LOCALLY ANALYTIC REPRESENTATIONS AND SHEAVES ON THE BRUHAT-TITS BUILDING

DEEPAM PATEL, TOBIAS SCHMIDT, AND MATTHIAS STRAUCH

Abstract. Let L be a finite field extension of $\mathbb{Q}_p$ and let G be the group of L-rational points of a split connected reductive group over L. We view G as a locally L-analytic group with Lie algebra $\mathfrak{g}$. The purpose of this work is to propose a construction which extends the localization of smooth G-representations of P. Schneider and U. Stuhler to the case of locally analytic G-representations. We define a functor from admissible locally analytic G-representations with prescribed infinitesimal character to a category of equivariant sheaves on the Bruhat-Tits building of G. For smooth representations, the corresponding sheaves are closely related to the sheaves of Schneider and Stuhler. The functor is also compatible, in a certain sense, with the localization of $\mathfrak{g}$-modules on the flag variety by A. Beilinson and J. Bernstein.

Contents

M. S. would like to acknowledge the support of the National Science Foundation (award numbers DMS-0902103 and DMS-1202303).

1. Introduction

Let L be a finite field extension of the field $\mathbb{Q}_p$ of p-adic numbers. Let $\mathbf{G}$ be a connected split reductive group over L and $\mathbf{B} \subset \mathbf{G}$ a Borel subgroup defined over L. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus contained in $\mathbf{B}$. Let $G := \mathbf{G}(L)$, $T := \mathbf{T}(L)$ denote the groups of rational points, viewed as locally L-analytic groups. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the corresponding Lie algebras.

The purpose of this work is to propose a construction which extends the localization theory for smooth G-representations of P. Schneider and U. Stuhler ([**?**]) to the case of locally analytic G-representations. In more concrete terms, we define an exact functor from admissible locally analytic G-representations with prescribed infinitesimal character to a category of equivariant sheaves on the Bruhat-Tits building of G. The functor is also compatible, in a certain sense, with the localization theory for $\mathfrak{g}$-modules on the flag variety of $\mathbf{G}$ by A. Beilinson and J. Bernstein ([**?**]), and J.-L. Brylinski and M. Kashiwara ([**?**], [**?**]).

To give more details, let $\mathscr{B}$ be the (semisimple) Bruhat-Tits building of G. The torus $\mathbf{T}$ determines an apartment A in $\mathscr{B}$. We fix a fundamental chamber $\mathscr{C} \subset A$ and a special vertex $x_0 \in \overline{\mathscr{C}}$ which will be used as an origin for the affine space A. In [?] the authors consider, for any facet $F \in \mathscr{B}$, a well-behaved filtration

$$P_F \supset U_F^{(0)} \supset U_F^{(1)} \supset \dots$$

of the pointwise stabilizer P_F of F in G by open pro-p subgroups $U_F^{(e)}$. For any point $z \in \mathscr{B}$, one sets $U_z^{(e)} := U_F^{(e)}$ where F is the unique facet containing z. It forms a fundamental system of neighborhoods of $1 \in P_z$ where P_z is the stabilizer of z. Let from now on $e \geq 0$ be a fixed number (called a *level* in loc.cit.).

Using the groups $U_z^{(e)}$, Schneider and Stuhler define in [?, sec. IV] an exact functor

$$V \mapsto \underset{\approx}{V}$$

from smooth complex G-representations to sheaves of complex vector spaces on $\mathscr{B}$. The stalk of the sheaf $\underset{\approx}{V}$ at a point z is given by the coinvariants $V_{U_z^{(e)}}$ and the restriction of $\underset{\approx}{V}$ to a facet $F \subset \mathscr{B}$ equals the constant sheaf with fibre $V_{U_F^{(e)}}$. The functor $V \mapsto \underset{\approx}{V}$ has particularly good properties when restricted to the subcategory of representations generated by their $U_{x_0}^{(e)}$-fixed vectors. It is a major tool in the proof of the Zelevinsky conjecture (loc.cit.).

¿From now on we fix a complete discretely valued field extension K of L. The functor $V \mapsto \underset{\approx}{V}$ can be defined in exactly the same way for smooth G-representations on K-vector spaces, and produces sheaves of K-vector spaces on $\mathscr{B}$. The naive extension of the functor $V \mapsto \underset{\approx}{V}$ to locally analytic representations, by taking coinvariants as above, does not have good properties. For instance, applying this procedure to an irreducible finite-dimensional algebraic representation, which is not the trivial representation, produces the zero sheaf. Moreover, if we aim at a picture which is related to the localization theory of $\mathfrak{g}$-modules, then localizing an irreducible algebraic representation should give a line bundle.

We consider the variety of Borel subgroups

$$X = \mathbf{G}/\mathbf{B}$$

of $\mathbf{G}$. We let $\mathcal{O}_X$ be its structure sheaf and $\mathcal{D}_X$ be its sheaf of differential operators. Deriving the left regular action of $\mathbf{G}$ on X yields an algebra homomorphism

$$\alpha : \underline{U}(\mathfrak{g}) \longrightarrow \mathcal{D}_X$$

where the source refers to the constant sheaf on X with fibre equal to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the center of the ring $U(\mathfrak{g})$.

The torus $\mathbf{T}$ determines a root system. Let ρ be half the sum over the positive roots with respect to $\mathbf{B}$. For any algebraic character $\chi - \rho$ of the torus $\mathbf{T}$ we have the sheaf

$\mathcal{D}_\chi$ of differential endomorphisms of the line bundle on X associated with $\chi - \rho$. Any trivialization of the line bundle induces a local isomorphism between $\mathcal{D}_\chi$ and $\mathcal{D}_X$, and we have $\mathcal{D}_\rho = \mathcal{D}_X$. More generally, if $\chi - \rho$ is an arbitrary character of $\mathfrak{t}$ there is a sheaf of so-called *twisted* differential operators $\mathcal{D}_\chi$ on X. As in the former case it comes equipped with a morphism $\mathcal{O}_X \hookrightarrow \mathcal{D}_\chi$ which is locally isomorphic to the canonical morphism $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$. Moreover, there is an algebra homomorphism $\underline{U}(\mathfrak{g}) \to \mathcal{D}_\chi$ locally isomorphic to α. The sheaf $\mathcal{D}_\chi$ was first introduced in [?] as a certain quotient sheaf of the skew tensor product algebra $\mathcal{O}_X \# U(\mathfrak{g})$. We use this notation ('#') to indicate that the multiplication on the tensor product $\mathcal{O}_X \otimes U(\mathfrak{g})$ involves the action of $U(\mathfrak{g})$ on $\mathcal{O}_X$.

Let χ be a character of $\mathfrak{t}$. Let θ be the character of $Z(\mathfrak{g})$ associated with χ via the classical Harish-Chandra homomorphism. The above map factors via a homomorphism

$$\underline{U}(\mathfrak{g})_\theta \longrightarrow \mathcal{D}_\chi$$

where $U(\mathfrak{g})_\theta = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\theta} L$. If χ is dominant and regular, a version of the localization theorem ([?]) asserts that the functor

$$\Delta_\chi : M \mapsto \mathcal{D}_\chi \otimes_{\underline{U}(\mathfrak{g})_\theta} \underline{M}$$

is an equivalence of categories between the (left) $U(\mathfrak{g})_\theta$-modules and the (left) $\mathcal{D}_\chi$-modules which are quasi-coherent as $\mathcal{O}_X$-modules. The underlined objects refer to the associated constant sheaves on X. We remark that a seminal application of this theorem (or rather its complex version) leads to a proof of the Kazhdan-Lusztig multiplicity conjecture (cf. [?], [?], [?]).

The starting point of our work is a result of V. Berkovich ([?], [?]) according to which the building $\mathscr{B}$ may be viewed as a locally closed subspace

$$\mathscr{B} \hookrightarrow X^{an}$$

of the Berkovich analytification X^{an} of X. This makes it possible to 'compare' sheaves on $\mathscr{B}$ and X^{an} in various ways. Most of what has been said above about the scheme X extends to the analytic space X^{an}. In particular, there is an analytic version $\mathcal{D}_\chi^{an}$ of $\mathcal{D}_\chi$ and an analytic version $\Delta_\chi(\cdot)^{an}$ of the functor Δ_χ (sec. 6).

For technical reasons we have to assume at some point in this paper that $L = \mathbb{Q}_p$, with $p > 2$ an odd prime. (However, we have no doubts that our results eventually extend to general L and p). To describe our proposed locally analytic 'localization functor' under this assumption we let $D(G)$ be the algebra of K-valued locally analytic distributions on G. It naturally contains $U(\mathfrak{g})$. Recall that the category of admissible locally analytic G-representations over K (in the sense of P. Schneider and J. Teitelbaum, cf. [?]) is anti-equivalent to a full abelian subcategory of the (left) $D(G)$-modules, the so-called coadmissible modules. A similar result holds over any compact open subgroup $U_z^{(e)}$.

¿From now on we fix a central character

$$\theta : Z(\mathfrak{g}_K) \longrightarrow K$$

and a toral character $\chi \in \mathfrak{t}_K^*$ associated to θ via the classical Harish-Chandra homomorphism. For example, the character $\chi = \rho$ corresponds to the trivial infinitesimal character θ_0 with $\ker \theta_0 = Z(\mathfrak{g}_K) \cap U(\mathfrak{g}_K)\mathfrak{g}_K$. The ring $Z(\mathfrak{g}_K)$ lies in the center of the ring $D(G)$, cf. [**?**, Prop. 3.7], so that we may consider the central reduction

$$D(G)_\theta := D(G) \otimes_{Z(\mathfrak{g}_K),\theta} K \ .$$

We propose to study the abelian category of (left) $D(G)_\theta$-modules which are coadmissible over $D(G)$. As remarked above it is anti-equivalent to the category of admissible locally analytic G-representations over K with infinitesimal character θ. We emphasize that *any* topologically irreducible admissible locally analytic G-representation admits, up to a finite extension of K, an infinitesimal character, cf. [**?**, Cor. 3.10].

To start with, consider a point $z \in \mathscr{B}$. The group $U_z^{(e)}$ carries a natural p-valuation in the sense of M. Lazard, cf. [**?**, III.2.1]. According to the general locally analytic theory ([**?**, sec. 4]), this induces a family of norms $\|\cdot\|_r$ on the distribution algebra $D(U_z^{(e)})$ for $r \in [r_0, 1)$ where $r_0 := p^{-1}$. We let $D_r(U_z^{(e)})$ be the corresponding completion of $D(U_z^{(e)})$ and $D_r(U_z^{(e)})_\theta$ its central reduction. In sec. 8.2 we introduce sheaves of distribution algebras $\underline{D}_r$ and $\underline{D}_{r,\theta}$ on $\mathscr{B}$ with stalks

$$(\underline{D}_r)_z = D_r(U_z^{(e)}),\ (\underline{D}_{r,\theta})_z = D_r(U_z^{(e)})_\theta$$

for all points $z \in \mathscr{B}$. The inclusions $U(\mathfrak{g}) \subset D_r(U_z^{(e)})$ sheafify to a morphism $\underline{U}(\mathfrak{g}_K)_\theta \rightarrow \underline{D}_{r,\theta}$. Similarly, for any coadmissible $D(G)_\theta$-module M we consider a $\underline{D}_{r,\theta}$-module $\underline{M}_r$ on $\mathscr{B}$ having stalks

$$(\underline{M}_r)_z = D_r(U_z^{(e)})_\theta \otimes_{D(U_z^{(e)})_\theta} M$$

for all points $z \in \mathscr{B}$. The formation of $\underline{M}_r$ is functorial in M. The sheaves $\underline{D}_{r,\theta}$, $\underline{M}_r$ are constructible and will formally replace the constant sheaves appearing in the definition of the functors Δ_χ, Δ_χ^{an}.[1]

Consider the restriction of the structure sheaf of X^{an} to $\mathscr{B}$, i.e.,

$$\mathcal{O}_{\mathscr{B}} = \mathcal{O}_{X^{an}}|_{\mathscr{B}} \ .$$

We then define a sheaf of noncommutative rings $\mathscr{D}_{r,\chi}$ on $\mathscr{B}$ which is also a module over $\mathcal{O}_{\mathscr{B}}$ and which is vaguely reminiscent of a 'sheaf of twisted differential operators'. It has a natural G-equivariant structure. It depends on the level e, but, following the usage of [**?**,

[1]We assume from now on that e is sufficiently large (later in the paper we require $e > e_{st}$, where e_{st} is defined in 6.2.6) and that the radius r is equal to $\sqrt[p^m]{1/p}$ for some $m \geq 0$, cf. 7.4.7.

sec. IV.1], we suppress this in our notation. More importantly, it depends on the 'radius' r which is genuine to the locally analytic situation and is related to a choice of completed distribution algebra $D_r(U_z^{(e)})$ at each point $z \in \mathscr{B}$. Completely analogous to constructing $\mathcal{D}_\chi$ out of the skew tensor product algebra $\mathcal{O}_X \# U(\mathfrak{g}_K)$ (cf. [?]) we obtain the sheaf $\mathscr{D}_{r,\chi}$ out of a skew tensor product algebra of the form $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$.

To describe the sheaf $\mathscr{D}_{r,\chi}$ we observe first that, for any point $z \in \mathscr{B}$, the inclusion $U_z^{(e)} \subset P_z$ implies that there is a locally analytic $U_z^{(e)}$-action on the analytic stalk $\mathcal{O}_{\mathscr{B},z}$. We therefore have the corresponding skew group ring $\mathcal{O}_{\mathscr{B},z} \# U_z^{(e)}$ as well as the skew enveloping algebra $\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g})$, familiar objects from noncommutative ring theory ([?]). In section 3 and in sections 6.3, 6.4, we explain how the completed tensor product

$$\mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)})$$

can be endowed with a unique structure of a topological K-algebra such that the $\mathcal{O}_{\mathscr{B},z}$-linear maps

$$\mathcal{O}_{\mathscr{B},z} \# U_z^{(e)} \to \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)}) \quad \text{and} \quad \mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g}) \to \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L D_r(U_z^{(e)}) \;, \tag{1.1.1}$$

induced by $U_z^{(e)} \subset D(U_z^{(e)})^\times$ and $U(\mathfrak{g}) \subset D(U_z^{(e)})$ respectively, become ring homomorphisms. To emphasize this skew multiplication we denote the target of the two maps in 1.1.1 by $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ keeping in mind that there is a *completed* tensor product involved. This process leads to a sheaf of K-algebras $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$ on $\mathscr{B}$ with stalks

$$(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z = \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$$

at points $z \in \mathscr{B}$. It comes equipped with a morphism $\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}) \to \mathcal{O}_{\mathscr{B}} \# \underline{D}_r$ giving back the second map in (1.1.1) at a point $z \in \mathscr{B}$.

