\mathbb{Q}_p}$ follows immediately from the elementary

Proposition 2.4.5. *Suppose $n \geqslant 1$ ($n \geqslant 2$ if $p = 2$), and let T be an indeterminate. For all $\nu \geqslant 0$, if one writes the polynomial $p^{n\nu}\binom{T}{\nu}$ as*

$$\sum_{j=1}^{\nu} c_{\nu,j} \frac{(p^n T)^j}{j!} \ ,$$

the coefficients $c_{\nu,j}$ are in $\mathbb{Z}_p$.

Proof. Let z be another indeterminate and consider the formal power series

$$\sum_{\nu \geqslant 0} p^{n\nu} \binom{T}{\nu} z^\nu \ .$$

This is equal to $(1 + p^n z)^T = \exp(T \log(1 + p^n z))$. Under the assumption $n \geqslant 1$ ($n \geqslant 2$ if $p = 2$), one can write $\log(1 + p^n z) = p^n z f(z)$ with a power series $f(z) \in \mathbb{Z}_p[[z]]$. Hence

$$\exp(T \log(1 + p^n z)) = \sum_{j \geqslant 0} (z f(z))^j \frac{(p^n T)^j}{j!} \ .$$

Now compare the coefficients of z^ν on both sides. ∎

3. Arithmetic differential operators on the smooth formal model

3.1. **Differential operators with divided powers.** We consider $\mathbb{X} \underset{\mathrm{df}}{=} \mathbb{P}^1_{\mathbb{Z}_p}$ as being glued together from the affine lines

$$U_x = \mathrm{Spec}(\mathbb{Z}_p[x]) \quad \text{and} \quad U_y = \mathrm{Spec}(\mathbb{Z}_p[y])$$

along the open subsets $\mathrm{Spec}(\mathbb{Z}_p[x, \frac{1}{x}])$ and $\mathrm{Spec}(\mathbb{Z}_p[y, \frac{1}{y}])$ according to the relation $xy = 1$. The formulas

$$x.\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{b+dx}{a+cx} \ , \quad y.\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{ay+c}{by+d} \ ,$$

describe a *right* action of $\mathbb{G} = \mathrm{GL}_{2,\mathbb{Z}_p} = \mathrm{Spec}\left(\mathbb{Z}_p\left[a,b,c,d,\frac{1}{\Delta}\right]\right)$ on $\mathbb{X}$. Put $\partial_x = \frac{d}{dx}$ and $\partial_y = \frac{d}{dy}$. These differential operators satisfy the relations

$$\partial_x = -y^2\partial_y \ , \ \ x\partial_x = -y\partial_y \ , \ \ x^2\partial_x = -\partial_y \ .$$

Denote by $\mathcal{T}_{\mathbb{X}}$ the tangent sheaf of $\mathbb{X}$ (over $\mathbb{Z}_p$). The action above gives rise to a homomorphism of Lie algebras

$$\mathfrak{g}_{\mathbb{Z}_p} \to H^0(\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \ , \tag{3.1.1}$$

which is explicitly given by

$$\begin{array}{rcl} e & \mapsto & \partial_x \\ h_1 & \mapsto & -x\partial_x \\ h_2 & \mapsto & x\partial_x \\ f & \mapsto & \partial_y \end{array}$$

On $\mathbb{X}$ we consider the sheaf of differential operators $\mathcal{D}^{(m)}_{\mathbb{X}}$ as defined in [4], [10]. Sections are locally given as finite sums

$$\sum_\nu \gamma_\nu \frac{q_\nu^{(m)}!}{\nu!}\partial_x^\nu \quad \text{or} \quad \sum_\nu \gamma'_\nu \frac{q_\nu^{(m)}!}{\nu!}\partial_y^\nu$$

with $\gamma_\nu \in \mathbb{Z}_p[x]$ and $\gamma'_\nu \in \mathbb{Z}_p[y]$, respectively. The sheaf $\mathcal{D}^{(m)}_{\mathbb{X}}$ is filtered by subsheaves $\mathcal{D}^{(m)}_{\mathbb{X},d}$ of differential operators of degree $\leqslant d$. Furthermore, for the symmetric algebra $\mathrm{Sym}(\mathcal{T}_{\mathbb{X}}) = \bigoplus_{d\geqslant 0} \mathcal{T}_{\mathbb{X}}^{\otimes d}$ there exists a divided power version

$$\mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)} = \bigoplus_{d\geqslant 0} (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)} \;,$$

cf. [10]. The sheaf $(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}$ in degree d is, as $\mathcal{O}_{\mathbb{X}}$-module, locally generated by

$$\frac{q_{i_1}^{(m)}!}{i_1!} s_1^{\otimes i_1} \cdot \ldots \cdot \frac{q_{i_r}^{(m)}!}{i_r!} s_r^{\otimes i_r} \;, \tag{3.1.2}$$

where $i_1 + \ldots + i_r = d$ and $s_1, \ldots, s_r$ are local sections of $\mathcal{T}_{\mathbb{X}}$. There is an obvious monomorphism of sheaves

$$\mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)} \hookrightarrow \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(0)}_{\mathbb{Q}} = \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(0)} \otimes_{\mathbb{Z}} \mathbb{Q} \;. \tag{3.1.3}$$

Lemma 3.1.4. *The image of the subsheaf*

$$(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)} \subset \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}$$

under the map 3.1.3 is equal to

$$\frac{q_d^{(m)}!}{d!} \mathcal{T}_{\mathbb{X}}^{\otimes d} \subset \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(0)}_{\mathbb{Q}} \;.$$

Therefore,

$$\mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)} = \bigoplus_{d\geqslant 0} \frac{q_d^{(m)}!}{d!} \mathcal{T}_{\mathbb{X}}^{\otimes d} \;.$$

Proof. Because $\mathcal{T}_{\mathbb{X}}^{\otimes d}$ is locally free of rank one, we can write the local sections s_i in 3.1.2 as $s_i = f_i \cdot s$ with a local generator s of $\mathcal{T}^{\otimes d}$ and local sections f_i of $\mathcal{O}_{\mathbb{X}}$. Hence we assume $s_i = s$ for $i = 1, \ldots, r$. Moreover, for any $i, j \geqslant 0$ one has that

$$\frac{(i+j)!}{i!j!} \left(\frac{q_{i+j}^{(m)}!}{q_i^{(m)}! q_j^{(m)}!} \right)^{-1} \in \mathbb{Z}_p \;, \tag{3.1.5}$$

cf. [10, sec. 1]. Applying this fact repeatedly shows that

$$\frac{q_{i_1}^{(m)}!}{i_1!} \cdot \ldots \cdot \frac{q_{i_r}^{(m)}!}{i_r!} \in \frac{q_d^{(m)}!}{d!} \mathbb{Z}_p \;,$$

and this proves the assertion of the lemma. □

Lemma 3.1.6. *Fix $d \geqslant 1$. The map sending $\frac{q_d^{(m)}!}{d!}\partial_x^d$ (resp. $\frac{q_d^{(m)}!}{d!}\partial_y^d$), considered as a local generator of $\mathcal{D}^{(m)}_{\mathbb{X},d}$ to $\frac{q_d^{(m)}!}{d!}\partial_x^{\otimes d}$ (resp. $\frac{q_d^{(m)}!}{d!}\partial_y^{\otimes d}$), considered as a local generator of $(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}$, induces a canonical exact sequence of sheaves*

$$0 \to \mathcal{D}^{(m)}_{\mathbb{X},d-1} \to \mathcal{D}^{(m)}_{\mathbb{X},d} \to (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)} \to 0\ . \tag{3.1.7}$$

Proof. This is [10, 1.3.7.3]. In the case considered here, it is also an immediate consequence of 3.1.4. □

Proposition 3.1.8. *(a) For all $d \geqslant 0$ one has $H^1(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d}) = 0$.*

(b) For all $d \geqslant 1$ the sequence

$$0 \to H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d-1}\right) \to H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d}\right) \to H^0\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right) \to 0 \tag{3.1.9}$$

induced by 3.1.7 is exact.

(c) The canonical map

$$\operatorname{gr}\left(H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}\right)\right) = \bigoplus_{d \geqslant 0} H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d}\right) \Big/ H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d-1}\right) \longrightarrow H^0\left(\mathbb{X}, \operatorname{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\right)$$

is an isomorphism.

Proof. (a) The proof proceeds by induction on d. We have $\mathcal{D}^{(m)}_{\mathbb{X},0} = \mathcal{O}_{\mathbb{X}}$, and the assertion is true for $d = 0$. Moreover, $\mathcal{T}_{\mathbb{X}}^{\otimes d} \simeq \mathcal{O}_{\mathbb{X}}(2d)$ and therefore $H^1(\mathbb{X}, \mathcal{T}_{\mathbb{X}}^{\otimes d}) = 0$. Using 3.1.4, we find that $H^1(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}) = 0$ for all $d, m \geqslant 0$. Now suppose $d \geqslant 1$. By 3.1.7 we get an exact sequence

$$H^1\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d-1}\right) \to H^1\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d}\right) \to H^1\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right)\ ,$$

and our induction hypothesis implies $H^1\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d}\right) = 0$.

(b) This assertion follows from (a) and the long exact cohomology sequence attached to 3.1.7.

(c) This follows immediately from (b). □

Remark 3.1.10. Assertion (c) of the previous proposition is as in [10, 2.3.6 (ii)], at least for large d. Though Noot-Huyghe's result would be good enough for our purposes, we have preferred to give a self-contained proof here. The proof given here proceeds along the same lines as the proof in [10]. □

In the following we consider the filtration of $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ whose submodule of elements of degree $\leqslant d$ is generated as a $\mathbb{Z}_p$-module by terms of the form 2.3.1 with $\nu_1+\nu_2+\nu_3+\nu_4 \leqslant d$.

Proposition 3.1.11. *(a) For all $\nu \geqslant 0$ one has the following identity of differential operators in $\mathcal{D}_{\mathbb{X}} \otimes_{\mathbb{Z}} \mathbb{Q}$: $\binom{x\partial_x}{\nu} = x^\nu \frac{\partial_x^\nu}{\nu!}$.*

(b) The canonical map $U(\mathfrak{g}_{\mathbb{Z}_p}) \to H^0(\mathbb{X}, \mathcal{D}^{(0)}_{\mathbb{X}})$ induced by 3.1.1 extends to a homomorphism

$$\xi^{(m)} : U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)} \longrightarrow H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}) \ , \tag{3.1.12}$$

of $\mathbb{Z}_p$-algebras which is compatible with the filtrations on both sides.