To generalize the formalism of *twisting* to this new situation we proceed similarly to [?]. Let $\mathcal{T}_{X^{an}}$ be the tangent sheaf of X^{an} and let $\alpha^{an} : \mathfrak{g} \to \mathcal{T}_{X^{an}}$ be the analytification of the map $\alpha|_{\mathfrak{g}}$. There is the sheaf of L-Lie algebras

$$\mathfrak{b}^{\circ,an} := \ker\, (\mathcal{O}_{X^{an}} \otimes_L \mathfrak{g} \xrightarrow{\alpha^{an}} \mathcal{T}_{X^{an}}) \;.$$

The inclusion $\mathbf{T} \subset \mathbf{B}$ induces an isomorphism of Lie algebras

$$\mathcal{O}_{X^{an}} \otimes_L \mathfrak{t} \xrightarrow{\cong} \mathfrak{b}^{\circ,an}/[\mathfrak{b}^{\circ,an}, \mathfrak{b}^{\circ,an}] \;.$$

We have thus an obvious $\mathcal{O}_{X^{an}}$-linear extension of the character $\chi - \rho$ of $\mathfrak{t}_K$ to $\mathfrak{b}^{\circ,an} \otimes_L K$. Its kernel, restricted to the building $\mathscr{B}$, generates a two-sided ideal $\mathscr{I}_\chi^{an}$ in $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$ and we set

$$\mathscr{D}_{r,\chi} := (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r) / \mathscr{I}_\chi^{an} \;.$$

Let $\mathcal{D}^{an}_{\mathscr{B},\chi}$ denote the restriction of $\mathcal{D}^{an}_{\chi}$ to the building $\mathscr{B}$. The sheaf $\mathscr{D}_{r,\chi}$ comes with an algebra homomorphism $\mathcal{D}^{an}_{\mathscr{B},\chi} \to \mathscr{D}_{r,\chi}$ induced from the inclusion $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$. Most importantly, the canonical morphism $\underline{D}_r \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ induces a canonical morphism $\underline{D}_{r,\theta} \to \mathscr{D}_{r,\chi}$ making the diagram

$$\begin{array}{ccc} \underline{U}(\mathfrak{g}_K)_\theta & \longrightarrow & \mathcal{D}^{an}_{\mathscr{B},\chi} \\ \downarrow & & \downarrow \\ \underline{D}_{r,\theta} & \longrightarrow & \mathscr{D}_{r,\chi} \end{array}$$

commutative. In this situation we prove that

$$M \mapsto \mathscr{L}_{r,\chi}(M) := \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

is an exact covariant functor from coadmissible $D(G)_\theta$ modules into G-equivariant (left) $\mathscr{D}_{r,\chi}$-modules. The stalk of the sheaf $\mathscr{L}_{r,\chi}(M)$ at a point $z \in \mathscr{B}$ with residue field $\kappa(z)$ equals the $(\chi - \rho)$-coinvariants of the $\mathfrak{t}_K$-module

$$(\kappa(z)\hat{\otimes}_L \underline{M}_{r,z})/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L \underline{M}_{r,z})$$

as it should ([?]). Here, $\mathfrak{n}_{\pi(z)}$ equals the nilpotent radical of the Borel subalgebra of $\kappa(z) \otimes_L \mathfrak{g}$ defined by the point $\pi(z) \in X$ where $\pi : X^{an} \to X$ is the canonical map. We tentatively call $\mathscr{L}_{r,\chi}$ a locally analytic 'localization functor'. We suppress the dependence of $\mathscr{L}_{r,\chi}$ on the level e in our notation.

We prove the following compatibilities with the Schneider-Stuhler and the Beilinson-Bernstein localizations. Suppose first that the coadmissible module M is associated to a *smooth* G-representation V. Since $\mathfrak{g}M = 0$ it has infinitesimal character $\theta = \theta_0$ and the natural choice of twisting is therefore $\chi = \rho$. We establish a canonical isomorphism (Thm. 9.2.4) of $\mathcal{O}_{\mathscr{B}}$-modules

$$\mathscr{L}_{r_0,\rho}(M) \xrightarrow{\cong} \mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx}{\check{V}}$$

where $\check{V}$ is the smooth dual of V and $\underset{\approx}{\check{V}}$ the sheaf associated to $\check{V}$ by Schneider-Stuhler. The isomorphism is natural in M.

Secondly, suppose the coadmissible module M is associated to a *finite dimensional algebraic* G-representation. The functor $\Delta_\chi(\cdot)^{an}$ may be applied to its underlying $\mathfrak{g}$-module and gives a $\mathcal{D}^{an}_{\chi}$-module on X^{an} and then, via restriction, a $\mathcal{D}^{an}_{\mathscr{B},\chi}$-module $\Delta_\chi(M)^{an}_{\mathscr{B}}$ on $\mathscr{B}$. We prove (Thm. 10.1.1) that there is a number $r(M) \in [r_0, 1)$ which is intrinsic to M and a canonical isomorphism of $\mathcal{D}^{an}_{\mathscr{B},\chi}$-modules

$$\mathscr{L}_{r,\chi}(M) \xrightarrow{\cong} \Delta_\chi(M)^{an}_{\mathscr{B}}$$

for $r \geq r(M)$. The isomorphism is natural in M.

As a class of examples we finally investigate the localizations of locally analytic representations in the image of the functor $\mathcal{F}^G_B$ introduced by S. Orlik and investigated in [**?**]. The image of $\mathcal{F}^G_B$ comprises a wide class of interesting representations and contains all principal series representations as well as all locally algebraic representations (e.g. tensor products of smooth with algebraic representations).

This paper is the first of a series of papers whose aim is to develop a localization theory for locally analytic representations. Here we only make a first step in this direction, focusing on the building and merging the theory of Schneider and Stuhler with the theory of Beilinson and Bernstein, resp. Brylinski and Kashiwara. One approach to get a more complete picture would be to extend the construction given here to a compactification $\overline{\mathscr{B}}$ of the building. The compactification which one would take here is, of course, the closure of $\mathscr{B}$ in X^{an}. Moreover, for intended applications like functorial resolutions and the computation of Ext groups, one has to develop a 'homological theory', in analogy to [**?**, sec. II]. However, the sheaves produced in this way (using a compactification) would still have too many global sections. For instance, the space of global sections would be a module for the ring of meromorphic functions on X^{an} with poles outside $\overline{\mathscr{B}}$, and this is a very large ring. The aim would be to produce sheaves whose global sections give back the $D(G)$-module one started with. In the paper ([**?**]) we explore an approach (in the case of $GL(2)$) which is based on the use of (a family of) semistable formal models $\mathfrak{X}$ of X^{an}, and we replace $\mathcal{O}_{\mathscr{B}}$ by the pull-back of $\mathcal{O}_{\mathfrak{X}} \otimes L$ via the specialization map $X^{an} \to \mathfrak{X}$, and the rôle of $\mathscr{D}_{r,\chi}$ is played by arithmetic logarithmic differential operators. In this regard we want to mention related works by C. Noot-Huyghe ([**?**]), and K. Ardakov and S. Wadsley ([**?**]). While Noot-Huyghe studies localizations of arithmetic $\mathscr{D}$-modules on smooth formal models of X, Ardakov and Wadsley define and study localizations of representations of Iwasawa algebras on smooth models. Our present paper is in some sense complementary to these papers, as our focus is on non-compact groups.

Despite the many aspects (like compactifications, homological theory, relation with formal models) that still have to be explored, given the many technical details that one has to take care of we thought it worthwhile to give an account of the constructions as developed up to this point.

Acknowledgments. We thank Vladimir Berkovich for helpful correspondence on p-adic symmetric spaces and buildings. Furthermore, we thank an anonymous referee for pointing out inaccuracies in an earlier version and many comments which helped improve the paper in several places. T. S. gratefully acknowledges travel support by the SFB 878 "Groups, Geometry & Actions" at the University of Münster. D. P. would like to thank Indiana University, Bloomington, for its support and hospitality.

Notations. Let p be an odd prime. Let $L/\mathbb{Q}_p$ be a finite extension and $K \subseteq \mathbb{C}_p$ a complete discretely valued extension of L. The absolute value $|.|$ on $\mathbb{C}_p$ is normalized by $|p| = p^{-1}$. Let $o_L \subset L$ be the ring of integers and $\varpi_L \in o_L$ a uniformizer. We denote by v_L always the normalized p-adic valuation on L, i.e. $v_L(\varpi) = 1$. Let n and $e(L/\mathbb{Q}_p)$ be the degree and the ramification index of the extension $L/\mathbb{Q}_p$ respectively. Similarly, $o_K \subset K$ denotes the integers in K and $\varpi_K \in o_K$ a uniformizer. Let $k := o_K/(\varpi_K)$ denote the residue field of K.

The letter $\mathbf{G}$ always denotes a connected reductive linear algebraic group over L which is split over L and $G = \mathbf{G}(L)$ denotes its group of rational points.

2. Distribution algebras and locally analytic representations

For notions and notation from non-archimedean functional analysis we refer to the book [**?**]. If not indicated otherwise, topological tensor products of locally convex vector spaces are always taken with respect to the projective tensor product topology.

2.1. **Distribution algebras.** In this section we recall some definitions and results about algebras of distributions attached to locally analytic groups ([**?**], [**?**]). We consider a locally L-analytic group H and denote by $C^{an}(H,K)$ the locally convex K-vector space of locally L-analytic functions on H as defined in [**?**]. The strong dual

$$D(H,K) := C^{an}(H,K)'_b$$

is the algebra of K-valued locally analytic distributions on H where the multiplication is given by the usual convolution product. This multiplication is separately continuous. However, if H is compact, then $D(H,K)$ is a K-Fréchet algebra. The algebra $D(H,K)$ comes equipped with a continuous K-algebra homomorphism

$$\Delta : D(H,K) \longrightarrow D(H,K)\hat{\otimes}_{K,\iota} D(H,K)$$

which has all the usual properties of a comultiplication ([**?**, §3 App.]). Here ι refers to the (complete) inductive tensor product[2]. If H is compact, then $D(H,K)$ is a Fréchet space and the inductive and projective trensor product topology on the right hand side coincide, cf. [**?**, 17.6]. Of course, $\Delta(\delta_h) = \delta_h \otimes \delta_h$ for $h \in H$.

The universal enveloping algebra $U(\mathfrak{h})$ of the Lie algebra $\mathfrak{h} := Lie(H)$ of H acts naturally on $C^{an}(H,K)$. On elements $\mathfrak{x} \in \mathfrak{h}$ this action is given by

$$(\mathfrak{x}f)(h) = \frac{d}{dt}(t \mapsto f(\exp_H(-t\mathfrak{x})h))|_{t=0}$$

where $\exp_H : \mathfrak{h} - - > H$ denotes the exponential map of H, defined in a small neighborhood of 0 in $\mathfrak{h}$. This gives rise to an embedding of $U(\mathfrak{h})_K := U(\mathfrak{h}) \otimes_L K$ into $D(H,K)$ via

$$U(\mathfrak{h})_K \hookrightarrow D(H,K),\ \mathfrak{x} \mapsto (f \mapsto (\dot{\mathfrak{x}}f)(1))\ .$$

[2] the only exception to our general convention to only consider the projective tensor product topology

Here $\mathfrak{x} \mapsto \dot{\mathfrak{x}}$ is the unique anti-automorphism of the K-algebra $U(\mathfrak{h})_K$ which induces multiplication by -1 on $\mathfrak{h}$. The comultiplication Δ restricted to $U(\mathfrak{g})_K$ gives the usual comultiplication of the Hopf algebra $U(\mathfrak{g})_K$, i.e. $\Delta(\mathfrak{x}) = \mathfrak{x} \otimes 1 + 1 \otimes \mathfrak{x}$ for all $\mathfrak{x} \in \mathfrak{h}$.

2.2. **Norms and completions of distribution algebras.**

2.2.1. *p-valuations.* Let H be a compact locally $\mathbb{Q}_p$-analytic group. Recall ([**?**]) that a *p-valuation* ω on H is a real valued function $\omega : H \setminus \{1\} \to (1/(p-1), \infty) \subset \mathbb{R}$ satisfying

(i) $\omega(gh^{-1}) \geq \min\,(\omega(g), \omega(h))$,
(ii) $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$,
(iii) $\omega(g^p) = \omega(g) + 1$

for all $g, h \in H$. As usual one puts $\omega(1) := \infty$ and interprets the above inequalities in the obvious sense if a term $\omega(1)$ occurs.

Let ω be a p-valuation on H. It follows from loc.cit., III.3.1.3/7/9 that the topology on H is defined by ω (loc.cit., II.1.1.5) and H is a pro-p group. Moreover, there is a topological generating system $h_1, ..., h_d$ of H such that the map

$$\mathbb{Z}_p^d \to H,\ (a_1, ..., a_d) \mapsto h_1^{a_1} \cdots h_d^{a_d}$$

is well-defined and a homeomorphism. Moreover,

$$\omega(h_1^{a_1} \cdots h_d^{a_d}) = \min\{\omega(h_i) + v_p(a_i) \mid i = 1, ..., d\}$$

where v_p denotes the p-adic valuation on $\mathbb{Z}_p$. The sequence $(h_1, ..., h_d)$ is called a *p-basis* (or an *ordered basis*, cf. [**?**, §4]) of the p-valued group (H, ω).

Finally, a p-valued group (H, ω) is called *p-saturated* if any $g \in H$ such that $\omega(g) > p/(p-1)$ is a p-th power in H.

2.2.2. *The canonical p-valuation on uniform pro-p groups.* We recall some definitions and results about pro-p groups ([**?**, ch. 3,4]) in the case $p \neq 2$. In this subsection H will be a pro-p group which is equipped with its topology of a profinite group. Then H is called *powerful* if H/H^p is abelian. Here, H^p is the closure of the subgroup generated by the p-th powers of its elements. If H is topologically finitely generated one can show that the subgroup H^p is open and hence automatically closed. The lower p-series $(P_i(H))_{i\geq 1}$ of an arbitrary pro-p group H is defined inductively by

$$P_1(H) := H,\ P_{i+1}(H) := \overline{P_i(H)^p[P_i(H), H]}\ .$$

If H is topologically finitely generated, then the groups $P_i(H)$ are all open in H and form a fundamental system of neighborhoods of 1 (loc.cit, Prop. 1.16). A pro-p group H is called *uniform* if it is topologically finitely generated, powerful and its lower p-series satisfies $(H : P_2(H)) = (P_i(H) : P_{i+1}(H))$ for all $i \geq 1$. If H is a topologically finitely generated powerful pro-p group then $P_i(H)$ is a uniform pro-p group for all sufficiently large i (loc.cit. 4.2). Moreover, any compact $\mathbb{Q}_p$-analytic group contains an open normal uniform pro-p subgroup (loc.cit., 8.34). According to loc.cit., Thm. 9.10, any uniform pro-p group H determines a powerful $\mathbb{Z}_p$-Lie algebra $\mathcal{L}(H)$.[3] Now let H be a uniform pro-p group. It carries a distinguished p-valuation ω^{can} which is associated to the lower p-series and which we call the *canonical p-valuation*. For $h \neq 1$, it is defined by $\omega^{can}(h) = \max\{i \geq 1 : h \in P_i(H)\}$.

2.2.3. *Norms arising from p-valuations.* In this section we let H be a compact $\mathbb{Q}_p$-analytic group endowed with a p-valuation ω that has rational values. For convenience of the reader we briefly recall ([**?**, §4]) the construction of a suitable family of submultiplicative norms $\|\cdot\|_r, r \in [1/p, 1)$ on the algebra $D(H, K)$.

Let $h_1, ..., h_d$ be an ordered basis for (H, ω). The homeomorphism $\psi : \mathbb{Z}_p^d \simeq H$ given by $(a_1, ..., a_d) \mapsto h_1^{a_1} \cdots h_d^{a_d}$ is a global chart for the $\mathbb{Q}_p$-analytic manifold H. By functoriality of $C^{an}(., K)$ it induces an isomorphism $\psi^* : C^{an}(H, K) \xrightarrow{\cong} C^{an}(\mathbb{Z}_p^d, K)$ of topological K-vector spaces. Using Mahler expansions ([**?**, III.1.2.4]) we may express elements of $C(\mathbb{Z}_p^d, K)$, the space of continuous K-valued functions on $\mathbb{Z}_p^d$, as series $f(x) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{x}{\alpha}$ where $c_\alpha \in K$ and $\binom{x}{\alpha} = \binom{x_1}{\alpha_1} \cdots \binom{x_d}{\alpha_d}$ for $x = (x_1, ..., x_d) \in \mathbb{Z}_p^d$ and multi-indices $\alpha = (\alpha_1, .., \alpha_d) \in \mathbb{N}_0^d$. Further, we have $|c_\alpha| \to 0$ as $|\alpha| = \alpha_1 + ... + \alpha_d \to \infty$. A continuous function $f \in C(\mathbb{Z}_p^d, K)$ is locally analytic if and only if $|c_\alpha| r^{|\alpha|} \to 0$ for some real number $r > 1$ (loc.cit. III.1.3.9).

Put $b_i := h_i - 1 \in \mathbb{Z}[H]$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$. Identifying group elements with Dirac distributions induces a K-algebra embedding $K[H] \hookrightarrow D(H, K),\ h \mapsto \delta_h$. In the light of the dual isomorphism $\psi_* : D(\mathbb{Z}_p^d, K) \xrightarrow{\cong} D(H, K)$ we see that any $\delta \in D(H, K)$ has a unique convergent expansion $\delta = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$ with $d_\alpha \in K$ such that the set $\{|d_\alpha| r^{|\alpha|}\}_\alpha$ is bounded for all $0 < r < 1$. Conversely, any such series is convergent in $D(H, K)$. By construction the value $\delta(f) \in K$ of such a series on a function $f \in C^{an}(H, K)$ equals $\delta(f) = \sum_\alpha d_\alpha c_\alpha$ where c_α are the Mahler coefficients of $\psi^*(f)$.

To take the original p-valuation ω into account we define $\tau\alpha := \sum_i \omega(h_i)\alpha_i$ for $\alpha \in \mathbb{N}_0^d$. The family of norms $\|\cdot\|_r$, $0 < r < 1$, on $D(H, K)$ defined on a series δ as above via $\|\delta\|_r := \sup_\alpha |d_\alpha| r^{\tau\alpha}$ defines the Fréchet topology on $D(H, K)$. Let $D_r(H, K)$ denote the norm completion of $D(H, K)$ with respect to $\|\cdot\|_r$. Thus we obtain

$$D_r(H, K) = \left\{ \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha \mid d_\alpha \in K, \lim_{|\alpha| \to \infty} |d_\alpha| r^{\tau\alpha} = 0 \right\} .$$

There is an obvious norm-decreasing linear map $D_{r'}(H, K) \to D_r(H, K)$ whenever $r \leq r'$.

[3]The adjective *powerful* refers here to the property $[\mathcal{L}(H), \mathcal{L}(H)] \subseteq p\mathcal{L}(H)$.

The norms $\|\cdot\|_r$ belonging to the subfamily $\frac{1}{p} \leq r < 1$ are submultiplicative (loc.cit., Prop. 4.2) and do not depend on the choice of ordered basis (loc.cit., before Thm. 4.11). In particular, each $D_r(H,K)$ is a K-Banach algebra in this case. If we equip the projective limit $\varprojlim_r D_r(H,K)$ with the projective limit topology the natural map

$$D(H,K) \xrightarrow{\cong} \varprojlim_r D_r(H,K)$$

is an isomorphism of topological K-algebras. Finally, it is easy to see that the comultiplication Δ completes to continuous 'comultiplications'

$$\Delta_r : D_r(H,K) \longrightarrow D_r(H,K)\hat{\otimes}_K D_r(H,K)$$

for any r in the above range. We make two final remarks in case H is a uniform pro-p group and ω is its canonical p-valuation, cf. sec. 2.2.2. In this case each group $P_m(H), m \geq 0$ is a uniform pro-p group.

(i) For $r = \frac{1}{p}$ there is a canonical isomorphism between $D_{1/p}(H,\mathbb{Q}_p)$ and the p-adic completion (with p inverted) of the universal enveloping algebra of the $\mathbb{Z}_p$-Lie algebra $\frac{1}{p}\mathcal{L}(H)$ ([**?**, Thm. 10.4/Remark 10.5 (c)]).

(ii) Let

$$r_m := \sqrt[p^m]{1/p}$$

for $m \geq 0$. In particular, $r_0 = 1/p$. Since $P_{m+1}(H)$ is uniform pro-p we may consider the corresponding $\|\cdot\|_{r_0}$- norm on its distribution algebra $D(P_{m+1}(H))$. In this situation the ring extension $D(P_{m+1}(H)) \subset D(H)$ completes in the $\|\cdot\|_{r_m}$-norm topology on $D(H)$ to a ring extension

$$D_{r_0}(P_{m+1}(H)) \subset D_{r_m}(H)$$

and $D_{r_m}(H)$ is a finite and free (left or right) module over $D_{r_0}(P_{m+1}(H)))$ with basis given by any system of coset representatives for the finite group $H/P_{m+1}(H)$ ([**?**, Lem. 5.11]).

2.3. **Coadmissible modules.** We keep all notations from the preceding section but suppose that the p-valuation ω on H satisfies additionally

(HYP) (H,ω) is $p-$saturated and the ordered basis $h_1,\ldots,h_d$ of H satisfies $\omega(h_i)+\omega(h_j) > \frac{p}{p-1}$ for any $1 \leq i \neq j \leq d$.

Remark. This implies that H is a uniform pro-p group. Conversely, the canonical p-valuation on a uniform pro-p group (p arbitrary) satisfies (HYP). For both statements we refer to [**?**, Rem. before Lem. 4.4] and [**?**, Prop. 2.1].

Suppose in the following $r \in (p^{-1}, 1)$ and $r \in p^{\mathbb{Q}}$. In this case the norm $\|\cdot\|_r$ on $D_r(H,K)$ is multiplicative and $D_r(H,K)$ is a (left and right) noetherian integral domain ([**?**, Thm. 4.5]). For two numbers $r \leq r'$ in the given range the ring homomorphism

$$D_{r'}(H,K) \to D_r(H,K)$$

makes the target a flat (left or right) module over the source (loc.cit., Thm. 4.9). The above isomorphism $D(H,K) \xrightarrow{\cong} \varprojlim_r D_r(H,K)$ realizes therefore a *Fréchet-Stein structure* on $D(H,K)$ in the sense of loc.cit. §3. The latter allows one to define a well-behaved abelian full subcategory $\mathcal{C}_H$ of the (left) $D(H,K)$-modules, the so-called *coadmissible modules*. By definition, an abstract (left) $D(H,K)$-module M is coadmissible if for all r in the given range

(i) $M_r := D_r(H,K) \otimes_{D(H,K)} M$ is finitely generated over $D_r(H,K)$,

(ii) the natural map $M \xrightarrow{\cong} \varprojlim_r M_r$ is an isomorphism.

The projective system $\{M_r\}_r$ is sometimes called the *coherent sheaf* associated to M. To give an example, any finitely presented $D(H,K)$-module is coadmissible.

More generally, for any compact locally L-analytic group H the ring $D(H,K)$ has the structure of a Frechet-Stein algebra ([**?**, Thm. 5.1]). In particular, we may define the notion of a coadmissible module over $D(H,K)$ for any compact L-analytic group in a similar manner. For a general locally L-analytic group G, a $D(G,K)$-module M is coadmissible if it is coadmissible as a $D(H,K)$-module for every compact open subgroup $H \subset G$. It follows from loc. cit. that it is sufficient to check this for a single compact open subgroup.

2.4. **Locally analytic representations.** A topological abelian group M which is a (left) module over a topological ring R is a *separately continuous* (left) module, if the map $R \times M \to M$ giving the action is separately continuous. Any separately continuous bilinear map between Fréchet spaces is jointly continuous [**?**, III.30. Cor.1].

After this preliminary remark, we recall some facts about locally analytic representations. A vector space V which equals a locally convex inductive limit $V = \varinjlim_{n\in\mathbb{N}} V_n$ over a countable system of Banach spaces V_n where the transition maps $V_n \to V_{n+1}$ are injective compact linear maps is called a vector space *of compact type*. We recall ([**?**, Thm. 1.1]) that such a space is Hausdorff, complete, bornological and reflexive. Its strong dual V_b' is a nuclear Fréchet space satisfying $V_b' = \varprojlim_n (V_n)_b'$.

Now let H be a locally L-analytic group, V a Hausdorff locally convex K-vector space and $\rho : H \to \mathrm{GL}(V)$ a homomorphism. Then V (or the pair (V,ρ)) is called a *locally analytic representation of* H if the topological K-vector space V is barrelled, each $h \in H$ acts K-linearly and continuously on V, and the orbit maps $\rho_v : H \to V, h \mapsto \rho(h)(v)$ are locally analytic maps for all $v \in V$, cf. [**?**, sec. 3]. If V is of compact type, then the contragredient G-action on its strong dual V_b' extends to a separately continuous left $D(H,K)$-module on a nuclear Fréchet space.

In this way the functor $V \mapsto V_b'$ induces an anti-equivalence of categories between locally analytic H-representations on K-vector spaces of compact type (with continuous linear H-maps as morphisms) and separately continuous $D(H,K)$-modules on nuclear Fréchet spaces (with continuous $D(H,K)$-module maps as morphisms).

A locally analytic H-representation V is said to be admissible if V_b' is a coadmissible $D(H,K)$-module. The above functor restricts to an anti-equivalence between the corresponding categories of admissible locally analytic representations and coadmissible $D(H,K)$-modules.

3. Completed skew group rings

In this section we will describe a general method of completing certain skew group rings. We recall our general convention that in this paper we only consider the completed tensor product of locally convex vector spaces with respect to the projective tensor product topology[4].

3.1. **Preliminaries.** Let H be a compact locally L-analytic group and let A be a locally convex L-algebra equipped with a locally analytic H-representation $\rho : H \to \mathrm{GL}(A)$. The H-action on A extends to $D(H,L)$ and makes A a separately continuous $D(H,L)$-module ([**?**, Prop. 3.2]). On the other hand, $D(H,L)$ is a topological module over itself via left multiplication. The completion $A\hat{\otimes}_L D(H,L)$ is thus a separately continuous $D(H,L)\hat{\otimes}_L D(H,L)$-module. We view it as a separately continuous $D(H,L)$-module by restricting scalars via the comultiplication Δ. This allows us to define the L-bilinear map

$$(A \otimes_L D(H,L)) \times (A\hat{\otimes}_L D(H,L)) \longrightarrow A\hat{\otimes}_L D(H,L)$$

given by $(\sum_i f_i \otimes \delta_i, b) \mapsto \sum_i f_i \cdot \delta_i(b)$. We consider the product topology on the source. In view of the separate continuity of all operations involved together with [**?**, Lem. 17.1] this map is separately continuous. Since the target is complete it extends in a bilinear and separately continuous manner to the completion of the source. In other words, $A\hat{\otimes}_L D(H,L)$ becomes a separately continuous L-algebra. Of course, $A\hat{\otimes}_L D(H,K)$ is then a separately continuous K-algebra. To emphasize its skew multiplication we denote it in the following by

$$A\#_L D(H,K)$$

or even by $A\#D(H,K)$. This should not cause confusion. However, one has to keep in mind that there is a *completed* tensor product involved. If A is a Fréchet algebra, then the multiplication on $A\#D(H,K)$ is jointly continuous, i.e. $A\#D(H,K)$ is a topological algebra in the usual sense.

[4]the only exception occurred in sec. 2.1

3.2. **Skew group rings, skew enveloping algebras and their completions.**

3.2.1. Using the action ρ we may form the abstract skew group ring $A\#H$ ([?, 1.5.4]). We remind the reader that it equals the free left A-module with elements of H as a basis and with multiplication defined by $(ag)\cdot(bh) := a(\rho(g)(b))gh$ for any $a, b \in A$ and $g, h \in H$. Each element of $A\#H$ has a unique expression as $\sum_{h\in H} a_h h$ with $a_h = 0$ for all but finitely many $h \in H$. Evidently, $A\#H$ contains H as a subgroup of its group of units and A as a subring. Furthermore, the inclusion $L[H] \subseteq D(H, L)$ gives rise to an A-linear map

$$A\#H = A \otimes_L L[H] \longrightarrow A\#D(H, L). \tag{3.2.2}$$

On the other hand, let $\mathfrak{h} := Lie(H)$. Differentiating the locally analytic action ρ gives a homomorphism of L-Lie algebras $\alpha : \mathfrak{h} \longrightarrow \mathrm{Der}_L(A)$ into the L-derivations of the algebra A making the diagram

$$\begin{array}{ccc} U(\mathfrak{h}) & \xrightarrow{\alpha} & \mathrm{End}_L(A) \\ \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle Id} \\ D(H,L) & \xrightarrow{\rho} & \mathrm{End}_L(A) \end{array}$$

commutative ([?, 3.1]). We may therefore form the *skew enveloping algebra* $A\#U(\mathfrak{h})$ ([?, 1.7.10]). We recall that this is an L-algebra whose underlying L-vector space equals the tensor product $A \otimes_L U(\mathfrak{h})$. The multiplication is defined by

$$(f_1 \otimes \mathfrak{x}_1)\cdot(f_2 \otimes \mathfrak{x}_2) = (f_1\alpha(\mathfrak{x}_1)(f_2)) \otimes \mathfrak{x}_2 \;+\; (f_1 f_2) \otimes (\mathfrak{x}_1\mathfrak{x}_2)\ ,$$

for $f_i \otimes \mathfrak{x}_i \in A \otimes_L \mathfrak{h}$. Moreover, the inclusion $U(\mathfrak{h}) \subseteq D(H, L)$ induces an A-linear map

$$A\#U(\mathfrak{h}) \longrightarrow A\#D(H, L). \tag{3.2.3}$$

Proposition 3.2.4. *The A-linear maps (3.2.2) and (3.2.3) are L-algebra homomorphisms. The first of these maps has dense image.*

Proof. The first statement follows from the identities

(i) $(1\hat{\otimes}\delta_g)\cdot(f\hat{\otimes}1) = (\rho(g)(f))\hat{\otimes}\delta_g$ for any $g \in H, f \in A$,

(ii) $(1\hat{\otimes}\mathfrak{x})\cdot(f\hat{\otimes}1) = (\alpha(\mathfrak{x})(f))\hat{\otimes}1 + f\hat{\otimes}\mathfrak{x}$ for any $\mathfrak{x} \in \mathfrak{h}, f \in A$

in $A\hat{\otimes}_L D(H, L)$. In turn these identities follow from $\Delta(\delta_g) = \delta_g\hat{\otimes}\delta_g$ and $\Delta(\mathfrak{x}) = \mathfrak{x}\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{x}$. The final statement follows from [?, Lem. 3.1]. □

3.2.5. In this paragraph we assume that $L = \mathbb{Q}_p$ and that the compact locally $\mathbb{Q}_p$-analytic group H is endowed with a p-valuation ω. Recall that we have defined $r_0 = p^{-1}$, cf. 2.2.3. Consider the norm completion $D_r(H, L)$ for some arbitrary but fixed $r \in [r_0, 1)$. Let us assume for a moment that the natural map $D(H, L) \to D_r(H, L)$ satisfies the following hypothesis:

($\star$) The separately continuous $D(H, L)-$module structure of A extends to a separately continuous $D_r(H, L)-$module structure.

If we replace in the above discussion the comultiplication Δ by its completion Δ_r we obtain in an entirely analogous manner a completion $A \hat{\otimes}_L D_r(H, K)$ of the skew group ring $A\#H$, base changed to K. It satisfies *mutatis mutandis* the statement of the preceding proposition. As before we will often abbreviate it by $A\#D_r(H, K)$.