(c) $\xi^{(m)}$ maps the center $Z(\mathfrak{g}_{\mathbb{Z}_p})$ of $U(\mathfrak{g}_{\mathbb{Z}_p}) \subset U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ to $\mathbb{Z}_p$. Let $\theta_0 = \xi^{(m)}|_{Z(\mathfrak{g}_{\mathbb{Z}_p})}$ be the restriction of $\xi^{(m)}$ to $Z(\mathfrak{g}_{\mathbb{Z}_p})$. Then $\ker(\xi^{(m)})$ is the (two-sided) ideal generated by $\ker(\theta_0)$.

Proof. (a) Is easily proved by induction.

(b) Using (a) we see that $\binom{h_i}{\nu}$, $i = 1, 2$, is mapped to $\pm x^\nu \frac{\partial_x^\nu}{\nu!}$. The assertion now follows directly from the definition of $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$.

(c) Tensor with $\mathbb{Q}$ and use the statement in characteristic zero, cf. [2]. □

Using the notations introduced in 3.1.11 we define

$$U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\theta_0} \underset{\mathrm{df}}{=} U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)} \otimes_{Z(\mathfrak{g}_{\mathbb{Z}_p}), \theta_0} \mathbb{Z}_p \ .$$

Therefore, $\xi^{(m)}$ induces an injective homomorphism of $\mathbb{Z}_p$-algebras

$$\xi^{(m)}_0 : U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\theta_0} \hookrightarrow H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}) \ . \tag{3.1.13}$$

Proposition 3.1.14. *(a) Via the homomorphism*

$$\mathrm{gr}\, \xi^{(m)} : \mathrm{gr}\Big(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\Big) \longrightarrow H^0\Big(\mathbb{X}, \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\Big) = \mathrm{gr}\,\Big(H^0\Big(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}\Big)\Big)$$

induced by $\xi^{(m)}$, the ring $H^0\Big(\mathbb{X}, \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\Big)$ is a finitely generated module over $\mathrm{gr}\Big(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\Big)$.

(b) Via $\xi_0^{(m)}$ the ring $H^0(\mathbb{X},\mathcal{D}_{\mathbb{X}}^{(m)})$ is a finitely generated $U(\mathfrak{g}_{\mathbb{Z}_p})_{\theta_0}^{(m)}$-module. Moreover, there is $N(m)\in\mathbb{N}$ such that the $\operatorname{coker}(\xi_0^{(m)})$ *is annihilated by $p^{N(m)}$.*

Proof. (a) By 3.1.4 we have

$$H^0\left(\mathbb{X},\operatorname{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\right)=\bigoplus_{d\geqslant 0}\frac{q_d^{(m)}!}{d!}H^0\left(\mathbb{X},\mathcal{T}_{\mathbb{X}}^{\otimes d}\right)\ ,$$

as submodules of $H^0\left(\mathbb{X},\operatorname{Sym}(\mathcal{T}_{\mathbb{X}})_{\mathbb{Q}}^{(m)}\right)=H^0\left(\mathbb{X},\operatorname{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\right)\otimes_{\mathbb{Z}}\mathbb{Q}$. Furthermore,

$$H^0\left(\mathbb{X},\mathcal{T}_{\mathbb{X}}^{\otimes d}\right)=\bigoplus_{k=0}^{2d}\mathbb{Z}_p x^k\partial_x^{\otimes d}\ .$$

Our goal is to show that $H^0\left(\mathbb{X},\operatorname{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\right)$ is generated as a module over $\operatorname{gr}\left(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\right)$ by the elements

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^{\otimes d}\quad\text{with}\quad 0\leqslant d<2p^m\ ,\quad 0\leqslant k\leqslant 2d\ .$$

To this end, consider an element $\frac{q_d^{(m)}!}{d!}x^k\partial_x^{\otimes d}$ with $k\leqslant 2d$. Write $d=p^mq+s$. We are going to use the elementary fact

$$\frac{q_d^{(m)}!}{d!}=\frac{u}{s!(p^m!)^q}\ ,$$

with a p-adic unit u, cf. [8, 5.2.2].

Case $k\leqslant d$. Writing $k=p^mq'+r$, we have $q'\leqslant q$. If $r\leqslant 2s$ then consider the equation[1]

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^d=u\left(\frac{(x\partial_x)^{p^m}}{p^m!}\right)^{q'}\cdot\left(\frac{\partial_x^{p^m}}{p^m!}\right)^{q-q'}\cdot\frac{1}{s!}x^r\partial_x^s\ .$$

Now suppose $r>2s$. Because $k=p^mq'+r\leqslant d=p^mq+s$ we must have $q'<q$ and hence $q-q'-1\geqslant 0$. Then we can write

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^d=u\left(\frac{(x\partial_x)^{p^m}}{p^m!}\right)^{q'}\cdot\left(\frac{\partial_x^{p^m}}{p^m!}\right)^{q-q'-1}\cdot\frac{1}{s!(p^m!)}x^r\partial_x^{p^m+s}\ .$$

Case $d<k$ ($\leqslant 2d$). Write $k=p^mq'+r$, and suppose $q'=2q''$ is even. Because $\frac{k}{2}=p^mq''+\frac{r}{2}\leqslant p^mq+s$ we must have $q''\leqslant q$. If $r\leqslant 2s$ then consider the equation

[1]This equation and the following formulas are to be considered in the commutative ring $H^0\left(\mathbb{X},\operatorname{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\right)$. To simplify notation we have dropped the superscript "$\otimes$".

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^d = u\left(\frac{(x^2\partial_x)^{p^m}}{p^m!}\right)^{q''}\cdot\left(\frac{\partial_x^{p^m}}{p^m!}\right)^{q-q''}\cdot\frac{1}{s!}x^r\partial_x^s\ .$$

Now suppose $r > 2s$. Then we must have $q'' < q$, hence $q - q'' - 1 \geqslant 0$ and we can write

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^d = u\left(\frac{(x^2\partial_x)^{p^m}}{p^m!}\right)^{q''}\cdot\left(\frac{\partial_x^{p^m}}{p^m!}\right)^{q-q''-1}\cdot\frac{1}{s!(p^m!)}x^r\partial_x^{p^m+s}\ .$$

Assume now that $q' = 2q'' + 1$ is odd. Because

$$p^m q'' + \frac{p^m + r}{2} = p^m(q'' + \frac{1}{2}) + \frac{r}{2} = \frac{k}{2} \leqslant d = p^m q + s\ ,$$

we must have $q'' \leqslant q$. If $p^m + r \leqslant 2s$ we consider

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^d = u\left(\frac{(x^2\partial_x)^{p^m}}{p^m!}\right)^{q''}\cdot\left(\frac{\partial_x^{p^m}}{p^m!}\right)^{q-q''}\cdot\frac{1}{s!}x^{p^m+r}\partial_x^s\ .$$

Finally, if $p^m + r > 2s$ we must have $q'' < q$. In this case we consider

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^d = u\left(\frac{(x^2\partial_x)^{p^m}}{p^m!}\right)^{q''}\cdot\left(\frac{\partial_x^{p^m}}{p^m!}\right)^{q-q''-1}\cdot\frac{1}{s!(p^m!)}x^{p^m+r}\partial_x^{p^m+s}\ .$$

(b) For $0 \leqslant d < 2p^m$ and $0 \leqslant k \leqslant 2d$ let $e_{d,k}$ be a representative in $H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X},d}\right)$ of the element $\frac{q_d^{(m)}!}{d!}x^k\partial_x^{\otimes d}$ in $H^0\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right)$. By part (a), $H^0\Big(\mathbb{X}, \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\Big)$ is generated over $\mathrm{gr}\Big(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\Big)$ by the elements $\frac{q_d^{(m)}!}{d!}x^k\partial_x^{\otimes d}$, for $0 \leqslant d < 2p^m$ and $0 \leqslant k \leqslant 2d$, it follows that $H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})$ is generated over $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ by the elements $e_{d,k}$. And then, obviously, $H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})$ is actually a finitely generated $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\theta_0}$-module. Moreover, we see that the generators

$$\frac{q_d^{(m)}!}{d!}x^k\partial_x^{\otimes d} \quad \text{with}\ \ 0 \leqslant d < 2p^m\ ,\ \ 0 \leqslant k \leqslant 2d\ ,$$

of $H^0\Big(\mathbb{X}, \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\Big)$ over $\mathrm{gr}\Big(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\Big)$ have the property that

$$(3.1.15)\quad (p^m - 1)! \cdot (p^m)! \cdot \frac{q_d^{(m)}!}{d!}x^k\partial_x^{\otimes d} \in \mathrm{im}\left(\mathrm{gr}\Big(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}\Big) \to H^0\Big(\mathbb{X}, \mathrm{Sym}(\mathcal{T}_{\mathbb{X}})^{(m)}\Big)\right)\ .$$

Because the generators $e_{d,k}$ are in degrees $< 2p^m$, repeating 3.1.15 finitely often shows that there is $N(m)$ such that $p^{N(m)}e_{d,k}$ is in the image of $U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{\theta_0}$ for $0 \leqslant d < 2p^m$, $0 \leqslant k \leqslant 2d$. Now assertion (b) follows. □

3.2. **$\mathscr{D}^\dagger$ and the distribution algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)$.** Denote by $\mathfrak{X}$ the completion of $\mathbb{X}$ along its special fiber $\mathbb{X}_{\mathbb{F}_p}$. Let $\mathscr{D}^{(m)}_{\mathfrak{X}}$ be the p-adic completion of the sheaf $\mathcal{D}^{(m)}_{\mathbb{X}}$, which we consider as a sheaf on $\mathfrak{X}$.

Lemma 3.2.1. *The canonical map*

$$H^0\left(\mathbb{X},\mathcal{D}^{(m)}_{\mathbb{X}}\right) \longrightarrow H^0\left(\mathfrak{X},\mathscr{D}^{(m)}_{\mathfrak{X}}\right)$$

extends to an isomorphism

$$H^0\left(\mathbb{X},\mathcal{D}^{(m)}_{\mathbb{X}}\right)^\wedge \longrightarrow H^0\left(\mathfrak{X},\mathscr{D}^{(m)}_{\mathfrak{X}}\right) ,$$

where the left hand side is the p-adic completion of $H^0\left(\mathbb{X},\mathcal{D}^{(m)}_{\mathbb{X}}\right)$.