4. Sheaves on the Bruhat-Tits building and smooth representations

4.1. **Filtrations of stabilizer subgroups.**

4.1.1. Let $\mathbf{T}$ be a maximal L-split torus in $\mathbf{G}$. Let $X^*(\mathbf{T})$ resp. $X_*(\mathbf{T})$ be the group of algebraic characters resp. cocharacters of $\mathbf{T}$. Let $\Phi = \Phi(\mathbf{G}, \mathbf{T}) \subset X^*(\mathbf{T})$ denote the root system determined by the adjoint action of $\mathbf{T}$ on the Lie algebra of $\mathbf{G}$. Let W denote the corresponding Weyl group. For each $\alpha \in \Phi$ we have the unipotent root subgroup $\mathbf{U}_\alpha \subseteq \mathbf{G}$. Since $\mathbf{G}$ is split the choice of a *Chevalley basis* determines a system of L-group isomorphisms

$$x_\alpha : \mathbb{G}_a \xrightarrow{\cong} \mathbf{U}_\alpha$$

for each $\alpha \in \Phi$ (an *épinglage*) satisfying Chevalley's commutation relations ([**?**, p. 27]). Let $X_*(\mathbf{C})$ denote the group of L-algebraic cocharacters of the connected center $\mathbf{C}$ of $\mathbf{G}$. We denote by G, T, U_α the groups of L-rational points of $\mathbf{G}, \mathbf{T}, \mathbf{U}_\alpha (\alpha \in \Phi)$ respectively. Recall the normalized p-adic valuation v_L on L, i.e. $v_L(\varpi) = 1$. For $\alpha \in \Phi$ we denote by $(U_{\alpha,r})_{r \in \mathbb{R}}$ the filtration of U_α arising from the valuation v_L on L via the isomorphism x_α. It is an exhaustive and separated discrete filtration by subgroups. Put $U_{\alpha,\infty} := \{1\}$.

4.1.2. Let $\mathscr{B} = \mathscr{B}(G)$ be the semisimple Bruhat-Tits building of G. The torus $\mathbf{T}$ determines an apartment A in $\mathscr{B}$. Recall that a point z in the Coxeter complex A is called *special* if for any direction of wall there is a wall of A actually passing through z ([**?**, 1.3.7]). As in [**?**, 3.5] we choose once and for all a special vertex x_0 in A and a chamber $\mathscr{C} \subset A$ containing it. We use the point x_0 to identify the affine space A with the real vector space

$$A = (X_*(\mathbf{T})/X_*(\mathbf{C})) \otimes_{\mathbb{Z}} \mathbb{R} \; .$$

Each root $\alpha \in \Phi$ induces therefore a linear form $\alpha : A \to \mathbb{R}$ in an obvious way. For any nonempty subset $\Omega \subseteq A$ we let $f_\Omega : \Phi \to \mathbb{R} \cup \{\infty\}, \; \alpha \mapsto -\inf_{x \in \Omega} \alpha(x)$. It is a *concave* function in the sense of [**?**, 6.4.1-5]. We emphasize that the concept of a concave function is developed in loc.cit. more generally for functions taking values in the set

$$\tilde{\mathbb{R}} := \mathbb{R} \cup \{r+ : r \in \mathbb{R}\} \cup \{\infty\} \ .$$

The latter naturally has the structure of a totally ordered commutative monoid extending the total order and the addition on $\mathbb{R}$. For any $\alpha \in \Phi$ and $r \in \mathbb{R}$ we define

$$U_{\alpha,r+} := \cup_{s\in\mathbb{R},s>r} U_{\alpha,s} \ .$$

For any concave function $f : \Phi \to \tilde{\mathbb{R}}$ we then have the group

(4.1.3) $$U_f := \text{subgroup of } G \text{ generated by all } U_{\alpha,f(\alpha)} \text{ for } \alpha \in \Phi.$$

4.1.4. For each nonempty subset $\Omega \subseteq \mathscr{B}$ we let

$$P_\Omega := \{g \in G : gz = z \text{ for any } z \in \Omega\}$$

be its pointwise stabilizer in G. For any facet $F \subseteq \mathscr{B}$ we will recall from [**?**, I.2] a certain decreasing filtration of P_F by open normal pro-p subgroups which will be most important for all that follows in this article. To do this we first consider a facet F in the apartment A. For $\alpha \in \Phi$ we put $f_F^*(\alpha) := f_F(\alpha)+$ if $\alpha|_F$ is constant and $f_F^*(\alpha) := f_F(\alpha)$ otherwise. This yields a concave function $f_F^* : \Phi \to \tilde{\mathbb{R}}$. With f_F^* also the functions $f_F^* + e$, for any integer $e \geq 0$, are concave. Hence there is the descending sequence of subgroups

$$U_{f_F^*} \supseteq U_{f_F^*+1} \supseteq U_{f_F^*+2} \supseteq \dots$$

4.1.5. On the other hand we let $\mathfrak{T} := Spec(o_L[X^*(\mathbf{T})])$ and

$$T^{(e)} := \ker(\mathfrak{T}(o_L) \longrightarrow \mathfrak{T}(o_L/\varpi_L^{e+1} o_L))$$

for any $e \geq 0$ (cf. [**?**, pf. of Prop. I.2.6]) and finally define

$$U_F^{(e)} := U_{f_F^*+e} \cdot T^{(e)}$$

for each $e \geq 0$ (loc.cit. p.21). This definition is extended to *any* facet F in $\mathscr{B}$ by putting $U_F^{(e)} := g U_{F'}^{(e)} g^{-1}$ if $F = gF'$ with $g \in G$ and F' a facet in A. We thus obtain a filtration

$$P_F \supseteq U_F^{(0)} \supseteq U_F^{(1)} \supseteq \dots$$

of the pointwise stabilizer P_F by normal subgroups. As in loc.cit. we define, for any point $z \in \mathscr{B}$,

$$U_z^{(e)} := U_F^{(e)}$$

where F is the unique facet of $\mathscr{B}$ that contains z. The group $U_z^{(e)}$ fixes the point z. By construction we have

$$U_{gz}^{(e)} = gU_z^{(e)}g^{-1} \tag{4.1.6}$$

for any $z \in \mathscr{B}$ and any $g \in G$.

Remark: We emphasize that the definition of the groups $\{U_F^{(e)}\}_{F\subset\mathscr{B},e\geq 0}$ depends on the choice of the special vertex x_0 as an origin for A. We also remark that the very same groups appear in the work of Moy-Prasad on unrefined minimal types ([**?**], [**?**]).

We will make use of the following basic properties of the groups $U_F^{(e)}$. To formulate them let

$$\Phi = \Phi^+ \cup \Phi^-$$

be any fixed decomposition of Φ into positive and negative roots.

Proposition 4.1.7. (i) *Let $F \subset A$ be a facet. For any $e \geq 0$ the product map induces a bijection*

$$\Big(\prod_{\alpha\in\Phi^-} U_{f_F^*+e} \cap U_\alpha\Big) \times T^{(e)} \times \Big(\prod_{\alpha\in\Phi^+} U_{f_F^*+e} \cap U_\alpha\Big) \xrightarrow{\cong} U_F^{(e)}$$

whatever ordering of the factors of the left hand side we choose. Moreover, we have

$$U_{f_F^*+e} \cap U_\alpha = U_{\alpha,f_F^*(\alpha)+e}$$

for any $\alpha \in \Phi$.

(ii) *For any facet $F \subset \mathscr{B}$ the $U_F^{(e)}$ for $e \geq 0$ form a fundamental system of compact open neighborhoods of 1 in G,*

(iii) *$U_{F'}^{(e)} \subseteq U_F^{(e)}$ for any two facets F, F' in $\mathscr{B}$ such that $F' \subseteq \overline{F}$.*

Proof. Cf. [**?**, Prop. I.2.7, Cor. I.2.9, Prop. I.2.11]. □

4.1.8. As an example and in view of later applications we give a more concrete description of the groups $\{U_{x_0}^{(e)}\}_{e\geq 0}$. The stabilizer $P_{\{x_0\}}$ in G of the vertex x_0 is a special, good, maximal compact open subgroup of G ([**?**, 3.5]). We let $\mathfrak{G}$ be the connected reductive o_L-group scheme with generic fibre $\mathbf{G}$ associated with the special vertex x_0 ([**?**, 3.4], [**?**, 4.6.22]). Its group of o_L-valued points $\mathfrak{G}(o_L)$ can be identified with $P_{\{x_0\}}$. For $e \geq 0$ we therefore have in $P_{\{x_0\}}$ the normal subgroup $\mathfrak{G}(\varpi^e) := \ker\ (\mathfrak{G}(o_L) \to \mathfrak{G}(o_L/\varpi^e o_L))$.
Now the concave function $f_{\{x_0\}}$ vanishes identically whence $f_{\{x_0\}}^*$ has constant value $0+$. Thus,

$$U_{\alpha,f_{\{x_0\}}^*(\alpha)+e} = \cup_{s>0}\{a \in L : v_L(a) \geq e+s\} = \varpi^{e+1}o_L$$

for any $e \geq 0$. By 4.1.7 (i) and the definition of $T^{(e)}$ we therefore have a canonical isomorphism $U_{x_0}^{(e)} \xrightarrow{\cong} \mathfrak{G}(\varpi^{e+1})$ for any $e \geq 0$.

4.2. **The Schneider-Stuhler construction.** We now review the construction of a certain 'localization' functor constructed by P. Schneider and U. Stuhler in [**?**, IV.1]. In fact, there will be a functor for each 'level' $e \geq 0$. Following loc.cit., we will suppress this dependence in our notation. In [**?**] only complex representations are considered. However, all results remain true over our characteristic zero field K ([**?**]).

4.2.1. Recall that a *smooth* representation V of G is a K-vector space V together with a linear action of G such that the stabilizer of each vector is open in G. A morphism between two such representations is simply a K-linear G-equivariant map.
Now let us fix an integer $e \geq 0$ and let V be a smooth representation. For any subgroup $U \subseteq G$ we have the K-vector space

$$V_U := \text{maximal quotient of } V \text{ on which the } U\text{-action is trivial}$$

of U-coinvariants of V. For any open subset $\Omega \subseteq \mathscr{B}$ we let

$$\underset{\approx}{V}(\Omega) := K\text{-vector space of all maps } s : \Omega \to \dot{\bigcup}_{z \in \Omega} V_{U_z^{(e)}} \text{ such that}$$

- $s(z) \in V_{U_z^{(e)}}$ for all $z \in \Omega$,
- there is an open covering $\Omega = \cup_{i \in I} \Omega_i$ and vectors $v_i \in V$ with

$$s(z) = \text{class of } v_i \ \in V_{U_z^{(e)}}$$

  for any $z \in \Omega_i$ and $i \in I$.

We summarize some properties of this construction in the following proposition. Recall that a sheaf on a polysimplicial space is called *constructible* if its restriction to a given geometric polysimplex is a constant sheaf ([**?**, 8.1]).

Proposition 4.2.2. (i) *The correspondence $\Omega \mapsto \underset{\approx}{V}(\Omega)$ is a sheaf of K-vector spaces,*

(ii) *for any $z \in \mathscr{B}$ the stalk of the sheaf $\underset{\approx}{V}$ at z equals $(\underset{\approx}{V})_z = V_{U_z^{(e)}}$,*

(iii) *$\underset{\approx}{V}$ is a constructible sheaf whose restriction to any facet F of $\mathscr{B}$ is constant with value $V_{U_F^{(e)}}$,*

(iv) *the correspondence $V \mapsto \underset{\approx}{V}$ is an exact functor from smooth G-representations to sheaves of K-vector spaces on $\mathscr{B}$.*

Proof. (i) follows from the local nature of the preceding definition. (ii) and (iii) are [**?**, Lem. IV.1.1]. (iv) follows from (ii) because of $char(K) = 0$. □

We recall that the smooth representation V is called *admissible* if the H-invariants V^H form a finite dimensional K-vector space for any compact open subgroup H of G. In this situation the natural projection map $V \to V_H$ induces a linear isomorphism $V^H \xrightarrow{\cong} V_H$. For an admissible representation V we may therefore deduce from 4.2.2 (ii) that the stalks of $\underset{\approx}{V}$ are finite dimensional K-vector spaces. We emphasize again that the functor $V \mapsto \underset{\approx}{V}$ depends on the level $e \geq 0$.

4.3. **p-valuations on certain stabilizer subgroups.** We keep the notations from the preceding paragraph and define certain p-valuations on the groups $U_F^{(e)}$. However, for the rest of this section we assume $L = \mathbb{Q}_p$.

Lemma 4.3.1. *Let F be a facet in $\mathscr{B}$ and $e, e' \geq 0$. The commutator group $(U_F^{(e)}, U_F^{(e')})$ satisfies*

$$(U_F^{(e)}, U_F^{(e')}) \subseteq U_F^{(e+e')} .$$

Proof. Choosing a facet F' in A and an element $g \in G$ such that $F' = gF$ we may assume that F lies in A. Define a function $h_F : \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ via $h_F|_\Phi := f_F^*$ and $h_F(0) := 0+$. Then $g := h_F + e$ and $f := h_F + e'$ are concave functions in the sense of [**?**, Def. 6.4.3]. Consider the function $h : \Phi \cup \{0\} \to \tilde{\mathbb{R}} \cup \{-\infty\}$ defined as

$$h(a) := \inf\{\sum_i f(a_i) + \sum_j g(b_j)\}$$

where the infimum is taken over the set of pairs of finite nonempty sets (a_i) and (b_j) of elements in $\Phi \cup \{0\}$ such that $a = \sum_i a_i + \sum_j b_j$. Using that the functions f and g are concave one finds

$$h_F(a) + e + e' \leq h(a)$$

for any $a \in \Phi \cup \{0\}$. By loc.cit., Prop. 6.4.44, the function h is therefore concave and has the property

$$(U_f, U_g) \subseteq U_h \subseteq U_{h_F+e+e'} .$$

Here, the groups involved are defined completely analogous to (4.1.3) (cf. loc.cit., Def. 6.4.42). It remains to observe that $U_{h_F+a} = U_F^{(a)}$ for any integer $a \geq 0$ ([**?**, p. 21]). $\square$

Let l be the rank of the torus $\mathbf{T}$. By construction of $\mathfrak{T}$ any trivialization $\mathbf{T} \simeq (\mathbb{G}_m)^l$ yields an identification $\mathfrak{T} \simeq (\mathbb{G}_{m/o_L})^l$ which makes the structure of the topological groups $T^{(e)}, e \geq 0$ explicit. Moreover, we assume in the following $e \geq 2$. For each $g \in U_F^{(e)} \setminus \{1\}$ let

$$\omega_F^{(e)}(g) := \sup\{n \geq 0 : g \in U_F^{(n)}\} .$$

The following corollary is essentially due to H. Frommer ([**?**, 1.3, proof of Prop. 6]). For sake of completeness we include a proof.

Corollary 4.3.2. *The function*

$$\omega_F^{(e)} : U_F^{(e)} \setminus \{1\} \longrightarrow (1/(p-1), \infty) \subset \mathbb{R}$$

is a p-valuation on $U_F^{(e)}$.

Proof. The first axiom (i) is obvious and (ii) follows from the lemma. Let $g \in U_F^{(e)}$ with $n := \omega_F^{(e)}(g)$. We claim $\omega_F^{(e)}(g^p) = n+1$. The root space decomposition 4.1.7

$$m : (\prod_{\alpha\in\Phi^-} U_{\alpha, f_F^*(\alpha)+n}) \times T^{(n)} \times (\prod_{\alpha\in\Phi^+} U_{\alpha, f_F^*(\alpha)+n}) \xrightarrow{\cong} U_F^{(n)}$$

is in an obvious sense compatible with variation of the level n. If $g \in T^{(n)}$ the claim is immediate. The same is true if $g \in U_{\alpha, f_F^*(\alpha)+n}$ for some $\alpha \in \Phi$: indeed the filtration of U_α is induced by the p-adic valuation on $\mathbb{Q}_p$ via $x_\alpha : \mathbb{Q}_p \simeq U_\alpha$. In general let $m(h_1, ..., h_d) = g$. By what we have just said there is $1 \le i \le d$ such that $\omega^{(e)}(h_i^p) = n+1$ and $\omega^{(e)}(h_j^p) \ge n+1$ for all $j \ne i$. Furthermore, $h_1^p \cdots h_d^p g' = g^p$ where $g' \in (U_F^{(n)}, U_F^{(n)}) \subseteq U_F^{(2n)}$. Since $n \ge 2$ we have $2n \ge n+2$ and hence $g^p \in U_F^{(n+1)}$. If $g^p \in U_F^{(n+2)}$ then $h_1^p \cdots h_d^p = g^p g'^{-1} \in U_F^{(n+2)}$ which contradicts the existence of h_i. Hence $\omega^{(e)}(g^p) = n+1$ which verifies axiom (iii). □

4.3.3. For a given root $\alpha \in \Phi$ let u_α be a topological generator for the group $U_{\alpha, f_F^*(\alpha)+e}$. Let $t_1, ..., t_l$ be topological generators for the group $T^{(e)}$. In the light of the decomposition of 4.1.7 (i) it is rather obvious that the set

$$\{u_\alpha\}_{\alpha\in\Phi^-} \cup \{t_i\}_{i=1,\dots,l} \cup \{u_\alpha\}_{\alpha\in\Phi^+}$$

arranged in the order suggested by loc.cit. is an ordered basis for the p-valued group $(U_F^{(e)}, \omega_F^{(e)})$. Of course, $\omega_F^{(e)}(h) = e$ for any element h of this ordered basis.

For technical reasons we will work in the following with the slightly simpler p-valuations

$$\mathring{\omega}_F^{(e)} := \omega_F^{(e)} - (e-1)$$

satisfying $\mathring{\omega}_F^{(e)}(h) = 1$ for any element h of the above ordered basis. If $z \in \mathscr{B}$ lies in the facet $F \subset \mathscr{B}$ we write $\mathring{\omega}_z^{(e)}$ for $\mathring{\omega}_F^{(e)}$.

Remark 4.3.4. The tangent map at $1 \in G$ corresponding to the p-power map equals multiplication by p and thus, is an isomorphism. It follows from 4.1.7 (ii) that there is $e(F) \ge 2$ such that for any $e \ge e(F)$ any element $g \in U_F^{(e+1)}$ is a p-th power h^p with $h \in G$. By axiom (iii) such an h lies in $U_F^{(e)}$ which implies that $(U_F^{(e)}, \mathring{\omega}_F^{(e)})$ is p-saturated. For $e \ge e(F)$ the group $U_F^{(e)}$ is therefore a uniform pro-p group (apply remark before Lemma 4.4 in [?] to $\mathring{\omega}_F^{(e)}$ and use $p \ne 2$). Since any facet in $\mathscr{B}$ is conjugated to a facet in $\mathscr{C}$ we deduce from (4.1.6) that there is a number $e_{uni} \ge 2$ such that all the groups $U_F^{(e)}$ for $F \subset \mathscr{B}$ are uniform pro-p groups whenever $e \ge e_{uni}$. In this situation [**?**, Prop. A1], asserts that the subgroups

$$U_F^{(e)} \supset U_F^{(e+1)} \supset U_F^{(e+2)} \supset \dots$$

form the lower p-series of the group $U_F^{(e)}$.

We may apply the discussion of 2.1 to $(U_F^{(e)}, \mathring{\omega}_F^{(e)})$ and the above ordered basis to obtain a family of norms $\|\cdot\|_r, r \in [1/p, 1)$ on $D(U_F^{(e)}, K)$ with completions $D_r(U_F^{(e)}, K)$ being K-Banach algebras. For facets F, F' in $\mathscr{B}$ such that $F' \subseteq \overline{F}$ we shall need a certain 'gluing' lemma for these algebras.

Lemma 4.3.5. *Let F, F' be two facets in $\mathscr{B}$ such that $F' \subseteq \overline{F}$. The inclusion $U_{F'}^{(e)} \subseteq U_F^{(e)}$ extends to a norm-decreasing algebra homomorphism*

$$\sigma_r^{F'F} : D_r(U_{F'}^{(e)}, K) \longrightarrow D_r(U_F^{(e)}, K) \ .$$

Moreover,

(i) $\sigma_r^{FF} = \mathrm{id}$,

(ii) $\sigma_r^{F'F} \circ \sigma_r^{F''F'} = \sigma_r^{F''F}$ *if F'' is a third facet in $\mathscr{B}$ with $F'' \subseteq \overline{F'}$.*

Finally, $\sigma_r^{F'F}$ restricted to $Lie(U_{F'}^{(e)})$ equals the map $Lie(U_{F'}^{(e)}) \simeq Lie(U_F^{(e)}) \subset D_r(U_F^{(e)}, K)$ where the first arrow is the canonical Lie algebra isomorphism from [**?**, III §3.8].

Proof. By functoriality ([**?**, 1.1]) of $D(\cdot, K)$ we obtain an algebra homomorphism

$$\sigma : D(U_{F'}^{(e)}, K) \longrightarrow D(U_F^{(e)}, K) \ .$$

Let $h'_1, ..., h'_d$ and $h_1, ..., h_d$ be the ordered bases of $U_{F'}^{(e)}$ and $U_F^{(e)}$ respectively. Let $b'_i = h'_i - 1 \in \mathbb{Z}[U_{F'}^{(e)}]$ and $\mathbf{b}'^m := b_1'^{m_1} \cdots b_d'^{m_d}$ for $m \in \mathbb{N}_0^d$. Given an element

$$\lambda = \sum_{m \in \mathbb{N}_0^d} d_m \mathbf{b}'^m \in D(U_{F'}^{(e)}, K)$$

we have $\|\lambda\|_r = \sup_m |d_m| \ \|b'_i\|_r$. Because of

$$\|\sigma(\lambda)\|_r \leq \sup_m \ |d_m| (\|\sigma(b'_1)\|_r)^{m_1} \cdots (\|\sigma(b'_d)\|_r)^{m_d})$$

it therefore suffices to prove $\|\sigma(b'_i)\|_r \leq \|b'_i\|_r$ for any i. If h'_i belongs to the toral part of the ordered basis of $U_{F'}^{(e)}$ then we may assume $\sigma(b'_i) = b'_i$ and we are done. Let therefore $\alpha \in \Phi$ and consider the corresponding elements h'_α and h_α in the ordered bases of $U_{F'}^{(e)}$ and $U_F^{(e)}$ respectively. By the root space decomposition we have

$$U_{\alpha, f_{F'}^*(\alpha)+e} \subseteq U_{\alpha, f_F^*(\alpha)+e} = (h_\alpha)^{\mathbb{Z}_p} \ .$$

Let therefore $a \in \mathbb{Z}_p$ such that $h'_\alpha = (h_\alpha)^a$. Since a change of ordered basis does not affect the norms in question (cf. 2.2.3) we may assume $a = p^s$ for some natural number $s \geq 0$. Then

$$h'_\alpha - 1 = (h_\alpha + 1 - 1)^{p^s} - 1 = \sum_{k=1,\dots,p^s} \binom{p^s}{k} (h_\alpha - 1)^k$$

and therefore

$$\|\sigma(h'_\alpha - 1)\|_r \le \max_{k=1,\dots,p^s} \left|\binom{p^s}{k}\right| \ \|(h_\alpha - 1)\|_r^k = \max_{k=1,\dots,p^s} \left|\binom{p^s}{k}\right| r^k \le r = \|h'_\alpha - 1\|_r$$

which shows the claim and the existence of $\sigma_r^{FF'}$. The properties (i),(ii) follow from functoriality of $D(\cdot, K)$ by passing to completions. Since $U_{F'}^{(e)} \subseteq U_F^{(e)}$ is an open immersion of Lie groups the final statement is clear. □

5. Sheaves on the flag variety and Lie algebra representations

5.1. Differential operators on the flag variety.

5.1.1. Let X denote the variety of Borel subgroups of $\mathbf{G}$. It is a smooth and projective L-variety. Let $\mathcal{O}_X$ be its structure sheaf. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. Differentiating the natural (left) action of $\mathbf{G}$ on X yields a homomorphism of Lie algebras

$$\alpha : \mathfrak{g} \longrightarrow \Gamma(X, \mathcal{T}_X)$$

into the global sections of the tangent sheaf $\mathcal{T}_X = \mathcal{D}er_L(\mathcal{O}_X)$ of X ([?, II §4.4.4]). In the following we identify an abelian group (algebra, module etc.) with the corresponding constant sheaf on X. This should not cause any confusion. Letting

$$\mathfrak{g}^\circ := \mathcal{O}_X \otimes_L \mathfrak{g}$$

the map α extends to a morphism of $\mathcal{O}_X$-modules $\alpha^\circ : \mathfrak{g}^\circ \longrightarrow \mathcal{T}_X$. Defining $[\mathfrak{x}, f] := \alpha(\mathfrak{x})(f)$ for $\mathfrak{x} \in \mathfrak{g}$ and a local section f of $\mathcal{O}_X$ makes $\mathfrak{g}^\circ$ a sheaf of L-Lie algebras[5]. Then α° is a morphism of L-Lie algebras. We let $\mathfrak{b}^\circ := \ker \alpha^\circ$, a subalgebra of $\mathfrak{g}^\circ$, and $\mathfrak{n}^\circ := [\mathfrak{b}^\circ, \mathfrak{b}^\circ]$ its derived algebra. At a point $x \in X$ with residue field $\kappa(x)$ the reduced stalks of the sheaves $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ equal the Borel subalgebra $\mathfrak{b}_x$ of $\kappa(x) \otimes_L \mathfrak{g}$ defined by x and its nilpotent radical $\mathfrak{n}_x \subset \mathfrak{b}_x$ respectively. Let $\mathfrak{h}$ denote the abstract Cartan algebra of $\mathfrak{g}$ ([?, §2]). We view the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_L \mathfrak{h}$ as an abelian L-Lie algebra. By definition of $\mathfrak{h}$ there is a canonical isomorphism of $\mathcal{O}_X$-modules and L-Lie algebras

$$\mathfrak{b}^\circ / \mathfrak{n}^\circ \xrightarrow{\cong} \mathcal{O}_X \otimes_L \mathfrak{h}. \tag{5.1.2}$$

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. The enveloping algebra of the Lie-algebra $\mathfrak{g}^\circ$ has the underlying $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_L U(\mathfrak{g})$. Its L-algebra of local sections over an open affine

[5] Following [?] we call such a sheaf simply a Lie algebra over X in the sequel. This abuse of language should not cause confusion.

$V \subseteq X$ is the skew enveloping algebra $\mathcal{O}_X(V)\#U(\mathfrak{g})$ relative to $\alpha : \mathfrak{g} \to \mathrm{Der}_L(\mathcal{O}_X(V))$ (in the sense of sec 3). To emphasize this skew multiplication we follow [?, 3.1.3] and denote the enveloping algebra of $\mathfrak{g}^\circ$ by

$$\mathcal{O}_X\#U(\mathfrak{g}) \ .$$

5.1.3. To bring in the torus $\mathbf{T}$ we choose a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ defined over L containing $\mathbf{T}$. Let $\mathbf{N} \subset \mathbf{B}$ be the unipotent radical of $\mathbf{B}$ and let $\mathbf{N}^-$ be the unipotent radical of the Borel subgroup opposite to $\mathbf{B}$. We denote by

$$q : \mathbf{G} \longrightarrow \mathbf{G}/\mathbf{B} = X$$

the canonical projection. Let $\mathfrak{b}$ be the Lie algebra of $\mathbf{B}$ and $\mathfrak{n} \subset \mathfrak{b}$ its nilpotent radical. If $\mathfrak{t}$ denotes the Lie algebra of $\mathbf{T}$ the map $\mathfrak{t} \subset \mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h}$ induces an isomorphism $\mathfrak{t} \simeq \mathfrak{h}$ of L-Lie algebras. We will once and for all identify these two Lie algebras via this isomorphism. Consequently, (5.1.2) yields a morphism of $\mathcal{O}_X$-modules and L-Lie algebras

$$\mathfrak{b}^\circ \longrightarrow \mathfrak{b}^\circ/\mathfrak{n}^\circ \xrightarrow{\cong} \mathcal{O}_X \otimes_L \mathfrak{t} \ .$$

Given a linear form $\lambda \in \mathfrak{t}^*$ it extends $\mathcal{O}_X$-linearly to the target of this morphism and may then be pulled-back to $\mathfrak{b}^\circ$. This gives a $\mathcal{O}_X$-linear morphism $\lambda^\circ : \mathfrak{b}^\circ \longrightarrow \mathcal{O}_X$.

5.1.4. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Given $\chi \in \mathfrak{t}^*$ we put $\lambda := \chi - \rho$. Denote by $\mathcal{I}_\chi$ the right ideal sheaf of $\mathcal{O}_X\#U(\mathfrak{g})$ generated by $\ker \lambda^\circ$, i.e., by the expressions

$$\xi - \lambda^\circ(\xi)$$

with ξ a local section of $\mathfrak{b}^\circ \subset \mathfrak{g}^\circ \subset \mathcal{O}_X\#U(\mathfrak{g})$. It is a two-sided ideal and we let

$$\mathcal{D}_\chi := (\mathcal{O}_X\#U(\mathfrak{g}))/\mathcal{I}_\chi$$

be the quotient sheaf. This is a sheaf of noncommutative L-algebras on X endowed with a natural algebra morphism $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_\chi)$ induced by $\mathfrak{x} \mapsto 1 \otimes \mathfrak{x}$ for $\mathfrak{x} \in U(\mathfrak{g})$. On the other hand $\mathcal{D}_\chi$ is an $\mathcal{O}_X$-module through the (injective) L-algebra morphism $\mathcal{O}_X \to \mathcal{D}_\chi$ induced by $f \mapsto f \otimes 1$. This allows to define the full subcategory $\mathcal{M}_{qc}(\mathcal{D}_\chi)$ of the (left) $\mathcal{D}_\chi$-modules consisting of modules which are quasi-coherent as $\mathcal{O}_X$-modules. It is abelian.

5.1.5. For future reference we briefly discuss a refinement of the above construction of the sheaf $\mathcal{D}_\chi$. The right ideal of $\mathcal{O}_X\#U(\mathfrak{g})$ generated by $\mathfrak{n}^\circ$ is a two-sided ideal and, following [?, §3] we let

$$\mathcal{D}_{\mathfrak{t}} := (\mathcal{O}_X\#U(\mathfrak{g}))/\mathfrak{n}^\circ(\mathcal{O}_X\#U(\mathfrak{g}))$$

be the quotient sheaf. We have the open subscheme $U_1 := q(\mathbf{N}^-)$ of X. Choose a representative $\dot{w} \in G$ for every $w \in W$ with $\dot{1} = 1$. The translates $U_w := \dot{w}U_1$ for all $w \in W$ form a Zariski covering of X. Let $\mathfrak{n}^-$ be the Lie algebra of $\mathbf{N}^-$ and put $\mathfrak{n}^{-,w} := \mathrm{Ad}(\dot{w})(\mathfrak{n}^-)$ for any $w \in W$.

For any $w \in W$ there are obvious canonical maps from $\mathcal{O}_X(U_w), U(\mathfrak{n}^{-,w})$ and $U(\mathfrak{t})$ to $\mathcal{O}_X(U_w)\#U(\mathfrak{g})$ and therefore to $\mathcal{D}_\mathfrak{t}(U_w)$. According to [?, Lem. C.1.3] they induce a K-algebra isomorphism

$$(\mathcal{O}_X(U_w)\#U(\mathfrak{n}^{-,w})) \otimes_L U(\mathfrak{t}) \xrightarrow{\cong} \mathcal{D}_\mathfrak{t}(U_w). \tag{5.1.6}$$

Note here that $\mathbf{N}^- \cong \mathbb{A}_L^{|\Phi^-|}$ implies that the skew enveloping algebra $\mathcal{O}_X(U_w)\#U(\mathfrak{n}^{-,w})$ is equal to the usual algebra of differential operators $\mathcal{D}_X(U_w)$ on the translated affine space $U_w = \dot{w}U_1$.