Proof. This is contained in [10, Prop. 3.2]. The key ingredient used in [10, Prop. 3.2] is that H^1 of the sheaf in question (here $\mathcal{D}^{(m)}_{\mathbb{X}}$) is annihilated by a finite power of p. Here we have seen $H^1(\mathbb{X},\mathcal{D}^{(m)}_{\mathbb{X}}) = 0$, cf. 3.1.8. Thus it would be possible to give a self-contained proof following the arguments given in the proof of [10, Prop. 3.2]. □

Put

$$\mathscr{D}^\dagger_{\mathfrak{X}} = \varinjlim_m \mathscr{D}^{(m)}_{\mathfrak{X}} ,$$

and

$$\mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}} = \varinjlim_m \mathscr{D}^{(m)}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q} .$$

Theorem 3.2.2. *(a) The homomorphism*

$$\xi^{(m)}_0 : U\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0} \to H^0\left(\mathbb{X},\mathcal{D}^{(m)}_{\mathbb{X}}\right) ,$$

cf. 3.1.13, induces a homomorphism

$$\widehat{\xi}^{(m)}_0 : \widehat{U}\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0} \to H^0\left(\mathfrak{X},\mathscr{D}^{(m)}_{\mathfrak{X}}\right) ,$$

which is injective and whose cokernel is annihilated by $p^{N(m)}$ where $N(m)$ is as in 3.1.14. Therefore, $\widehat{\xi}^{(m)}_0$ induces an isomorphism

$$\widehat{U}\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0,\mathbb{Q}} \xrightarrow{\simeq} H^0\left(\mathfrak{X}, \mathscr{D}^{(m)}_{\mathfrak{X},\mathbb{Q}}\right) .$$

(b) The isomorphisms in (a) give rise to a canonical isomorphism

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)_{\theta_0} \simeq \varinjlim_m \widehat{U}\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0,\mathbb{Q}} \xrightarrow{\simeq} H^0(\mathfrak{X}, \mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}}) .$$

Proof. (a) We consider the exact sequence induced by $\xi_0^{(m)}$

$$0 \longrightarrow \left(U\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0} \cap p^k H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}\right)\right) \Big/ p^k U\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0}$$
$$\longrightarrow U\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0} \Big/ p^k U\left(\mathfrak{g}_{\mathbb{Z}_p}\right)^{(m)}_{\theta_0} \longrightarrow H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}\right) \Big/ p^k H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}\right) .$$

Because the projective limit functor is left-exact, and as $H^0\left(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}\right)$ is separated for the p-adic topology, we deduce that the homomorphism $\widehat{\xi}_0^{(m)}$ between the completions is injective as well. The assertion about the cokernel is an immediate consequence of 3.1.14. Hence the isomorphism after extending scalars to $\mathbb{Q}$.

(b) This assertion follows from (a) and the fact that cohomology commutes with direct limits. □

As already indicated in the introduction, after having obtained this result we have been informed by C. Noot-Huyghe that she has proved the general case of this theorem, for an arbitrary split reductive group and the corresponding flag variety, in an unpublished manuscript.
The isomorphism in (a) for an arbitrary split semisimple group and the corresponding flag variety has appeared, in the case $m = 0$ and with some restrictions on the prime number p, in [1].

4. The semistable models $\mathbb{X}_n$ and their completions $\mathfrak{X}_n$

4.1. **The construction via blowing-up.**

4.1.1. In the following, all closed subsets of a scheme are considered as closed subschemes with their reduced induced subscheme structure. Put $\mathbb{X}_0 = \mathbb{X} = \mathbb{P}^1_{\mathbb{Z}_p}$. Blowing up $\mathbb{X}_0$ in the $\mathbb{F}_p$-rational points of its special fiber $\mathbb{X}_{0,\mathbb{F}_p}$ produces a scheme $\mathbb{X}_1$. The irreducible components of the special fiber of $\mathbb{X}_1$ are all projective lines over $\mathbb{F}_p$, and there are $p+2$ of them: on the one hand we have the strict transform of $\mathbb{X}_{0,\mathbb{F}_p}$, which we can and will identify with $\mathbb{X}_{0,\mathbb{F}_p}$, and then there is for any $\mathbb{F}_p$-rational point P of $\mathbb{X}_0$ the corresponding component $E_P \simeq \mathbb{P}^1_{\mathbb{F}_p}$ of the exceptional divisor. No two components E_P intersect each other, but any one of these intersects $\mathbb{X}_{0,\mathbb{F}_p}$ in a unique point which corresponds to the point P that has been blown up. We call the components E_P the 'end components' or 'ends' of the special fiber of $\mathbb{X}_1$.

Then blow up $\mathbb{X}_1$ in the smooth $\mathbb{F}_p$-rational points of its special fiber. There are p such points on each component E_P. Call the resulting scheme $\mathbb{X}_2$. The special fiber of $\mathbb{X}_2$ consists of the strict transform of the special fiber of $\mathbb{X}_1$, which we identify with $\mathbb{X}_{1,\mathbb{F}_p}$, and, for each of the components E_P of $\mathbb{X}_{1,\mathbb{F}_p}$ there are p irreducible components $E_{P,P'}$ of the exceptional divisor, and $E_{P,P'}$ intersects E_P in the point P' that has been blown up. Again, we call the irreducible components $E_{P,P'}$ the 'end components' or 'ends' of the special fiber of $\mathbb{X}_2$.

Inductively one defines $\mathbb{X}_n$ by blowing up $\mathbb{X}_{n-1}$ in the smooth $\mathbb{F}_p$-rational points of the special fiber of $\mathbb{X}_{n-1}$. The irreducible components of the exceptional divisor are called the 'end components' or 'ends' of the special fiber of $\mathbb{X}_n$. It is easy to see that the intersection graph of the special fiber of $\mathbb{X}_n$ is a tree. There are $p+1$ edges meeting at every vertex, except for the vertices which correspond to the end components: these are only connected to the rest of the tree by a single edge.

Remark 4.1.2.[2] Because the group $\mathbb{G}(\mathbb{Z}_p) = \mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\mathbb{X}_0$ and preserves the closed subscheme $\mathbb{X}_0(\mathbb{F}_p)$, the group $\mathbb{G}(\mathbb{Z}_p)$ acts also on $\mathbb{X}_1$. It is easy to see that $\mathbb{G}(\mathbb{Z}_p)$ preserves the subscheme of $\mathbb{X}_1$ which gives rise to $\mathbb{X}_2$. Inductively we find that $\mathbb{G}(\mathbb{Z}_p)$ acts on $\mathbb{X}_n$ for all n. Furthermore, one can show that the group scheme $\mathbb{G}(n)$ acts on the scheme $\mathbb{X}_n$.

4.2. **An open affine covering of $\mathbb{X}_n$.** Here we describe an open affine covering of the scheme $\mathbb{X}_n$, and a coherent system of local coordinates[3]. This will be used later in sec. 5.1.

4.2.1. *Outline.* We will first describe the general shape of this covering and the procedure by which it is obtained. Let $\mathcal{R} \subset \mathbb{Z}_p$ be any system of representatives for $\mathbb{Z}_p/p\mathbb{Z}_p$ and put $\mathcal{R}_\infty = \mathcal{R} \cup \{\infty\}$. Let $n \geqslant 1$. Inductively we will define an open subset $\mathbb{X}^\circ_{n-1} \subset \mathbb{X}_{n-1}$ and open affine 'residual disc schemes' $\mathbb{D}^{(n-1)}_{\underline{a}}$ for any tuple $\underline{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathcal{R}_\infty \times \mathcal{R}^{n-1}$. Each scheme $\mathbb{D}^{(n-1)}_{\underline{a}}$ has a unique $\mathbb{F}_p$-rational point and $\mathbb{X}_n$ is obtained from $\mathbb{X}_{n-1}$ by blowing up all these points. The open subset $\mathbb{X}^\circ_{n-1} \subset \mathbb{X}_{n-1}$ is not affine (except if $n-1 = 0$) but it is equipped with an open affine covering. Moreover, the special fiber of $\mathbb{X}^\circ_{n-1}$ does not contain any smooth $\mathbb{F}_p$-rational point of the special fiber of $\mathbb{X}_{n-1}$. The blow-up morphism $pr_{n,n-1} : \mathbb{X}_n \to \mathbb{X}_{n-1}$ is thus an isomorphism over $\mathbb{X}^\circ_{n-1}$, and the preimage $pr^{-1}_{n,n-1}(\mathbb{X}^\circ_{n-1}) \subset \mathbb{X}_n$ is then equipped with the open affine covering of $\mathbb{X}^\circ_{n-1}$. In the following we identify $pr^{-1}_{n,n-1}(\mathbb{X}^\circ_{n-1})$ with $\mathbb{X}^\circ_{n-1}$.

[2]The content of this remark will not be used later on.

[3]By this we mean a set of local coordinates together with transition formulas for the local coordinates on 'neighboring' open affine subsets. The meaning of 'neighboring' in our context will become clear in the sequel.

Next we define for any such $\underline{a}$ open affine subschemes $\mathbb{X}_{\underline{a}}^{(n)}$ and, for all $a_n \in \mathcal{R}$, 'residual disc schemes' $\mathbb{D}_{\underline{a},a_n}^{(n)}$ of $\mathbb{X}_n$. These open affine subschemes, together with the open affine covering of $\mathbb{X}_{n-1}^\circ$ constitute then the open affine covering of $\mathbb{X}_n$. The open subset $\mathbb{X}_n^\circ$ is defined as

$$\mathbb{X}_n^\circ \underset{\mathrm{df}}{=} \mathbb{X}_{n-1}^\circ \cup \bigcup_{\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{n-1}} \mathbb{X}_{\underline{a}}^{(n)} .$$

4.2.2. *When* $n = 0$. We start with the affine covering $\mathbb{X}_0 = U_x \cup U_y$ of $\mathbb{X}_0$, cf. 3, where $U_x = \mathrm{Spec}(\mathbb{Z}_p[x])$ and $U_y = \mathrm{Spec}(\mathbb{Z}_p[y])$ and these open subschemes are glued together according to the relation $xy = 1$. For $a \in \mathcal{R}$ put $x_a^{(0)} = x - a$, and consider this as a local coordinate at $x = a$, and set $x_\infty^{(0)} = y = \frac{1}{x}$. For $a \in \mathcal{R}_\infty$ put

$$R_a^{(0)} = \mathbb{Z}_p[x_a^{(0)}] \left[\frac{1}{x_b^{(0)}} \;\middle|\; b \in \mathcal{R}, b \neq a \right] ,$$

and view this as a subring of the rational function field $\mathbb{Q}_p(x)$. It is immediate that for all $a \in \mathcal{R}_\infty$ the ring

$$R^{(0)} \underset{\mathrm{df}}{=} R_a^{(0)} \left[\frac{1}{x_a^{(0)}} \right] ,$$

as a subring of $\mathbb{Q}_p(x)$, is independent of a. Set $\mathbb{X}_0^\circ = \mathrm{Spec}(R^{(0)})$. The special fiber of $\mathbb{X}_0^\circ$ is $\mathbb{P}^1_{\mathbb{F}_p} \backslash \mathbb{P}^1(\mathbb{F}_p)$. Furthermore, for $a \in \mathcal{R}_\infty$ put

$$\mathbb{D}_a^{(0)} = \mathrm{Spec}(R_a^{(0)}) .$$

The special fiber of $\mathbb{D}_a^{(0)}$ is $\mathbb{X}_{0,\mathbb{F}_p}^\circ \cup \{\overline{a}\}$, where $\overline{a}$ is the 'mod-p reduction of a'. This is the unique $\mathbb{F}_p$-rational point which corresponds to the ideal $(p, x_a^{(0)})$. We call $\mathbb{D}_a^{(0)}$ a 'residual disc scheme'. For later use we fix the coordinate function $x_a^{(0)}$ on $\mathbb{D}_a^{(0)}$. For any two distinct $a, a' \in \mathcal{R}_\infty$ we have $\mathbb{D}_a^{(0)} \cap \mathbb{D}_{a'}^{(0)} = \mathbb{X}_0^\circ$. Then we consider the covering of $\mathbb{X}_0$ by the open subschemes $\mathbb{D}_a^{(0)}$, $a \in \mathcal{R}_\infty$, together with $\mathbb{X}_0^\circ$.