The above discussion implies that the canonical homomorphism

$$U(\mathfrak{t}) \mapsto \mathcal{O}_X\#U(\mathfrak{g}), \mathfrak{x} \mapsto 1 \otimes \mathfrak{x}$$

induces a central embedding $U(\mathfrak{t}) \hookrightarrow \mathcal{D}_\mathfrak{t}$. In particular, the sheaf $(\ker\lambda)\mathcal{D}_\mathfrak{t}$ is a two-sided ideal in $\mathcal{D}_\mathfrak{t}$. According to the discussion before Thm. 3.2 in [?], p. 138, the canonical map $\mathcal{D}_\mathfrak{t} \to \mathcal{D}_\chi$ coming from $\mathfrak{n}^\circ \subset \ker\lambda^\circ$ induces

$$\mathcal{D}_\mathfrak{t} \otimes_{U(\mathfrak{t})} L_\lambda = \mathcal{D}_\mathfrak{t}/(\ker\lambda)\mathcal{D}_\mathfrak{t} \xrightarrow{\cong} \mathcal{D}_\chi \ ,$$

an isomorphism of sheaves of K-algebras.

Remark: According to the above we may view the formation of the sheaf $\mathcal{D}_\chi$ as a two-step process. In a first step on constructs the sheaf $\mathcal{D}_\mathfrak{t}$ whose sections over the Weyl translates of the big cell U_1 are explicitly computable. Secondly, one performs a central reduction $\mathcal{D}_\mathfrak{t} \otimes_{U(\mathfrak{t})} L_\lambda$ via the chosen character $\lambda = \chi - \rho$. This point of view will be useful in later investigations.

5.2. **The Beilinson-Bernstein localization theorem.**

5.2.1. We recall some notions related to the classical *Harish-Chandra isomorphism*. To begin with let $S(\mathfrak{t})$ be the symmetric algebra of $\mathfrak{t}$ and let $S(\mathfrak{t})^W$ be the subalgebra of Weyl invariants. Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The classical Harish-Chandra map is an algebra isomorphism $Z(\mathfrak{g}) \xrightarrow{\cong} S(\mathfrak{t})^W$ relating central characters and highest weights of irreducible highest weight $\mathfrak{g}$-modules in a meaningful way ([?, 7.4]). Given a linear form $\chi \in \mathfrak{t}^*$ we let

$$\sigma(\chi) : Z(\mathfrak{g}) \to L$$

denote the central character associated with χ via the Harish-Chandra map. Recall that $\chi \in \mathfrak{t}^*$ is called *dominant* if $\chi(\check{\alpha}) \notin \{-1, -2, ...\}$ for any coroot $\check{\alpha}$ with $\alpha \in \Phi^+$. It is called *regular* if $w(\chi) \neq \chi$ for any $w \in W$ with $w \neq 1$.

Let $\theta := \sigma(\chi)$ and put $U(\mathfrak{g})_\theta := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\theta} L$ for the corresponding central reduction.

Theorem 5.2.2. *(Beilinson/Bernstein)*

(i) *The algebra morphism* $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_\chi)$ *induces an isomorphism* $U(\mathfrak{g})_\theta \simeq \Gamma(X, \mathcal{D}_\chi)$.

(ii) *If χ is dominant and regular the functor $M \mapsto \mathcal{D}_\chi \otimes_{U(\mathfrak{g})_\theta} M$ is an equivalence of categories between the (left) $U(\mathfrak{g})_\theta$-modules and $\mathcal{M}_{qc}(\mathcal{D}_\chi)$.*

(iii) *Let M be a $U(\mathfrak{g})_\theta$-module. The reduced stalk of the sheaf $\mathcal{D}_\chi \otimes_{U(\mathfrak{g})_\theta} M$ at a point $x \in X$ equals the λ-coinvariants of the $\mathfrak{h}$-module $(\kappa(x) \otimes_L M)/\mathfrak{n}_x(\kappa(x) \otimes_L M)$.*

Proof. This is the main theorem of [**?**]. □

Remarks:

(i) In [**?**] the theorem is proved under the assumption that the base field is algebraically closed. However, all proofs of loc. cit. go through over an arbitrary characteristic zero field in the case where the $\mathfrak{g}$ is split over the base field. In the following, this is the only case we shall require.

(ii) If $\lambda := \chi - \rho \in X^*(\mathbf{T}) \subset \mathfrak{t}^*$ and if $\mathcal{O}(\lambda)$ denotes the associated invertible sheaf on X then $\mathcal{D}_\chi$ can be identified with the sheaf of differential endomorphisms of $\mathcal{O}(\lambda)$ ([**?**, p. 138]). It is therefore a *twisted sheaf of differential operators* on X in the sense of [**?**, §1]. In particular, if $\chi = \rho$ the map α° induces an isomorphism $\mathcal{D}_\rho \xrightarrow{\cong} \mathcal{D}_X$ with the usual sheaf of differential operators on X ([**?**, §16.8]). In this case, $\mathcal{M}_{qc}(\mathcal{D}_\chi)$ equals therefore the usual category of algebraic D-modules on X in the sense of [**?**].

6. Berkovich analytifications

6.1. Differential operators on the analytic flag variety.

6.1.1. For the theory of Berkovich analytic spaces we refer to [**?**], [**?**]. We keep the notations introduced in the preceding section. In particular, X denotes the variety of Borel subgroups of $\mathbf{G}$. Being a scheme of finite type over L we have an associated Berkovich analytic space X^{an} over L ([**?**, Thm. 3.4.1]). In the preceding section we recalled a part of the algebraic Beilinson-Bernstein localization theory over X. It admits the following 'analytification' over X^{an}.

By construction X^{an} comes equipped with a canonical morphism

$$\pi : X^{an} \to X$$

of locally ringed spaces. Let π^* be the associated inverse image functor from $\mathcal{O}_X$-modules to $\mathcal{O}_{X^{an}}$-modules. Here $\mathcal{O}_{X^{an}}$ denotes the structure sheaf of the locally ringed space X^{an}. As with any morphism of locally ringed spaces we have the sheaf

$$\mathcal{T}_{X^{an}} := \mathcal{D}er_L(\mathcal{O}_{X^{an}})$$

of L-derivations of $\mathcal{O}_{X^{an}}$ ([**?**, 16.5.4]). By definition $\Gamma(X^{an}, \mathcal{T}_{X^{an}}) = \mathrm{Der}_L(\mathcal{O}_{X^{an}})$. Since X^{an} is smooth over L the results of [**?**, 3.3/3.5] imply that the stalk of this sheaf at a point $x \in X^{an}$ equals $\mathcal{T}_{X^{an},x} = \mathrm{Der}_L(\mathcal{O}_{X^{an},x})$.

Let $\mathbf{G}^{an}$ denote the analytic space associated to the variety $\mathbf{G}$ and let $\pi_{\mathbf{G}} : \mathbf{G}^{an} \to \mathbf{G}$ be the canonical morphism. The space $\mathbf{G}^{an}$ is a group object in the category of L-analytic spaces (a *L-analytic group* in the terminology of [**?**, 5.1]). The unit sections of $\mathbf{G}$ and $\mathbf{G}^{an}$ correspond via $\pi_{\mathbf{G}}$ which allows us to canonically identify the Lie algebra of $\mathbf{G}^{an}$ with $\mathfrak{g}$ (loc.cit., Thm. 3.4.1 (ii)). By functoriality the group $\mathbf{G}^{an}$ acts on X^{an}. The following result is proved as in the scheme case.

Lemma 6.1.2. *The group action induces a Lie algebra homomorphism*

$$\mathfrak{g} \to \Gamma(X^{an}, \mathcal{T}_{X^{an}}) \ .$$

We define

$$\mathfrak{g}^{\circ,an} := \mathcal{O}_{X^{an}} \otimes_L \mathfrak{g} = \rho^*(\mathfrak{g}^\circ) \ .$$

The preceding lemma allows on the one hand, to define a structure of L-Lie algebra on $\mathfrak{g}^{\circ,an}$. The respective enveloping algebra will be denoted by $\mathcal{O}_{X^{an}}\#U(\mathfrak{g})$. On the other hand, the map from the lemma extends to a $\mathcal{O}_{X^{an}}$-linear morphism of L-Lie algebras

$$\alpha^{\circ,an} : \mathfrak{g}^{\circ,an} \longrightarrow \mathcal{T}_{X^{an}} \ . \tag{6.1.3}$$

As in the algebraic case we put $\mathfrak{b}^{\circ,an} := \ker \alpha^{\circ,an}$ and $\mathfrak{n}^{\circ,an} := [\mathfrak{b}^{\circ,an}, \mathfrak{b}^{\circ,an}]$. Again, we obtain a morphism $\mathfrak{b}^{\circ,an} \to \mathcal{O}_{X^{an}} \otimes_L \mathfrak{t}$. Given $\chi \in \mathfrak{t}^*$ and $\lambda := \chi - \rho$ we denote by $\mathcal{I}^{an}$ resp. $\mathcal{I}^{an}_\chi$ the right ideal sheaf of $\mathcal{O}_{X^{an}}\#U(\mathfrak{g})$ generated by $\mathfrak{n}^{\circ,an}$ resp. $\ker \lambda^{\circ,an}$ where $\lambda^{\circ,an}$ equals the $\mathcal{O}_{X^{an}}$-linear form of $\mathfrak{b}^{\circ,an}$ induced by λ. These are two-sided ideals. We let

$$\mathcal{D}^{an}_{\mathfrak{t}} := (\mathcal{O}_{X^{an}}\#U(\mathfrak{g}))\,/\mathcal{I}^{an} \quad \text{and} \quad \mathcal{D}^{an}_{\chi} := (\mathcal{O}_{X^{an}}\#U(\mathfrak{g}))\,/\mathcal{I}^{an}_\chi$$

be the quotient sheaves. We view $\mathcal{D}^{an}_\chi$ as a sheaf of twisted differential operators on X^{an}.

All these constructions are compatible with their algebraic counterparts via the functor π^*. For example, using the fact that $\pi^*(\mathcal{T}_X) = \mathcal{T}_{X^{an}}$ it follows from the above proof that $\alpha^{\circ,an} = \pi^*(\alpha^\circ)$. Moreover, all that has been said in sec. 5 on the relation between the sheaves $\mathcal{D}_{\mathfrak{t}}$ and $\mathcal{D}_\chi$ remains true for its analytifications. In particular, $\mathcal{D}^{an}_\chi$ is a central reduction of $\mathcal{D}^{an}_{\mathfrak{t}}$ via the character $\lambda : U(\mathfrak{t}) \to L$:

$$\mathcal{D}^{an}_{\mathfrak{t}}/(\ker \lambda)\mathcal{D}^{an}_{\mathfrak{t}} \xrightarrow{\cong} \mathcal{D}^{an}_\chi \ . \tag{6.1.4}$$

6.2. **The Berkovich embedding and analytic stalks.** Recall our chosen Borel subgroup $\mathbf{B} \subset \mathbf{G}$ containing $\mathbf{T}$ and the quotient morphism $q : \mathbf{G} \to \mathbf{G}/\mathbf{B} = X$. We will make heavy use of the following result of V. Berkovich which was taken up and generalized in a conceptual way in [**?**]. Let $\eta \in X$ be the generic point of X.

Theorem 6.2.1. *(Berkovich,Rémy/Thuillier/Werner) There exists a G-equivariant injective map*

$$\vartheta_{\mathbf{B}} : \mathscr{B} \longrightarrow X^{an}$$

which is a homeomorphism onto its image. The latter is a locally closed subspace of X^{an} contained in the preimage $\pi^{-1}(\eta)$ of the generic point of X.

Proof. This is [**?**, 5.5.1]. We sketch the construction in the language of [**?**]. The map is constructed in three steps. First one attaches to any point $z \in \mathscr{B}$ an L-affinoid subgroup $\mathbf{G}_z$ of $\mathbf{G}^{an}$ whose rational points coincide with the stabilizer of z in G. In a second step one attaches to $\mathbf{G}_z$ the unique point in its Shilov boundary (the *sup-norm* on $\mathbf{G}_z$) which defines a map $\vartheta : \mathscr{B} \to \mathbf{G}^{an}$. In a final step one composes this map with the analytification of the orbit map $\mathbf{G} \to X, g \mapsto g.\mathbf{B}$. The last assertion follows from the next lemma. □

Lemma 6.2.2. *Let $z \in \mathscr{B}$. The local rings $\mathcal{O}_{X^{an},\vartheta_{\mathbf{B}}(z)}$ and $\mathcal{O}_{X,\pi(\vartheta_{\mathbf{B}}(z))}$ are fields. In particular, $\pi(\vartheta_{\mathbf{B}}(z))) = \eta$, the generic point of X.*

Proof. This is a direct consequence of ([**?**], Corollary 2.18) and the sentence immediately following that corollary. □

Since X^{an} is a compact Hausdorff space by [**?**, 3.4.8], the closure of the image of $\vartheta_{\mathbf{B}}$ in X^{an} is a compactification of $\mathscr{B}$ (loc.cit., Remark 3.31). It is called the *Berkovich compactification* of $\mathscr{B}$ of type $\emptyset$ (loc.cit., Def. 3.30). We will in the following often identify $\mathscr{B}$ with its image under $\vartheta_{\mathbf{B}}$ and hence, view $\mathscr{B}$ as a locally closed subspace of X^{an}.

6.2.3. By [**?**, 1.5] the space X^{an} is a *good* analytic space (in the sense of loc.cit., Rem. 1.2.16) which means that any point of X^{an} lies in the topological interior of an affinoid domain. In particular, given $x \in X^{an}$ the stalk $\mathcal{O}_{X^{an},x}$ may be written as

$$\mathcal{O}_{X^{an},x} = \varinjlim_{x \in V} \mathcal{A}_V$$

where the inductive limit ranges over the affinoid neighborhoods V of x and where $\mathcal{A}_V$ denotes the associated affinoid algebra. As usual a subset of neighborhoods of x will be called *cofinal* if it is cofinal in the directed partially ordered set of all neighborhoods of x. If V is an affinoid neighborhood of x, the corresponding affinoid algebra $\mathcal{A}_V$ carries its Banach topology. We endow $\mathcal{O}_{X^{an},x}$ with the locally convex final topology ([**?**, §5.E]) arising from the above inductive limit. This topology makes $\mathcal{O}_{X^{an},x}$ a topological L-algebra. We need another, rather technical, property of this topology.