4.2.3. *When* $n = 1$. $\mathbb{X}_1$ is obtained by blowing up $\mathbb{X}_0$ in the points corresponding to the ideals $(p, x_{a_0}^{(0)}) \subset R_{a_0}^{(0)}$, $a_0 \in \mathcal{R}_\infty$. In order to describe $\mathbb{X}_1$, we introduce new indeterminates $z_{a_0}^{(1)}$ and $x_{a_0}^{(1)}$ satisfying

$$x_{a_0}^{(0)} z_{a_0}^{(1)} = p \quad \text{and} \quad z_{a_0}^{(1)} x_{a_0}^{(1)} = 1 .$$

Set also $x_{a_0,a_1}^{(1)} = x_{a_0}^{(1)} - a_1$ for $a_1 \in \mathcal{R}$. Then define

$$R^{(1)}_{a_0} = R^{(0)}_{a_0}[z^{(1)}_{a_0}]\left[\frac{1}{x^{(1)}_{a_0,a_1}} \;\middle|\; a_1 \in \mathcal{R}\right] \Big/ (x^{(0)}_{a_0} z^{(1)}_{a_0} - p) \;,$$

and put $\mathbb{X}^{(1)}_{a_0} = \mathrm{Spec}(R^{(1)}_{a_0})$. For $a_1 \in \mathcal{R}$ set

$$R^{(1)}_{a_0,a_1} = R^{(0)}_{a_0}[x^{(1)}_{a_0,a_1}]\left[\frac{1}{x^{(1)}_{a_0,b}} \;\middle|\; b \in \mathcal{R}\backslash\{a_1\}\right] \;,$$

and define

$$\mathbb{D}^{(1)}_{a_0,a_1} = \mathrm{Spec}\left(R^{(1)}_{a_0,a_1}\right) \;.$$

The special fiber of each $\mathbb{D}^{(1)}_{a_0,a_1}$ is isomorphic to an affine line over $\mathbb{F}_p$ all of whose $\mathbb{F}_p$-rational points have been removed, except one. Again, in order to obtain a coherent system of coordinates, we fix the coordinate function $x^{(1)}_{a_0,a_1}$ on $\mathbb{D}^{(1)}_{a_0,a_1}$. For any $a_1 \in \mathcal{R}$ one has

$$R^{(1)}_{a_0}\left[\frac{1}{z^{(1)}_{a_0}}\right] = R^{(1)}_{a_0,a_1}\left[\frac{1}{x^{(1)}_{a_0,a_1}}\right] \;,$$

and this ring is thus independent of a_1. For any two distinct $a_1, a_1' \in \mathcal{R}$ one has

$$\mathbb{D}^{(1)}_{a_0,a_1} \cap \mathbb{D}^{(1)}_{a_0,a_1'} = \mathbb{D}^{(1)}_{a_0,a_1} \cap \mathbb{X}^{(1)}_{a_0} = \mathrm{Spec}\left(R^{(1)}_{a_0}\left[\frac{1}{z^{(1)}_{a_0}}\right]\right) \;,$$

and the special fiber of this scheme is isomorphic (via the coordinate $x^{(1)}_{a_0}$, say) to $\mathbb{P}^1_{\mathbb{F}_p}\backslash\mathbb{P}^1(\mathbb{F}_p)$. Furthermore, for any two distinct $a_0, a_0' \in \mathcal{R}_\infty$ one has

$$\mathbb{X}^{(1)}_{a_0} \cap \mathbb{X}^{(1)}_{a_0'} = \mathbb{X}^\circ_0 \;.$$

Let $\mathbb{X}^\circ_1$ be the union of the schemes $\mathbb{X}^{(1)}_{a_0}$, $a_0 \in \mathcal{R}_\infty$, and $\mathbb{X}^\circ_0$.

4.2.4. *From $n-1$ to n.* Firstly, we use the preimages of the affine covering of $\mathbb{X}^\circ_{n-1}$ under the blow-up map $\mathbb{X}_n \to \mathbb{X}_{n-1}$. Then we consider a 'residue disc scheme'

$$\mathbb{D}^{(n-1)}_{\underline{a}} = \mathrm{Spec}\left(R^{(n-1)}_{\underline{a}}\right)$$

of $\mathbb{X}_{n-1}$, where $\underline{a} = (a_0, a_1, \ldots, a_{n-1})$. It is equipped with a coordinate function $x^{(n-1)}_{\underline{a}}$ and has a unique $\mathbb{F}_p$-rational point which corresponds to the ideal $(p, x^{(n-1)}_{\underline{a}}) \subset R^{(n-1)}_{\underline{a}}$. $\mathbb{X}_n$ is obtained from $\mathbb{X}_{n-1}$ by blowing up these $\mathbb{F}_p$-rational points, for all $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{n-1}$.

To describe the blow-up process, we introduce indeterminates $z_{\underline{a}}^{(n)}$ and $x_{\underline{a}}^{(n)}$ satisfying

$$x_{\underline{a}}^{(n-1)} z_{\underline{a}}^{(n)} = p \quad \text{and} \quad z_{\underline{a}}^{(n)} x_{\underline{a}}^{(n)} = 1 \, .$$

For $a_n \in \mathcal{R}$ set $x_{\underline{a},a_n}^{(n)} = x_{\underline{a}}^{(n)} - a_n$ and define

$$R_{\underline{a}}^{(n)} = R_{\underline{a}}^{(n-1)}[z_{\underline{a}}^{(n)}] \left[\frac{1}{x_{\underline{a},b}^{(n)}} \;\middle|\; b \in \mathcal{R} \right] \Big/ (x_{\underline{a}}^{(n-1)} z_{\underline{a}}^{(n)} - p) \, , \tag{4.2.5}$$

and put

$$\mathbb{X}_{\underline{a}}^{(n)} = \operatorname{Spec}\left(R_{\underline{a}}^{(n)}\right) \, .$$

For $a_n \in \mathcal{R}$ define

$$R_{\underline{a},a_n}^{(n)} = R_{\underline{a}}^{(n-1)}[x_{\underline{a},a_n}^{(n)}] \left[\frac{1}{x_{\underline{a},b}^{(n)}} \;\middle|\; b \in \mathcal{R} \backslash \{a_n\} \right] \, ,$$

and put

$$\mathbb{D}_{\underline{a},a_n}^{(n)} = \operatorname{Spec}\left(R_{\underline{a},a_n}^{(n)}\right) \, .$$

Again, in order to obtain a coherent system of coordinates, we fix the coordinate function $x_{\underline{a},a_n}^{(n)}$ on $\mathbb{D}_{\underline{a},a_n}^{(n)}$. For any $a_n \in \mathcal{R}$ one has

$$R_{\underline{a}}^{(n)} \left[\frac{1}{z_{\underline{a}}^{(n)}} \right] = R_{\underline{a},a_n}^{(n)} \left[\frac{1}{x_{\underline{a},a_n}^{(n)}} \right] \, ,$$

and this ring is thus independent of a_n. For any two distinct $a_n, a_n' \in \mathcal{R}$ one has

$$\mathbb{D}_{\underline{a},a_n}^{(n)} \cap \mathbb{D}_{\underline{a},a_n'}^{(n)} = \mathbb{D}_{\underline{a},a_n}^{(n)} \cap \mathbb{X}_{\underline{a}}^{(n)} = \operatorname{Spec}\left(R_{\underline{a}}^{(n)} \left[\frac{1}{z_{\underline{a}}^{(n)}} \right] \right) \, ,$$

and the special fiber of this scheme is isomorphic to (via the coordinate $x_{\underline{a}}^{(n)}$, say) to $\mathbb{P}^1_{\mathbb{F}_p} \backslash \mathbb{P}^1(\mathbb{F}_p)$. Let $\mathbb{X}_n^\circ$ be the union of the schemes $\mathbb{X}_{\underline{a}}^{(n)}$, $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{n-1}$, and $\mathbb{X}_{n-1}^\circ$. One obtains an open affine cover for $\mathbb{X}_n^\circ$ from the union of the open affine cover from $\mathbb{X}_{n-1}^\circ$ and the collection of all $\mathbb{X}_{\underline{a}}^{(n)}$. Finally $\mathbb{X}_n$ is then covered by $\mathbb{X}_n^\circ$ and the open affine subschemes $\mathbb{D}_{\underline{a},a_n}^{(n)}$, $(\underline{a}, a_n) = (a_0, \ldots, a_{n-1}, a_n) \in \mathcal{R}_\infty \times \mathcal{R}^n$. Writing out the open affine covering of $\mathbb{X}_n^\circ$ explicitly gives:

$$\mathbb{X}_n = \mathbb{X}_0^\circ \ \cup \bigcup_{1\leqslant\nu\leqslant n} \ \bigcup_{\underline{a}\in\mathcal{R}_\infty\times\mathcal{R}^{\nu-1}} \mathbb{X}_{\underline{a}}^{(\nu)} \ \cup \bigcup_{\underline{b}\in\mathcal{R}_\infty\times\mathcal{R}^{n}} \mathbb{D}_{\underline{b}}^{(n)} \ . \tag{4.2.6}$$