Lemma 6.2.4. *Let $x \in X^{an}$. There is a sequence $V_1 \supset V_2 \supset V_3 \supset \ldots$ of irreducible reduced strictly affinoid neighborhoods of x which is cofinal and has the property: the homomorphism of affinoid algebras $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ associated with the inclusion $V_{i+1} \subset V_i$ is flat and an injective compact linear map between Banach spaces. In particular, the stalk $\mathcal{O}_{X^{an},x}$ is a vector space of compact type.*

Proof. Being an analytification of a variety over L, the analytic space X^{an} is closed (in the sense of [?, p. 49]), cf. [?, 3.4.1]. Since the valuation on L is nontrivial, X^{an} is strictly k-analytic (loc.cit., Prop. 3.1.2). Let V be a strictly affinoid neighborhood of x in X^{an} so that x lies in the topological interior of V. In the following we will use basic results on the relative interior $\mathrm{Int}(Y/Z)$ of an analytic morphism $Y \to Z$ (loc.cit., 2.5/3.1). As usual we write $\mathrm{Int}(Y)$ in case of the structure morphism $Y \to \mathscr{M}(L)$. Since X^{an} is closed we have by definition $\mathrm{Int}(X^{an}) = X^{an}$. Moreover, loc.cit., Prop. 3.1.3 (ii), implies $\mathrm{Int}(V) = \mathrm{Int}(V/X^{an})$. By part (i) of the same proposition the topological interior of V is equal to $\mathrm{Int}(V/X^{an})$ and, thus, $x \in \mathrm{Int}(V)$. Now the residue field of L being finite, there is a countable basis $\{W_n\}_{n\in\mathbb{N}}$ of neighborhoods of x (cf. discussion after [?, 3.2.8]) which consists of strictly affinoid subdomains (even Laurent domains) of V ([?, Prop. 3.2.9]). By smoothness of X^{an} the local ring $\mathcal{O}_{X^{an},x}$ is noetherian regular and hence an integral domain. We may therefore assume that all W_n are reduced and irreducible (loc.cit., last sentence of 2.3). Consider $V_1 := W_{n_1}$ for some $n_1 \in \mathbb{N}$. As we have just seen $x \in \mathrm{Int}(V_1)$. Since $\mathrm{Int}(V_1)$ is an open neighborhood of x there is $n_2 > n_1$ such that $W_{n_2} \subseteq \mathrm{Int}(V_1)$. We put $V_2 := W_{n_2}$ and repeat the above argument with V_1 replaced by V_2. In this way we find a cofinal sequence $V_1 \supset V_2 \supset V_3 \ldots$ of strictly irreducible reduced affinoid neighborhoods of x with the property $\mathrm{Int}(V_i) \supseteq V_{i+1}$ for all $i \geq 1$. According to loc.cit., Prop. 2.5.9, the bounded homomorphism of L-Banach algebras $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ associated with the inclusion $V_{i+1} \subset V_i$ is inner with respect to L (in the sense of loc.cit., Def. 2.5.1). The arguments in [?, Prop. 2.1.16] now show that $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ is a compact linear map between Banach spaces. Finally, this latter map is injective because V_i is irreducible and V_{i+1} contains a nonempty open subset of V_i. It is also flat since, by construction, V_{i+1} is an affinoid subdomain of V_i ([?, Prop. 2.2.4 (ii)]). □

6.2.5. In this paragraph and the next lemma we assume $L = \mathbb{Q}_p$. Consider for a given $z \in \mathscr{B}$ the group $U_z^{(e)} \subset G$, cf. 4.1.5. For $e \geq e_{uni}$ the group $U_z^{(e)}$ is uniform pro-p, cf. 4.3.4. As such, it has a $\mathbb{Z}_p$-Lie algebra $\mathcal{L}(U_z^{(e)})$, which is powerful, and the exponential map $\exp_{U_z^{(e)}} : \mathcal{L}(U_z^{(e)}) \to U_z^{(e)}$ is well-defined and a bijection, cf. [?, sec. 9.4]. Using the Baker-Campbell-Hausdorff series one can then associate to the lattice $\mathcal{L}(U_z^{(e)})$ a $\mathbb{Q}_p$-analytic affinoid subgroup $\mathbb{U}_z^{(e)} \subset \mathbf{G}^{an}$ which has the property that $\mathbb{U}_z^{(e)}(\mathbb{Q}_p) = U_z^{(e)}$.[6] ($U_z^{(e)}$ is a good analytic open subgroup of G in the sense of [?, sec. 5.2].) Let $V \subset X^{an}$ be an affinoid domain. We say that $U_z^{(e)}$ acts *analytically* on V, if there is an action of the

[6]Only here do we use that $L = \mathbb{Q}_p$. For general L it would be necessary to show that $U_z^{(e)}$ is actually an L-uniform pro-p group, cf. [?, 2.2.5]. This can be done, but we do not work here in this generality.

affinoid group $\mathbb{U}_z^{(e)}$ on V which is compatible with the action of $\mathbb{U}_z^{(e)}$ on X^{an}, i.e., if there is a commutative diagram of group operations

$$\begin{array}{ccc} \mathbb{U}_z^{(e)} \times V & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathbf{G}^{an} \times X^{an} & \longrightarrow & X^{an} \end{array}$$

where the vertical maps are inclusions (and the products on the left are taken in the category of L-analytic spaces).

Lemma 6.2.6. *There exists a number $e_{st} \geq e_{uni}$ with the following property. For any point $z \in \mathscr{B}$, viewed as a point in X^{an}, there is a fundamental system of strictly affinoid neighborhoods $\{V_n\}_{n\geq 0}$ of z with the properties as in 6.2.4, and such that for all $n \geq 0$ and $e \geq e_{st}$ the group $U_z^{(e)}$ acts analytically on V_n.*

The *proof*, which is lengthy, is given in section 12, in order to not interrupt the discussion at this point.

6.3. **A structure sheaf on the building.**

6.3.1. To be able to compare the localization of Schneider-Stuhler and Beilinson-Bernstein we equip the topological space $\mathscr{B}$ with a sheaf of commutative and topological L-algebras. Recall that a subset $V \subset X^{an}$ is called a *special domain* if it is a finite union of affinoid domains, and to any special domain V there is associated an L-Banach algebra $\mathcal{A}_V$, cf. [**?**, 2.2.6]. The sheaf $\mathcal{O}_{X^{an}}$ is naturally a sheaf of locally convex algebras as follows: given an open subset $U \subset X^{an}$ we have

$$\mathcal{O}_{X^{an}}(U) = \varprojlim_{V \subset U} \mathcal{A}_V \ ,$$

where the limit is taken over all special domains (or affinoid domains) of X^{an} which are contained in U. Here, $\mathcal{A}_V$ is the L-Banach algebra corresponding to V and the projective limit is equipped with the projective limit topology. Because the residue field of L is finite, X^{an} has a countable basis of open subsets, cf. [**?**, 3.2.9]. Therefore, one can cover U with a countable set of special domains and $\mathcal{O}_{X^{an}}(U)$ is thus a countable projective limit of Banach algebras, hence a Fréchet algebra over L.

We then consider the exact functor $\vartheta_{\mathbf{B}}^{-1}$ from abelian sheaves on X^{an} to abelian sheaves on $\mathscr{B}$ given by restriction along $\vartheta_{\mathbf{B}} : \mathscr{B} \hookrightarrow X^{an}$. Let

$$\mathcal{O}_{\mathscr{B}} := \vartheta_{\mathbf{B}}^{-1}(\mathcal{O}_{X^{an}}) \ .$$

For any subset $C \subset X^{an}$ we can consider $\mathcal{O}_{X^{an}}(C)$, the vector space of sections of $\mathcal{O}_{X^{an}}$ over C, i.e., the global sections of the restriction of the sheaf $\mathcal{O}_{X^{an}}$ to C.

Proposition 6.3.2. *For any subset $C \subset X^{an}$ we have*

$$\mathcal{O}_{X^{an}}(C) = \varinjlim_{C \subset U} \mathcal{O}_{X^{an}}(U)$$

where U runs through all open neighborhoods of C in X^{an}.

Proof. As was pointed out in 6.3.1, the compact Hausdorff topological space X^{an} has a countable basis of open subsets. By Urysohn's metrization theorem, it is therefore metrizable, and we may apply [**?**, II.3.3 Cor. 1]. ☐

In particular, given an open set $\Omega \subseteq \mathscr{B}$ we have

$$\mathcal{O}_{\mathscr{B}}(\Omega) = \varinjlim_{\Omega \subset U} \mathcal{O}_{X^{an}}(U)$$

where U runs through the open neighborhoods of Ω in X^{an}. Using the locally convex inductive limit topology on the right hand side, the sheaf $\mathcal{O}_{\mathscr{B}}$ becomes in this way a sheaf of locally convex algebras. We point out that the stalk $\mathcal{O}_{\mathscr{B},z} = \mathcal{O}_{X^{an},z}$ for any point $z \in \mathscr{B}$ is in fact a field, cf. 6.2.2. We summarize the following properties of $\vartheta_{\mathbf{B}}^{-1}$:

(1) $\vartheta_{\mathbf{B}}^{-1}$ preserves (commutative) rings, L-algebras, L-Lie algebras and G-equivariance,
(2) $\vartheta_{\mathbf{B}}^{-1}$ maps $\mathcal{O}_{X^{an}}$-modules into $\mathcal{O}_{\mathscr{B}}$-modules,
(3) $\vartheta_{\mathbf{B}}^{-1}$ induces a Lie algebra homomorphism $\mathrm{Der}_L(\mathcal{O}_{X^{an}}) \to \mathrm{Der}_L(\mathcal{O}_{\mathscr{B}})$,
(4) $\mathcal{O}_{\mathscr{B}}$ is a sheaf of *locally convex* L-algebras. For every $z \in \mathscr{B}$ the stalk $\mathcal{O}_{\mathscr{B},z}$ is of compact type with a defining system $\mathcal{A}_{V_n}$ of Banach algebras, where $(V_n)_n$ is a fundamental system of affinoid neighborhoods as in 6.2.6.

Composing the map $\mathfrak{g} \to \mathrm{Der}_L(\mathcal{O}_{X^{an}})$ from 6.1.2 with (3) yields a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}_L(\mathcal{O}_{\mathscr{B}})$ and the associated skew enveloping algebra $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})$. By (1),(2) we have the L-Lie algebras and $\mathcal{O}_{\mathscr{B}}$-modules $\mathfrak{n}_{\mathscr{B}}^{\circ,an} := \vartheta_{\mathbf{B}}^{-1}(\mathfrak{n}^{\circ,an})$ and $\mathfrak{b}_{\mathscr{B}}^{\circ,an} := \vartheta_{\mathbf{B}}^{-1}(\mathfrak{b}^{\circ,an})$. Similarly,

$$\lambda_{\mathscr{B}}^{\circ,an} := \vartheta_{\mathbf{B}}^{-1}(\lambda^{\circ,an}) : \mathfrak{b}_{\mathscr{B}}^{\circ,an} \longrightarrow \mathcal{O}_{\mathscr{B}}$$

is a morphism of L-Lie algebras and $\mathcal{O}_{\mathscr{B}}$-modules. Let $\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{an}$ resp. $\mathcal{I}_{\mathscr{B},\chi}^{an}$ be the right ideal sheaf of $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})$ generated by $\mathfrak{n}_{\mathscr{B}}^{\circ,an}$ resp. $\ker(\lambda_{\mathscr{B}}^{\circ,an})$. One checks that these are two-sided ideals. We let

$$\mathcal{D}_{\mathscr{B},\mathfrak{t}}^{an} := (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))/\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{an} \quad \text{and} \quad \mathcal{D}_{\mathscr{B},\chi}^{an} := (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))/\mathcal{I}_{\mathscr{B},\chi}^{an} .$$

Note that by exactness of $\vartheta_{\mathbf{B}}^{-1}$ we have

$$\mathcal{D}_{\mathscr{B},\mathfrak{t}}^{an} = \vartheta_{\mathbf{B}}^{-1}(\mathcal{D}_{\mathfrak{t}}^{an}) \quad \text{and} \quad \mathcal{D}_{\mathscr{B},\chi}^{an} = \vartheta_{\mathbf{B}}^{-1}(\mathcal{D}_{\chi}^{an}).$$

6.3.3. The sheaf $\mathcal{D}^{an}_{\mathscr{B},\chi}$ of twisted differential operators on $\mathscr{B}$ is formed with respect to the Lie algebra action of $\mathfrak{g}$ on the ambient space $\mathscr{B} \subset X^{an}$. In an attempt to keep track of the whole analytic G-action on X^{an} we will produce in the following a natural injective morphism of sheaves of algebras

$$\mathcal{D}^{an}_{\mathscr{B},\chi} \longrightarrow \mathscr{D}_{r,\chi}$$

with target a sheaf of what we tentatively call *twisted distribution operators* on $\mathscr{B}$. Actually, there will be one such sheaf for each 'radius' $r \in [r_0, 1)$ in $p^{\mathbb{Q}}$ and each sufficiently large 'level' $e > 0$. Again, following [?] we suppress the dependence on the level in our notation.

6.4. Mahler series and completed skew group rings.

6.4.1. Suppose for a moment that $\mathcal{A}$ is an arbitrary L-Banach algebra. Since $\mathbb{Q}_p \subset \mathcal{A}$ the completely valued $\mathbb{Z}_p$-module $(\mathcal{A}, |.|)$ is saturated in the sense of [?, I.2.2.10]. Consequently, we have the theory of Mahler expansions over $\mathcal{A}$ at our disposal (loc.cit., III.1.2.4 and III.1.3.9). In this situation we prove a version of the well-known relation between decay of Mahler coefficients and overconvergence.

Proposition 6.4.2. *Let $f = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha x^\alpha$ be a d-variable power series over $\mathcal{A}$ converging on the disc $|x_i| \leq R$ for some $R > 1$. Let $c > 0$ be a constant such that $|a_\alpha| \leq cR^{-|\alpha|}$ for all α. Let*

$$f(\cdot) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{\cdot}{\alpha} ,$$

$c_\alpha \in \mathcal{A}$, be the Mahler series expansion of f. Then $|c_\alpha| \leq cs^{|\alpha|}$ for all α where $s = \tilde{r}R^{-1}$ with $\tilde{r} = p^{-\frac{1}{p-1}}$.

Proof. We prove the lemma in case $d = 1$. The general case follows along the same lines but with more notation. We define the following series of polynomials over $\mathbb{Z}$

$$(x)_0 = 1, \ldots, (x)_k = x(x-1)\cdots(x-k+1)$$

for $k \geq 1$. The $\mathbb{Z}$-module $\mathbb{Z}[x]$ has the $\mathbb{Z}$-bases $\{x^k\}_{k\geq 0}$ and $\{(x)_k\}_{k\geq 0}$ and the transition matrices are unipotent upper triangular. We may therefore write

$$x^n = \sum_{k=0,\ldots,n} s(n,k)(x)_k \tag{6.4.3}$$

with $s(n,k) \in \mathbb{Z}$. Put $b_k := c_k/k!$. Then

$$\sum_{k\geq 0} c_k \binom{x}{k} = \sum_{k \geq 0} b_\kappa (x)_k$$

is a uniform limit of continuous functions (even polynomials) on $\mathbb{Z}_p$ (cf. [?, Thm. VI.4.7]). We now proceed as in (the proof of) [?, Prop. 5.8]. Fix $i \geq 1$ and write

$$\sum_{n\leq i} a_n x^n = \sum_{k\leq i} b_{k,i}(x)_k$$

as polynomials over $\mathcal{A}$ with some elements $b_{k,i} \in \mathcal{A}$. Inserting (6.4.3) and comparing coefficients yields $b_{k,i} = \sum_{k\leq n\leq i} a_n s(n,k)$ and consequently,

$$|b_{k,i}| \leq \max_{k\leq n\leq i} |a_n| \leq \max_{k\leq n\leq i} (cR^{-n}) \leq cR^{-k}$$

since $R^{-1} < 1$. It follows that, for $j \geq i$, we have

$$|b_{k,j} - b_{k,i}| = |\Sigma_{n=i+1}^{j} a_n s(n,k)| \leq R^{-(i+1)} \ .$$

We easily deduce from this that $\{b_{k,i}\}_{i\geq 0}$ is a Cauchy sequence in the Banach space $\mathcal{A}$. Let $\tilde{b}_k$ be its limit. Clearly, $|\tilde{b}_k| \leq cR^{-k}$. Put $\tilde{c}_k := k! \cdot \tilde{b}_k$. Since $|k!| \leq \tilde{r}^k$ we obtain $|\tilde{c}_k| \leq c(\tilde{r}R^{-1})^k = cs^k$ for all k. By definition of $\tilde{b}_k$ the series of polynomials

$$\sum_{k\geq 0} \tilde{c}_k \binom{x}{k} = \sum_{k\geq 0} \tilde{b}_\kappa(x)_k$$

converges pointwise to the limit

$$\lim_{i\to\infty} \sum_{k\leq i} b_{k,i}(x)_k = \lim_{i\to\infty} \sum_{n\leq i} a_n x^n = f(x) \ .$$

By [?, IV.2.3, p. 173] this convergence is uniform and so uniqueness of Mahler expansions implies $\tilde{c}_k = c_k$ for all k. This proves the lemma. □

Corollary 6.4.4. *Let $L = \mathbb{Q}_p$, $z \in \mathscr{B}$, and $e > e_{uni}$. (i) Consider an affinoid domain V of X^{an} on which $U_z^{(e-1)}$ acts analytically in the sense of 6.2.5, and let $\mathcal{A}_V$ be the corresponding Banach algebra.*

(a) *For any p-basis $(h_1, \ldots, h_d)$ of $U_z^{(e-1)}$, cf. 2.2, and for any $f \in \mathcal{A}_V$ the orbit map $U^{(e-1)} \to \mathcal{A}_V$, $h = h_1^{x_1} \cdots h_d^{x_d} \mapsto h.f$, can be expanded as a strictly convergent power series $\sum_{\nu\in\mathbb{N}^d} f_\nu x_1^{\nu_1} \cdots x_d^{\nu_d}$ with $f_\nu \in \mathcal{A}_V$ and $|f_\nu|_V \to 0$ as $|\nu| \to \infty$. ($|\cdot|_V$ denotes the supremum norm on $\mathcal{A}_V$.)*

(b) *The representation $\rho : U_z^{(e)} \to \mathrm{GL}(\mathcal{A}_V)$, $(\rho(h).f)(w) = f(h^{-1}.w)$, satisfies the assumption $(\star)$ of section 3.2.5 for any $r \in [r_0, 1)$. In particular, the ring*

$$\mathcal{A}_V \# D_r(U_z^{(e)}, K)$$

exists for all $r \in [r_0, 1)$.