4.2.7. Going through the successive definitions of the local coordinates $x_{a_0}^{(0)}, x_{a_0,a_1}^{(1)}, \ldots, x_{\underline{a}}^{(n)}$, $\underline{a} = (a_0, \ldots, a_n)$, one finds the relations, for $a_0 \neq \infty$,

$$\begin{aligned} x &= a_0 + a_1 p + \ldots + a_{n-1}p^{n-1} + a_n p^n + p^n x_{\underline{a}}^{(n)} \\ &= a_0 + a_1 p + \ldots + a_{n-1}p^{n-1} + p^n x_{(a_0,a_1,\ldots,a_{n-1})}^{(n)} \\ &= a_0 + a_1 p + \ldots + a_{n-1}p^{n-1} + p^{n-1} x_{(a_0,a_1,\ldots,a_{n-1})}^{(n-1)} \ , \end{aligned} \tag{4.2.8}$$

where we have used $x_{(a_0,a_1,\ldots,a_{n-1})}^{(n-1)} z_{(a_0,a_1,\ldots,a_{n-1})}^{(n)} = p$. Similarly we have for $\underline{a} = (\infty, a_1, \ldots, a_{n-1}, a_n)$ and y the relations

$$\begin{aligned} y &= a_1 p + \ldots + a_{n-1}p^{n-1} + a_n p^n + p^n x_{\underline{a}}^{(n)} \\ &= a_1 p + \ldots + a_{n-1}p^{n-1} + p^n x_{(a_0,a_1,\ldots,a_{n-1})}^{(n)} \\ &= a_1 p + \ldots + a_{n-1}p^{n-1} + p^{n-1} x_{(a_0,a_1,\ldots,a_{n-1})}^{(n-1)} \ . \end{aligned} \tag{4.2.9}$$

4.3. The formal schemes $\mathfrak{X}_n$.

4.3.1. We denote by $\mathfrak{X}_n$ the completion of $\mathbb{X}_n$ along its special fiber. One can also obtain $\mathfrak{X}_n$ directly from $\mathfrak{X}$ by the same procedure as in 4.1. Assuming we have constructed $\mathfrak{X}_{n-1}$, we define $\mathfrak{X}_n$ by blowing up (in the sense of formal geometry) the smooth $\mathbb{F}_p$-rational points of the special fiber of $\mathfrak{X}_{n-1}$.

Furthermore, the open affine covering described in 4.2 gives rise upon completion to a covering of $\mathfrak{X}_n$ by open affine subschemes. The explicit description of the formal completion $\widehat{\mathbb{X}}_{\underline{a}}^{(n)}$ of $\mathbb{X}_{\underline{a}}^{(n)}$, $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{n-1}$ is in fact simpler than the corresponding description for $\mathbb{X}_{\underline{a}}^{(n)}$. One can show

$$\widehat{\mathbb{X}}_{\underline{a}}^{(n)} = \mathrm{Spf}\left(\mathbb{Z}_p\langle x_{\underline{a}}^{(n-1)}, z_{\underline{a}}^{(n)}\rangle\left[\frac{1}{(x_{\underline{a}}^{(n-1)})^{p-1}-1}, \frac{1}{(z_{\underline{a}}^{(n)})^{p-1}-1}\right]\Big/(x_{\underline{a}}^{(n-1)}z_{\underline{a}}^{(n)} - p)\right) \ .$$

See [21][4] and [12, I.3] for details. Similarly, the formal completion $\widehat{\mathbb{D}}_{\underline{a},a_n}^{(n)}$ of $\mathbb{D}_{\underline{a},a_n}^{(n)}$, $a_n \in \mathcal{R}$, can be described by

[4] The relevant material is in the section "The formal scheme $\widehat{\mathscr{H}_p}$ – the naive construction".

$$\widehat{\mathbb{D}}^{(n)}_{\underline{a},a_n} = \mathrm{Spf}\left(\mathbb{Z}_p\langle x^{(n)}_{\underline{a},a_n}\rangle\left[\frac{1}{(x^{(n)}_{\underline{a},a_n})^{p-1}-1}\right]\right) .$$

Remark 4.3.2. Denote by $\mathfrak{X}^\circ_n$ the completion of $\mathbb{X}^\circ_n$ along its special fiber. The open embedding $\mathbb{X}^\circ_{n-1} \hookrightarrow \mathbb{X}^\circ_n$ induces an open embedding $\mathfrak{X}^\circ_{n-1} \hookrightarrow \mathfrak{X}^\circ_n$. ($\mathfrak{X}^\circ_n$ can also defined intrinsically, and more straightforwardly, without the use of $\mathbb{X}^\circ_n$.) The inductive limit $\varinjlim_n \mathfrak{X}^\circ_n$ is then a formal model of the *p-adic upper half plane*, cf. [12, I.3]. This links the objects studied here with the Bruhat-Tits building and the Berkovich embedding of the Bruhat-Tits building into the analytification of the flag variety. The present paper was motivated by this connection and the study done in [18].

5. Logarithmic differential operators on $\mathbb{X}_n$

We refer to [16] for a systematic discussion of sheaves of logarithmic differential operators. For $n \geqslant 1$ we equip $\mathbb{X}_n$ with the log structure defined by its normal crossings divisor $\{p = 0\}$. However, here we will not use the theory as developed in [16], but rather work with a more elementary approach.

5.1. The logarithmic tangent sheaf on $\mathbb{X}_n$.

5.1.1. For the purposes of this paper we consider the sheaf $\mathcal{D}_{\mathbb{X}_n,\log}$ of logarithmic differential operators on $\mathbb{X}_n$ as being generated as a subsheaf of $\mathcal{E}nd_{\mathbb{Z}_p}(\mathcal{O}_{\mathbb{X}_n},\mathcal{O}_{\mathbb{X}_n})$ by the logarithmic tangent sheaf $\mathcal{T}_{\mathbb{X}_n,\log}$. (This is as in [14, 1.3].) The restriction of $\mathcal{T}_{\mathbb{X}_n,\log}$ to an open affine subset $\mathbb{X}^{(\nu)}_{\underline{a}}$, $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{\nu-1}$, cf. 4.2.6, is generated by a differential operator D (over $\mathbb{Z}_p$) with the properties

$$D(x^{(\nu-1)}_{\underline{a}}) = x^{(\nu-1)}_{\underline{a}} \ , \quad D(z^{(\nu)}_{\underline{a}}) = -z^{(\nu)}_{\underline{a}} \ ,$$

cf. 4.2.5. D has the property that

$$D(x^{(\nu-1)}_{\underline{a}} z^{(\nu)}_{\underline{a}}) = x^{(\nu-1)}_{\underline{a}} D(z^{(\nu)}_{\underline{a}}) + z^{(\nu)}_{\underline{a}} D(x^{(\nu-1)}_{\underline{a}}) = 0 \ ,$$

and hence $D(x^{(\nu-1)}_{\underline{a}} z^{(\nu)}_{\underline{a}} - p) = 0$, so that D preserves the ideal generated by $x^{(\nu-1)}_{\underline{a}} z^{(\nu)}_{\underline{a}} - p$. Intuitively we may write

$$D = x^{(\nu-1)}_{\underline{a}} \partial_{x^{(\nu-1)}_{\underline{a}}} = -z^{(\nu)}_{\underline{a}} \partial_{z^{(\nu)}_{\underline{a}}} \ .$$

To put it another way, we may say that $\mathcal{T}_{\mathbb{X}_n,\log}$ is locally on an open subscheme $\mathbb{X}^{(\nu)}_{\underline{a}}$ generated by

$$x^{(\nu-1)}_{\underline{a}} \partial_{x^{(\nu-1)}_{\underline{a}}} \text{ and } z^{(\nu)}_{\underline{a}} \partial_{z^{(\nu)}_{\underline{a}}} \ , \text{ with the relation } \ x^{(\nu-1)}_{\underline{a}} \partial_{x^{(\nu-1)}_{\underline{a}}} = -z^{(\nu)}_{\underline{a}} \partial_{z^{(\nu)}_{\underline{a}}} \ .$$

Denote by

$$pr_n : \mathbb{X}_n \longrightarrow \mathbb{X}_0 = \mathbb{X}$$

the canonical projection. Write, as in sec. 3, $\mathbb{X}_0 = \mathbb{X} = U_x \cup U_y$, where $U_x = \mathrm{Spec}(\mathbb{Z}_p[x])$, $U_y = \mathrm{Spec}(\mathbb{Z}_p[y])$, with x and y satisfying the relation $xy = 1$. Let $\mathcal{I}_{n,d} \subset \mathcal{O}_{\mathbb{X}}$ be the ideal sheaf which is on U_x associated to the ideal

$$\bigcap_{a \in \mathbb{Z}_p/(p^n)} (x - a, p^n)^d \subset \mathbb{Z}_p[x] = \mathcal{O}_{\mathbb{X}}(U_x) \ ,$$

and on U_y associated to the ideal

$$\bigcap_{a \in \mathbb{Z}_p/(p^n)} (y - a, p^n)^d \subset \mathbb{Z}_p[y] = \mathcal{O}_{\mathbb{X}}(U_y) \ .$$

Obviously, $\mathcal{I}_{0,d} = \mathcal{O}_{\mathbb{X}}$ for all d. In the following proposition, if $n = 0$, we put $\mathcal{T}_{\mathbb{X}_0,\log} = \mathcal{T}_{\mathbb{X}}$.

Proposition 5.1.2. *(a) $\mathcal{T}_{\mathbb{X}_n,\log}$ is a subsheaf of the invertible sheaf $pr_n^*(\mathcal{T}_{\mathbb{X}})$.*

(b) $(pr_n)_*(\mathcal{O}_{\mathbb{X}_n}) = \mathcal{O}_{\mathbb{X}}$.

(c) For all $n, d \geqslant 0$ one has $(pr_n)_*(\mathcal{T}_{\mathbb{X}_n,\log}^{\otimes d}) = \mathcal{I}_{n,d}\mathcal{T}_{\mathbb{X}}^{\otimes d}$.

Proof. (a) In order to see this we express the coordinate x by the local coordinates $x_{\underline{a}}^{(\nu-1)}$ introduced in sec. 4.2, and deduce a corresponding relation for ∂_x and $\partial_{x_{\underline{a}}^{(\nu-1)}}$. (By symmetry it suffices to consider x.) To be precise, fix $1 \leqslant \nu \leqslant n$, $\underline{a} = (a_0, \ldots, a_{\nu-1}) \in \mathcal{R}_\infty \times \mathcal{R}^{\nu-1}$, and consider the open subset $\mathbb{X}_{\underline{a}}^{(\nu)} \subset \mathbb{X}_n$, cf. 4.2.4. Without loss of generality we may assume $a_0 \neq \infty$. Then we have $x - a = p^{\nu-1}x_{\underline{a}}^{(\nu-1)}$ where $a = a_0 + \ldots + a_{\nu-1}p^{\nu-1}$, cf. 4.2.8. Hence

$$\partial_{x_{\underline{a}}^{(\nu-1)}} = p^{\nu-1}\partial_x \ , \text{ and thus } \ x_{\underline{a}}^{(\nu-1)}\partial_{x_{\underline{a}}^{(\nu-1)}} = p^{\nu-1}x_{\underline{a}}^{(\nu-1)}\partial_x = (x-a)\partial_x \ . \tag{5.1.3}$$

This proves the assertion.

(b) The morphism $pr_n : \mathbb{X}_n \to \mathbb{X}_0$ is a birational projective morphism of noetherian integral schemes, and $\mathbb{X}_0$ is normal. The assertion then follows exactly as in the proof of Zariski's Main Theorem as given in [9, ch. III, Cor. 11.4].

(c) *1. The inclusion $(pr_n)_*(\mathcal{T}_{\mathbb{X}_n,\log}^{\otimes d}) \subset \mathcal{I}_{n,d}\mathcal{T}_{\mathbb{X}}^{\otimes d}$.* Put $\mathbb{X}_n' = \mathbb{X}_n - pr_n^{-1}(\mathbb{X}(\mathbb{F}_p))$. This scheme is smooth over $\mathbb{Z}_p$. The restriction of pr_n induces an isomorphism

$$\mathbb{X}_n' \xrightarrow{\simeq} \mathbb{X}' = \mathbb{X} - \mathbb{X}(\mathbb{F}_p) \ ,$$

and the restriction of $\mathcal{T}_{\mathbb{X}_n,\log}$ to $\mathbb{X}'_n$ is the relative tangent sheaf of $\mathbb{X}'_n$ over $\mathbb{Z}_p$ whose direct image under pr_n is the relative tangent sheaf of $\mathbb{X}'$ over $\mathbb{Z}_p$. Therefore, in order to understand $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})$ we need to investigate the stalks of this sheaf at the points in $\mathbb{X}(\mathbb{F}_p)$. We consider the point P_0 in $U_x = \mathrm{Spec}(\mathbb{Z}_p[x]) \subset \mathbb{X}$ corresponding to the ideal $(x-a_0,p)$. Our aim is to understand the stalk of $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})$ at P_0.

By (a) we can consider the stalk of $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})$ at P_0 as a $\mathcal{O}_{\mathbb{X},P_0}$-submodule of the stalk of $\mathcal{T}^{\otimes d}_{\mathbb{X}}$ at P_0. We consider thus an element

$$D = f(x)\partial_x^{\otimes d} \in \left(\mathcal{T}^{\otimes d}_{\mathbb{X}}\right)_{P_0} \ , \tag{5.1.4}$$

$f(x) \in \mathcal{O}_{\mathbb{X},P_0} = \mathbb{Z}_p[x-a_0]_{(x-a_0,p)}$, and want to find necessary and sufficient conditions for this element to be in the stalk of $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})$ at P_0. To this end, consider an open subset of $\mathbb{X}_n$ of the form

$$\mathbb{X}^{(1)}_{a_0} \cup \mathbb{X}^{(2)}_{a_0,a_1} \cup \ldots \cup \mathbb{X}^{(n)}_{a_0,\ldots,a_{n-1}} \cup \mathbb{D}^{(n)}_{a_0,\ldots,a_n} \ ,$$

for a sequence $\underline{a} = (a_0,\ldots,a_n) \in \mathcal{R}^{n+1}$. Consider the local coordinate $x^{(n)}_{a_0,\ldots,a_{n-1},a_n}$ on $\mathbb{D}^{(n)}_{a_0,\ldots,a_n}$ which we denote henceforth by $x^{(n)}$. Put $a = a_0 + a_1p + \ldots + a_{n-1}p^{n-1} + a_np^n$. The equation 4.2.8 shows that

$$x^{(n)} = \frac{1}{p^n}(x-a) \ , \text{ hence } \ \partial_{x^{(n)}} = p^n\partial_x \ , \text{ and thus } \ x^{(n)}\partial_{x^{(n)}} = (x-a)\,\partial_{x-a} \ . \tag{5.1.5}$$

If D is in the stalk of $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})$ at P_0 then D extends to the stalk of $\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log}$ at the point $P_n \in \mathbb{D}^{(n)}_{a_0,\ldots,a_n}$ corresponding to the ideal $(x^{(n)},p)$. Therefore, D can be written as

$$g(x^{(n)})\partial^{\otimes d}_{x^{(n)}} \ , \tag{5.1.6}$$

with a function $g(x^{(n)}) \in \mathcal{O}_{\mathbb{X}_n,P_n} = \mathbb{Z}_p[x^{(n)}]_{(x^{(n)},p)}$. Completing this latter ring with respect to its maximal ideal gives $\mathbb{Z}_p[[x^{(n)}]]$, and so we can consider $g(x^{(n)}) = \sum_{k\geqslant 0} c_k(x^{(n)})^k$ as an element in $\mathbb{Z}_p[[x^{(n)}]]$. Now we write 5.1.6 as

$$g\left(\frac{1}{p^n}(x-a)\right)p^{nd}\partial_x^{\otimes d} \ .$$

Using the power series expansion for $g(x^{(n)})$ gives

$$g\left(\frac{1}{p^n}(x-a)\right)p^{nd} = \sum_{k\geqslant 0} c_k p^{-nk+nd}(x-a)^k \ .$$

For $k \leqslant d$ we have $p^{n(d-k)}(x-a)^k \in (x-a,p^n)^d$. And for $k > d$ we must have $c_k p^{-nk+nd} \in \mathbb{Z}_p$ and so $c_k p^{-nk+nd}(x-a)^k$ is in $(x-a,p^n)^d$ too. The function $f(x)$ in 5.1.4 is then contained in the ideal $(x-a,p^n)^d$ for all $a = a_0 + \ldots + a_{n-1}p^{n-1}$. Hence we see that the stalk of $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})$ at P_0 is contained in the stalk of $\mathcal{I}_{n,d}\mathcal{T}^{\otimes d}_{\mathbb{X}}$ at P_0. This is then true for all $\mathbb{F}_p$-rational points of $\mathbb{X}$. For the point at infinity one uses the equation 4.2.9.

2. The inclusion $(pr_n)_*(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log}) \supset \mathcal{I}_{n,d}\mathcal{T}^{\otimes d}_{\mathbb{X}}$. As above, we consider the point P_0 corresponding to the ideal $(x-a_0,p) \subset \mathbb{Z}_p[x] = \mathcal{O}_{\mathbb{X}}(U_x)$. For $1 \leqslant \nu \leqslant n$ consider an open affine subset $\mathbb{X}^{(\nu)}_{\underline{a}}$ of $\mathbb{X}_n$, as introduced in 4.2.4 (cf. also 4.2.6), where $\underline{a} = (a_0,a_1,\ldots,a_{\nu-1}) \in \mathcal{R}_\infty \times \mathcal{R}^{\nu-1}$. On $\mathbb{X}^{(\nu)}_{\underline{a}}$ we have the coordinate function $x^{(\nu-1)}_{\underline{a}}$, cf. 4.2.5, which is related to x by

$$x = a_0 + \ldots + a_{\nu-1}p^{\nu-1} + p^{\nu-1}x^{(\nu-1)}_{\underline{a}} \ , \ \text{ i.e. }, \quad x-a = p^{\nu-1}x^{(\nu-1)}_{\underline{a}} \ ,$$

cf. 4.2.8, where $a = a_0 + \ldots + a_{\nu-1}p^{\nu-1}$. Suppose $0 \leqslant k \leqslant d$ and consider the differential operator

$$D = p^{n(d-k)}(x-a)^k\partial_x^{\otimes d} \in (x-a,p^n)^d(\mathcal{T}^{\otimes d}_{\mathbb{X}})_{P_0} \ .$$

We have $\partial_x = \frac{1}{p^{\nu-1}}\partial_{x^{(\nu-1)}_{\underline{a}}}$ and thus

$$\begin{aligned} D &= p^{n(d-k)}p^{k(\nu-1)-d(\nu-1)}(x^{(\nu-1)}_{\underline{a}})^k(\partial_{x^{(\nu-1)}_{\underline{a}}})^{\otimes d} \\ &= p^{(n-\nu+1)(d-k)}(x^{(\nu-1)}_{\underline{a}})^k(\partial_{x^{(\nu-1)}_{\underline{a}}})^{\otimes d} \\ &= (z^{(\nu)}_{\underline{a}})^{(n-\nu+1)(d-k)}(x^{(\nu-1)}_{\underline{a}})^{(n-\nu+1)(d-k)}(x^{(\nu-1)}_{\underline{a}})^k(\partial_{x^{(\nu-1)}_{\underline{a}}})^{\otimes d} \\ &= (z^{(\nu)}_{\underline{a}})^{(n-\nu+1)(d-k)}(x^{(\nu-1)}_{\underline{a}})^{(n-\nu)(d-k)}(x^{(\nu-1)}_{\underline{a}})^d(\partial_{x^{(\nu-1)}_{\underline{a}}})^{\otimes d} \ . \end{aligned} \tag{5.1.7}$$

Because of the term $(x^{(\nu-1)}_{\underline{a}})^d(\partial_{x^{(\nu-1)}_{\underline{a}}})^{\otimes d}$ on the last line of 5.1.7, this shows that D extends to $\mathbb{X}^{(\nu)}_{\underline{a}}$. Here we have used the equation $z^{(\nu)}_{\underline{a}}x^{(\nu-1)}_{\underline{a}} = p$, cf. 4.2.5. To see that D also extends to $\mathbb{D}^{(n)}_{\underline{b}}$, where here $\underline{b} = (a_0,a_1,\ldots,a_n)$, we use the coordinate $x^{(n)}_{\underline{b}}$ on $\mathbb{D}^{(n)}_{\underline{b}}$. The equations 5.1.5 give then

$$D = p^{n(d-k)}p^{kn-dn}(x^{(n)}_{\underline{b}})^k(\partial_{x^{(n)}_{\underline{b}}})^{\otimes d} = (x^{(n)}_{\underline{b}})^k(\partial_{x^{(n)}_{\underline{b}}})^{\otimes d} \ ,$$

and this shows that D extends to $\mathbb{D}_{\underline{b}}^{(n)}$. If, more generally, we consider an element of the form $f(x)D$, where $f(x) \in \mathbb{Z}_p[x]_{(x-a_0,p)}$ and D is as before, then this will extend to a neighborhood of the special fiber of $\mathbb{X}_{\underline{a}}^{(\nu)}$ and $\mathbb{D}_{\underline{b}}^{(n)}$, respectively. □

Corollary 5.1.8. *For all $n, d, m \geqslant 0$ one has*

$$(pr_n)_* \left((\mathcal{T}_{\mathbb{X}_n,\log}^{\otimes d})^{(m)}\right) = \mathcal{I}_{n,d}(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)} = \frac{q_d^{(m)}!}{d!}\mathcal{I}_{n,d}\mathcal{T}_{\mathbb{X}}^{\otimes d} .$$

Proof. The sheaf $\mathcal{T}_{\mathbb{X}_n,\log}^{\otimes d}$ is a line bundle and the same reasoning as in the proof of 3.1.4 applies, i.e., $(\mathcal{T}_{\mathbb{X}_n,\log}^{\otimes d})^{(m)} = \frac{q_d^{(m)}!}{d!}\mathcal{T}_{\mathbb{X}_n,\log}^{\otimes d}$. This equality is to be understood as in 3.1.4. The statement then follows from 5.1.2. □

Consider $\mathcal{I}_{n,d}(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}$ as a subsheaf of $(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}$. The global sections of the former are thus contained in the global sections of the latter.

Proposition 5.1.9. *For all $n, d, m \geqslant 0$ one has the following inclusions*

$$p^{nd}H^0\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right) \subset H^0\left(\mathbb{X}, \mathcal{I}_{n,d}(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right) \subset p^{nc}H^0\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right) , \tag{5.1.10}$$

as submodules of $H^0\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right)$, where $c = \lceil d\frac{p-1}{p+1}\rceil$ is the smallest integer greater or equal to $d\frac{p-1}{p+1}$. In particular, for $d = 1$ and any $n, m \geqslant 0$ we have

$$H^0\left(\mathbb{X}, \mathcal{I}_{n,d}(\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right) = p^nH^0\left(\mathbb{X}, (\mathcal{T}_{\mathbb{X}}^{\otimes d})^{(m)}\right) . \tag{5.1.11}$$

Proof. Because of 3.1.4 it suffices to treat the case $m = 0$. By the very definition of $\mathcal{I}_{n,d}$ one has $p^{nd}\mathcal{O}_{\mathbb{X}} \subset \mathcal{I}_{n,d}$ and thus $p^{nd}\mathcal{T}_{\mathbb{X}}^{\otimes d} \subset \mathcal{I}_{n,d}\mathcal{T}_{\mathbb{X}}^{\otimes d}$. The inclusion on the left follows from this. Furthermore, the statement is trivial for $n = 0$ or $d = 0$ (when $c = 0$), and so we may assume that n and d are both positive.

To show the inclusion on the right we write global sections of $\mathcal{T}_{\mathbb{X}}^{\otimes d}$ in the form $f(x)\partial_x^{\otimes d}$ with a polynomial $f(x) \in \mathbb{Z}_p[x]$ of degree $\leqslant 2d$. Suppose $n \geqslant 1$ and $f(x)\partial_x^{\otimes d}$ is a global section of $\mathcal{I}_{n,d}\mathcal{T}_{\mathbb{X}}^{\otimes d}$. Note that the reduction modulo p of $\mathcal{I}_{n,d}$ is an ideal sheaf on $\mathbb{P}_{\mathbb{F}_p}^1$ of degree $-(p+1)d$, which we denote by $\mathcal{I}_{n,d,\mathbb{F}_p}$. Now, if f is not divisible by p, then $(f \mod p)\partial_x^{\otimes d}$ would be a non-zero section of $\mathcal{I}_{n,d,\mathbb{F}_p}\mathcal{T}_{\mathbb{P}_{\mathbb{F}_p}^1}^{\otimes d}$ and this sheaf has degree $-(p+1)d + 2d = (1-p)d < 0$ (because $d > 0$), hence a contradiction. Fix $a \in \mathbb{Z}_p$ and write

$$f(x) = \sum_{i=0}^{d} g_i(x) p^{ni} (x-a)^{d-i} \in (x-a, p^n)^d \ , \tag{5.1.12}$$

with polynomials $g_i(x) \in \mathbb{Z}_p[x]$. We have seen that f is divisible by p, hence g_0 is divisible by p. Consider $\frac{1}{p} f(x) = \frac{g_0(x)}{p}(x-a)^d + \sum_{i=1}^{d} g_i(x) p^{ni-1}(x-a)^{d-i}$ and apply the previous reasoning. Doing this repeatedly shows that $g_0(x)$ is in fact divisible by p^n, and we find

$$f_1(x) \underset{\mathrm{df}}{=} \frac{1}{p^n} f(x) = \frac{g_0(x)}{p^n}(x-a)^d + \sum_{i=1}^{d} g_i(x) p^{n(i-1)} (x-a)^{(d-1)-(i-1)} \ ,$$

and this polynomial is in $(x-a, p^n)^{d-1}$. This shows that $f_1(x)\partial_x^d$ is a global section of $\mathcal{I}_{n,d-1}\mathcal{T}_{\mathbb{X}}^{\otimes d}$. If f_1 is not divisible by p, then the same reasoning as above shows that (f_1 mod p)∂_x^d gives rise to a non-zero global section of $\mathcal{I}_{n,d-1,\mathbb{F}_p}\mathcal{T}_{\mathbb{P}^1_{\mathbb{F}_p}}^{\otimes d}$ and this sheaf has degree $-(p+1)(d-1)+2d = (1-p)d+p+1$. If this number is negative we arrive at a contradiction. Suppose this number is non-negative. Arguing as above shows then that f_1 must be divisible by p^n, and hence f is divisible by p^{2n}. Running the same arguments repeatedly proves that if $(1-p)d + j(p+1) < 0$ we must have that f is divisible by $p^{n(j+1)}$. Now the assertion follows because $c-1$ is the largest possible value for j. □

Remark 5.1.13. The exponent nc of p on the right side of 5.1.10 is likely not the largest possible exponent for all n and d. While it is interesting to find the largest possible exponent of p for the inclusion on the right side of 5.1.10, the most optimistic guess that it be nd is in general false. Consider for instance the case when $n=1$ and $d=p$. Then $p^{p-1}(x^p - x)\partial_x^{\otimes p}$ is a global section of $\mathcal{I}_{1,p}\mathcal{T}_{\mathbb{X}}^{\otimes p}$ as can be checked easily. We thus see that the optimal exponent would be at least $p-1$ and this is indeed equal to $\lceil p\frac{p-1}{p+1}\rceil$ for all p. Moreover, $p^{k(p-1)}(x^p-x)^k\partial_x^{\otimes kp}$ is a global section of $\mathcal{I}_{1,kp}\mathcal{T}_{\mathbb{X}}^{\otimes kp}$ for all k and p, and we thus see that the exponent is at most $k(p-1) = kp\frac{p-1}{p}$. As a consequence, we see that the ratio $\frac{\text{optimal exponent}}{nd}$ is bounded by $\frac{p-1}{p}$ for $n=1$. Similar examples probably exist for arbitrary n.

5.2. **Differential operators on $\mathbb{X}_n$ and distribution algebras.** Let $\mathcal{D}^{(m)}_{\mathbb{X}_n} = \mathcal{D}^{(m)}_{\mathbb{X}_n,\log}$ be the sheaf of logarithmic differential operators on $\mathbb{X}_n$ of level m. As an $\mathcal{O}_{\mathbb{X}_n}$-module it is on an open affine subset $\mathbb{X}^{(\nu)}_{\underline{a}} \subset \mathbb{X}_n$, cf. 4.2.6, locally generated by logarithmic differential operators

$$q^{(m)}_d! \binom{D}{d}$$

where

$$D = x_{\underline{a}}^{(\nu-1)}\partial_{x_{\underline{a}}^{(\nu-1)}} = -z_{\underline{a}}^{(\nu)}\partial_{z_{\underline{a}}^{(\nu)}} \ .$$

is a local section of the logarithmic tangent sheaf $\mathcal{T}_{\mathbb{X}_n,\log}$, cf. 5.1. On the open subscheme $\mathbb{D}_{\underline{b}}^{(n)}$ with coordinate function $x_{\underline{b}}^{(n)}$ it is generated by

$$\frac{q_d^{(m)}!}{d!}\partial^d_{x_{\underline{b}}^{(n)}} \ .$$

Denote by $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge$ the p-adic completion of $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})$ and put $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge_{\mathbb{Q}} = H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 5.2.1. *Given n let $n' = \lfloor n\frac{p-1}{p+1}\rfloor$ be the greatest integer less or equal to $n\frac{p-1}{p+1}$. Then we have natural inclusions*

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0} \ \hookrightarrow \ \varinjlim_m H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^\wedge_{\mathbb{Q}} \ \hookrightarrow \ \mathcal{D}^{\mathrm{an}}(\mathbb{G}(n')^\circ)_{\theta_0} \ .$$

Proof. 1. The inclusion on the left side. The inclusion $\mathbb{G}(n)^\circ \subset \mathbb{G}(0)^\circ$ induces an embedding

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0} \hookrightarrow \mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)_{\theta_0} \ ,$$

and the right hand side is canonically isomorphic to

$$\varinjlim_m H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})^\wedge_{\mathbb{Q}} \ ,$$

by 3.2.1 and 3.2.2. On the other hand, arguing as in the proof of 5.1.2, part (a), one sees that $\mathcal{D}^{(m)}_{\mathbb{X}_n}$ is naturally a subsheaf of $pr^*\left(\mathcal{D}^{(m)}_{\mathbb{X}}\right)$, and so $H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n}) \hookrightarrow H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})$. The inclusion in question is thus understood to be an inclusion inside $\varinjlim_m H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})^\wedge_{\mathbb{Q}}$.

Now use 2.4.3 and the explicit form of the generators of $U(p^n\mathfrak{g}_{\mathbb{Z}_p})^{(m)}$ in 2.4.1. Consider such an element

$$q_{\nu_1}^{(m)}!\frac{(p^ne)^{\nu_1}}{\nu_1!} \cdot q_{\nu_2}^{(m)}!p^{n\nu_2}\binom{h_1}{\nu_2} \cdot q_{\nu_3}^{(m)}!p^{n\nu_3}\binom{h_2}{\nu_3} \cdot q_{\nu_4}^{(m)}!\frac{(p^nf)^{\nu_4}}{\nu_4!} \ .$$

Its image under the canonical map

$$\xi^{(m)} : U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)} \longrightarrow H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})$$

cf. 3.1.12, is

$$\frac{q^{(m)}_{\nu_1}!}{\nu_1!}(p^n\partial_x)^{\nu_1}\cdot\frac{q^{(m)}_{\nu_2}!}{\nu_2!}p^{n\nu_2}(-x)^{\nu_2}\partial_x^{\nu_2}\cdot\frac{q^{(m)}_{\nu_3}!}{\nu_3!}p^{n\nu_3}x^{\nu_3}\partial_x^{\nu_3}\cdot\frac{q^{(m)}_{\nu_4}!}{\nu_4!}(-p^nx^2\partial_x)^{\nu_4}\ .$$

The first and last term are of the form

$$\frac{q^{(m)}_{\nu}!}{\nu!}\left(p^n(\text{global section of }\mathcal{T}_{\mathbb{X}})\right)^{\nu}\ .$$

Because $H^0(\mathbb{X}_n,\mathcal{T}_{\mathbb{X}_n,\log}) = p^nH^0(\mathbb{X},\mathcal{T}_{\mathbb{X}})$, cf. 5.1.11, we see that these terms are in $H^0(\mathbb{X}_n,\mathcal{D}^{(m)}_{\mathbb{X}_n})$. For the second and third term we consider an open affine subset $\mathbb{X}^{(\mu)}_{\underline{a}}$. Let $x^{(\mu-1)} = x^{(\mu-1)}_{\underline{a}}$ be the coordinate on $\mathbb{X}^{(\mu)}_{\underline{a}}$ as in 4.2.5. Use 5.1.3, i.e., $\partial_{x^{(\mu-1)}} = p^{\mu-1}\partial_x$, and write

$$\begin{aligned}\frac{q^{(m)}_{\nu}!}{\nu!}p^{n\nu}x^{\nu}\partial_x^{\nu} &= \frac{q^{(m)}_{\nu}!}{\nu!}p^{n\nu}(x-a+a)^{\nu}\partial_x^{\nu}\\ &= \sum_{k=0}^{\nu}\frac{q^{(m)}_{\nu}!}{(q^{(m)}_{k}!)(q^{(m)}_{\nu-k}!)}p^{nk}\cdot a^{\nu-k}\cdot\frac{q^{(m)}_{k}!}{k!}(x-a)^k\partial_x^k\cdot\frac{q^{(m)}_{\nu-k}!}{(\nu-k)!}(p^n\partial_x)^{\nu-k}\end{aligned}\tag{5.2.2}$$

By what we have observed before we find that the term $\frac{q^{(m)}_{\nu-k}!}{(\nu-k)!}(p^n\partial_x)^{\nu-k}$ is a global section of $\mathcal{D}^{(m)}_{\mathbb{X}_n}$. The relation $p^{\mu-1}x^{(\mu-1)} = x-a$ together with 5.1.3 gives

$$\frac{q^{(m)}_{k}!}{k!}(x-a)^k\partial_x^k = \frac{q^{(m)}_{k}!}{k!}(x^{(\mu-1)})^k\partial^k_{x^{(\mu-1)}}\ ,$$

and so extends to a section of $\mathcal{D}^{(m)}_{\mathbb{X}_n}$ over $\mathbb{X}^{(\mu)}_{\underline{a}}$. It is a straightforward exercise to see that $\frac{q^{(m)}_{\nu}!}{(q^{(m)}_{k}!)(q^{(m)}_{\nu-k}!)}$ is always an integer, and $\frac{q^{(m)}_{\nu}!}{\nu!}p^{n\nu}x^{\nu}\partial_x^{\nu}$ therefore extends to a section of $\mathcal{D}^{(m)}_{\mathbb{X}_n}$ over $\mathbb{X}^{(\mu)}_{\underline{a}}$. Finally, we consider the subscheme $\mathbb{D}^{(n)}_{\underline{b}}$. Let $x^{(n)} = x^{(n)}_{\underline{b}}$ be the coordinate on $\mathbb{D}^{(n)}_{\underline{b}}$, as in 4.2.4, where $\underline{b} = (a_0,\ldots,a_n)$. Put $b = a_0+\ldots+a_np^n$. Writing $x = (x-b)+b$, we can perform exactly the same calculation 5.2.2 as above, using 5.1.5, and find that $\frac{q^{(m)}_{\nu}!}{\nu!}p^{n\nu}x^{\nu}\partial_x^{\nu}$ extends to a section of $\mathcal{D}^{(m)}_{\mathbb{X}_n}$ over $\mathbb{D}^{(n)}_{\underline{b}}$. So we can conclude that the terms $\frac{q^{(m)}_{\nu}!}{\nu!}p^{n\nu}x^{\nu}\partial_x^{\nu}$ are in $H^0(\mathbb{X}_n,\mathcal{D}^{(m)}_{\mathbb{X}_n})$.

The image of $\xi^{(m)}$ thus lies in $H^0(\mathbb{X}_n,\mathcal{D}^{(m)}_{\mathbb{X}_n})$. Passing to the completions and the direct limit over m we find that $\xi^{(m)}$ induces a map

$$D^{an}(\mathbb{G}(n)^{\circ})\longrightarrow\varinjlim_m H^0(\mathbb{X}_n,\mathcal{D}^{(m)}_{\mathbb{X}_n})^{\wedge}_{\mathbb{Q}}\ ,$$

which makes the diagram

$$\begin{array}{ccc} D^{an}(\mathbb{G}(n)^\circ) & \longrightarrow & \varinjlim_m H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^{\wedge}_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ D^{an}(\mathbb{G}(0)^\circ)_{\theta_0} & \longrightarrow & \varinjlim_m H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})^{\wedge}_{\mathbb{Q}} \end{array}$$

commute. The lower horizontal arrow is an isomorphism and the right vertical arrow is injective. The assertion now follows.

2. The inclusion on the right side. For this inclusion consider the diagram

$$\begin{array}{ccc} \varinjlim_m H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n})^{\wedge}_{\mathbb{Q}} & \dashrightarrow & D^{an}(\mathbb{G}(n')^\circ)_{\theta_0} \\ \downarrow & & \downarrow \\ \varinjlim_m H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})^{\wedge}_{\mathbb{Q}} & \longrightarrow & D^{an}(\mathbb{G}(0)^\circ)_{\theta_0} \end{array}$$

where the vertical arrows are injective and we have to show the existence of the dashed arrow. Let $N(m)$ be such that the cokernel of the canonical map

$$U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)} \to H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}})$$

is annihilated by $p^{N(m)}$, cf. 3.1.14 (b). Furthermore, consider the subsheaf $\mathcal{D}^{(m)}_{\mathbb{X}_n,d}$ of logarithmic differential operators of level m and degree $\leqslant d$. Similarly, let $U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d$ be the submodule of elements of degree $\leqslant d$ as defined right before 3.1.11. Then, in order to prove the existence of the dashed arrow in the diagram above, it suffices to prove the existence of a map

$$p^{N(m)}H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n,d}) \dashrightarrow U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d ,$$

which makes the corresponding diagram

$$\begin{array}{ccc} p^{N(m)}H^0(\mathbb{X}_n, \mathcal{D}^{(m)}_{\mathbb{X}_n,d}) & \dashrightarrow & U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d \\ \downarrow & & \downarrow \\ p^{N(m)}H^0(\mathbb{X}, \mathcal{D}^{(m)}_{\mathbb{X}}) & \longrightarrow & U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d \end{array}$$

commute. We do this by induction over d. This is obvious for $d = 0$. For the induction step we can pass to the corresponding graded object in degree d and thus consider

$$\begin{array}{ccc} p^{N(m)}H^0(\mathbb{X}_n,(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})^{(m)}) & \dashrightarrow & U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d/U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{d-1} \\ \downarrow & & \downarrow \\ p^{N(m)}H^0(\mathbb{X},(\mathcal{T}^{\otimes d}_{\mathbb{X}})^{(m)}) & \longrightarrow & U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d/U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{d-1} \end{array}$$

Note that

$$U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d/U(p^{n'}\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{d-1} \quad = \quad p^{dn'}\left(U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_d/U(\mathfrak{g}_{\mathbb{Z}_p})^{(m)}_{d-1}\right) \ .$$

By 5.1.2 and 5.1.9 we have an inclusion

$$H^0(\mathbb{X}_n,(\mathcal{T}^{\otimes d}_{\mathbb{X}_n,\log})^{(m)}) \subset p^{nc(d)}H^0(\mathbb{X},(\mathcal{T}^{\otimes d}_{\mathbb{X}})^{(m)}) \ ,$$

where $c(d)=\left\lceil d\frac{p-1}{p+1}\right\rceil$. The assertion now follows from the following inequalities:

$$nc(d) = n\left\lceil d\frac{p-1}{p+1}\right\rceil \geqslant nd\frac{p-1}{p+1} \geqslant d\left\lfloor n\frac{p-1}{p+1}\right\rfloor = dn' \ .$$

□

Remark 5.2.3. We recall that $\mathfrak{X}_n$ denotes the completion of $\mathbb{X}_n$ along its special fiber, and we let $\mathscr{D}^{(m)}_{\mathfrak{X}_n}=\widehat{\mathcal{D}}^{(m)}_{\mathbb{X}_n}$ be the p-adic completion of the sheaf $\mathcal{D}^{(m)}_{\mathbb{X}_n}$. Consider these as sheaves on $\mathfrak{X}_n$. Put $\mathscr{D}^{(m)}_{\mathfrak{X}_n,\mathbb{Q}}=\mathscr{D}^{(m)}_{\mathfrak{X}_n}\otimes_{\mathbb{Z}}\mathbb{Q}$ and

$$\mathscr{D}^{\dagger}_{\mathfrak{X}_n,\mathbb{Q}} \underset{\mathrm{df}}{=} \varinjlim_m \mathscr{D}^{(m)}_{\mathfrak{X}_n,\mathbb{Q}} \ .$$

Then, as is not difficult to see, there is a canonical injective ring homomorphism

$$\varinjlim_m H^0(\mathbb{X}_n,\mathcal{D}^{(m)}_{\mathbb{X}_n})^{\wedge}_{\mathbb{Q}} \ \hookrightarrow \ H^0(\mathfrak{X}_n,\mathscr{D}^{\dagger}_{\mathfrak{X}_n,\mathbb{Q}}) \ . \tag{5.2.4}$$

The same reasoning as in [10, Prop. 3.2] shows that this map is an isomorphism, if $H^1(\mathbb{X}_n,\mathcal{D}^{(m)}_{\mathbb{X}_n})$ is annihilated by some fixed power of p. This question in turn is closely connected to the question whether $\mathfrak{X}_n$ is $\mathscr{D}^{\dagger}_{\mathfrak{X}_n,\mathbb{Q}}$-affine, a problem we plan to discuss in a future paper.

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