(ii) More generally, let $V = V_1 \cup \ldots \cup V_m$ be a special domain of X^{an} (cf. 6.3.1), where V_i is affinoid for $1 \le i \le m$, and suppose that $U^{(e-1)}$ acts analytically on each V_i, $1 \le i \le m$. Then the representation $\rho : U_z^{(e)} \to \mathrm{GL}(\mathcal{A}_V)$, $(\rho(h).f)(w) = f(h^{-1}.w)$, satisfies the assumption $(\star)$ of section 3.2.5 for any $r \in [r_0, 1)$. In particular, the ring

$$\mathcal{A}_V \# D_r(U_z^{(e)}, K)$$

exists for all $r \in [r_0, 1)$.

Proof. (i)(a) To simplify notation put $U = U_z^{(e-1)}$ and $\mathbb{U} = \mathbb{U}_z^{(e-1)}$, cf. 6.2.5. Let $\mathcal{A}_{\mathbb{U}}$ be the affinoid algebra of $\mathbb{U}$. The p-basis gives rise to an isomorphism $\mathcal{A}_{\mathbb{U}} \simeq \mathbb{Q}_p\langle x_1, \ldots, x_d\rangle$, where the latter denotes strictly convergent power series. The action of $\mathbb{U}$ on V corresponds to a morphism of affinoid algebras

$$\mathcal{A}_V \to \mathcal{A}_{\mathbb{U}} \hat{\otimes}_L \mathcal{A}_V \simeq \mathcal{A}_V\langle x_1, \ldots, x_d\rangle \ .$$

On the right we have the algebra of strictly convergent power series over $\mathcal{A}_V$. This proves the first assertion.

(i)(b) By 4.3.4 we have that $U_z^{(e)}$ is the second member of the lower p-series of $U_z^{(e-1)}$. Therefore, if $(h_1, \ldots, h_d)$ is the p-basis for $U_z^{(e-1)}$ used in (i), it follows that $(h_1^p, \ldots, h_d^p)$ is a p-basis for $U_z^{(e)}$. Denote by $(y_1, \ldots, y_d)$ the coordinates on $U_z^{(e)}$ corresponding to this p-basis. Then, applying (i) to the group $U_z^{(e)}$, we find that

$$\rho(h).f = \sum_{\nu \in \mathbb{N}^d} f_\nu y_1^{\nu_1} \cdots y_d^{\nu_d} \tag{6.4.5}$$

when $h = (h_1^p)^{y_1} \cdots (h_d^p)^{y_d} \in U_z^{(e)}$ and $f \in \mathcal{A}_V$. Therefore, the right hand side of 6.4.5 converges on the disc $|y_i| \le p$. Next consider the Mahler expansion

$$\rho\big((h_1^p)^{y_1} \cdots (h_d^p)^{y_d}\big).f = \sum_{\alpha \in \mathbb{N}^d} c_{f,\alpha} \binom{y}{\alpha} \ .$$

By 6.4.2 we have $|c_{f,\alpha}|_V \le c s^{|\alpha|}$ with some $c > 0$ and $s = r_1 p^{-1} < p^{-1} = r_0$.

Now write $\delta \in D_r(U_z^{(e)}, L)$ as a series $\delta = \sum_{\alpha \in \mathbb{N}^d} d_\alpha \mathbf{b}^\alpha$ with $\mathbf{b}^\alpha = (h_1^p - 1)^{\alpha_1} \cdots (h_d^p - 1)^{\alpha_d}$ and $d_\alpha \in L$ such that $|d_\alpha| r^{|\alpha|} \to 0$. Since $s < r_0 \le r$ and $|c_{f,\alpha}|_V \le c s^{|\alpha|}$ the sum

$$\delta.f = \delta\big(h \mapsto \rho(h).f\big) = \sum_{\alpha \in \mathbb{N}^d} d_\alpha c_{f,\alpha} \tag{6.4.6}$$

converges in the Banach space $\mathcal{A}_V$. The map $(\delta, f) \mapsto \delta.f$ makes $\mathcal{A}_V$ a topological module over $D_r(U_z^{(e)}, L)$ in a way compatible with the map $D(U_z^{(e)}) \to D_r(U_z^{(e)})$. The last assertion is contained in 3.2.5.

(ii) As X is separated, X^{an} is Hausdorff and therefore separated, cf. [**?**, 3.4.8, 3.1.5]. This implies that the intersection of any two affinoid domains in X^{an} is again an affinoid domain, cf. [**?**, 3.1.6]. In this case,

$$\mathcal{A}_V = \ker\left(\prod_{i=1}^m \mathcal{A}_{V_i} \rightrightarrows \prod_{i,j} \mathcal{A}_{V_i\cap V_j}\right) \ ,$$

cf. [**?**, 2.2.6, 3.3]. We can now apply the assertions in (i)(b) to each factor $\mathcal{A}_{V_i}$ and $\mathcal{A}_{V_i\cap V_j}$, and to the corresponding products, and deduce statement (ii). □

6.4.7. Later on we will sometimes need to consider the action of $U_z^{(e)}$ on rings of the form $\mathcal{A}_V$, where V is a special domain in X^{an} as in 6.4.4 (ii). In order to conveniently refer to this situation, we will say that $U_z^{(e)}$ acts *analytically* on a special domain V, if one can write $V = V_1 \cup \ldots \cup V_m$ as finite union of affinoid domains $V_i \subset X^{an}$, $1 \le i \le m$, with the property that $U_z^{(e)}$ acts analytically on each V_i, $1 \le i \le m$.

Until the end of this section we will assume $L = \mathbb{Q}_p$, $e > e_{st}$ and $r \in [r_0, 1)$.

Proposition 6.4.8. *Let $z \in \mathscr{B}$, and let $(V_n)_n$ be a descending sequence of affinoid neighborhoods of z as in 6.2.6. Then the stalk $\mathcal{O}_{\mathscr{B},z}$ is equal to the inductive limit of the Banach algebras $\mathcal{A}_{V_n}$, the completed skew group rings $\mathcal{A}_{V_n}\#D_r(U_z^{(e)}, K)$ and $\mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}, K)$ exist, and the natural map*

$$\varinjlim_n \left(\mathcal{A}_{V_n}\#D_r(U_z^{(e)}, K)\right) \longrightarrow \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}, K)$$

is an isomorphism of topological K-algebras.

Proof. By 6.2.4 and 6.2.6, the stalk $\mathcal{O}_{\mathscr{B},z}$ is the inductive limit of the Banach algebras $\mathcal{A}_{V_n}$, and the transition maps $\mathcal{A}_{V_n} \to \mathcal{A}_{V_{n+1}}$ are compact and injective. By the main result of section 13, the natural map

$$(6.4.9)\quad \varinjlim \left(\mathcal{A}_{V_n}\hat{\otimes}_L D_r(U_z^{(e)}, K)\right) \longrightarrow \left(\varinjlim \mathcal{A}_{V_n}\right)\hat{\otimes}_L D_r(U_z^{(e)}, K) = \mathcal{O}_{\mathscr{B},z}\hat{\otimes}_L D_r(U_z^{(e)}, K)$$

is an isomorphism of topological vector spaces. By 6.4.4 (i)(b) the ring $\mathcal{A}_{V_n}$ is a $D_r(U_z^{(e)}, L)$-module for every n and $r \in [r_0, 1)$, and the transition maps are homomorphisms of $D_r(U_z^{(e)}, L)$-modules. This shows that $\mathcal{O}_{\mathscr{B},z}$ is naturally a $D_r(U_z^{(e)}, L)$-module, hence $\mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}, L)$ and $\mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}, K)$ exist. The natural map $\mathcal{A}_{V_n}\#D_r(U_z^{(e)}, K) \to \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}, K)$ is a ring homomorphism, and the map 6.4.9 is an isomorphism of topological K-algebras. □

The following corollary is immediate and recorded only for future reference.

Corollary 6.4.10. *Let V be a neighborhood of z which is a special subset of X^{an}. Suppose $U^{(e-1)}$ acts analytically on V in the sense of 6.4.7. Let ι_z be the natural map $\mathcal{A}_V \to \mathcal{O}_{\mathscr{B},z}$ sending a function to its germ at z. The map $\iota_z \hat{\otimes} \mathrm{id}$ is an algebra homomorphism*

$$\mathcal{A}_V \# D_r(U_z^{(e)}, K) \longrightarrow \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)\ .$$

Corollary 6.4.11. *Let V be a neighborhood of z which is a special subset of X^{an}. Suppose $U^{(e-1)}$ acts analytically on V in the sense of 6.4.7. Then he inclusions*

$$L[U_z^{(e)}] \subseteq D_r(U_z^{(e)}, K)$$

and $U(\mathfrak{g})_K \subseteq D_r(U_z^{(e)}, K)$ induce algebra homomorphisms

(i) $\mathcal{A}_V \# U_z^{(e)} = \mathcal{A}_V \otimes_L L[U_z^{(e)}] \longrightarrow \mathcal{A}_V \# D_r(U_z^{(e)}, K)$,

(ii) $\mathcal{A}_V \# U(\mathfrak{g})_K \longrightarrow \mathcal{A}_V \# D_r(U_z^{(e)}, K)$.

If V runs through a sequence of affinoid neighborhoods of z as in 6.2.6 these maps assemble to algebra homomorphisms

(i) $\mathcal{O}_{\mathscr{B},z} \# U_z^{(e)} = \mathcal{O}_{\mathscr{B},z} \otimes_L L[U_z^{(e)}] \longrightarrow \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$,

(ii) $\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g})_K \longrightarrow \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$.

Proof. Consider the case of $\mathcal{A}_V$. The existence of the map (i) follows from 3.2.4. The same is true for the map (ii) once we convince ourselves that there is a commutative diagram of algebra homomorphisms

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\alpha^{\circ,an}} & \mathrm{End}_L(\mathcal{A}_V) \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle Id} \\ D_r(U_z^{(e)}, L) & \longrightarrow & \mathrm{End}_L(\mathcal{A}_V) \end{array}$$

where the upper horizontal arrow is derived from (6.1.3) and the lower horizontal arrow describes the $D_r(U_z^{(e)}, L)$-module structure of $\mathcal{A}_V$ as given by 6.4.4. Restricting the lower horizontal arrow to $\mathfrak{g}$ amounts to differentiating the analytic $U_z^{(e)}$-action on $\mathcal{A}_V$. This action comes from the algebraic action of $\mathbf{G}$ on X. The diagram commutes by the remark following 6.1.2. Having settled the case $\mathcal{A}_V$ the case of $\mathcal{O}_{\mathscr{B},z}$ now follows by passage to the inductive limit. □

As a result of this discussion we have associated to each point $z \in \mathscr{B} \subset X^{an}$ the (noncommutative) topological K-algebra $\mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)$. As we have seen, it comes together with an injective algebra homomorphism

$$\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g})_K \hookrightarrow \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}, K)\ . \tag{6.4.12}$$

In the next section we will sheafify this situation and obtain a sheaf of noncommutative K-algebras $\mathcal{O}_{\mathscr{B}}\#D_r$ on $\mathscr{B}$ together with an injective morphism of sheaves of algebras

$$\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})_K \hookrightarrow \mathcal{O}_{\mathscr{B}}\#D_r$$

inducing the map (6.4.12) at all points $z \in \mathscr{B}$. To do this we shall need a simple 'gluing property' of the algebras $\mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}, K)$.

Lemma 6.4.13. *Let F, F' be facets in $\mathscr{B}$ such that $F' \subseteq \overline{F}$ and let $\sigma_r^{F'F} : D_r(U_{F'}^{(e)}, K) \to D_r(U_F^{(e)}, K)$ be the corresponding algebra homomorphism. Suppose V and V' are two special domains in X^{an} on which $U_F^{(e-1)}$ and $U_{F'}^{(e-1)}$, respectively, act analytically (cf. 6.4.7). If $V \subset V'$ the map $res_V^{V'} \hat{\otimes} \sigma_r^{F'F}$ is a continuous algebra homomorphism*

$$res_V^{V'} \hat{\otimes} \sigma_r^{F'F} : \mathcal{A}_{V'}\#D_r(U_{F'}^{(e)}, K) \longrightarrow \mathcal{A}_V\#D_r(U_F^{(e)}, K) \ .$$

Proof. Since the map $\sigma_r^{F'F}$ is induced from the inclusion $U_{F'}^{(e)} \subseteq U_F^{(e)}$ there is a commutative diagram

$$\begin{array}{ccc} D_r(U_{F'}^{(e)}, L) \times \mathcal{A}_{V'} & \longrightarrow & \mathcal{A}_{V'} \\ \downarrow{\scriptstyle \sigma_r^{F'F} \times res} & & \downarrow{\scriptstyle res} \\ D_r(U_F^{(e)}, L) \times \mathcal{A}_V & \longrightarrow & \mathcal{A}_V \end{array} \ .$$

where the horizontal arrows describe the module structures of $\mathcal{A}_{V'}$ and $\mathcal{A}_V$ over $D_r(U_{F'}^{(e)}, L)$ and $D_r(U_F^{(e)}, L)$ respectively, cf. 6.4.4. The assertion follows now from the construction of the skew multiplication of the source and target of $res_V^{V'} \hat{\otimes} \sigma_r^{F'F}$ (cf. sec. 3). ☐

6.4.14. For any subset $C \subset X^{an}$ we have by 6.3.2

$$\mathcal{O}_{X^{an}}(C) = \varinjlim_U \mathcal{O}_{X^{an}}(U) \ ,$$

where U runs over all open neighborhoods of C in X^{an}. Obviously, if C is contained in $\mathscr{B}$ we have $\mathcal{O}_{\mathscr{B}}(C) = \mathcal{O}_{X^{an}}(C)$. We recall that the *star* of a facet F' in $\mathscr{B}$ is the subset of $\mathscr{B}$ defined by

$$St(F) := \text{ union of all facets } F' \subseteq \mathscr{B} \text{ such that } F \subseteq \overline{F'} \ .$$

These stars form a locally finite open covering of $\mathscr{B}$.

Proposition 6.4.15. *Let $F \subset \mathscr{B}$ be a facet, and let $C \subset St(F)$ be a compact set.*

(i) There is a countable fundamental system of neighborhoods $V_1 \supset V_2 \supset \dots$ of C in X^{an} with the following properties:

- *for all i the neighborhood V_i is a special subdomain on which $U_F^{(e)}$ acts analytically,*
- *for all $i<j$ the induced map $\mathcal{A}_{V_i} \to \mathcal{A}_{V_j}$ is compact and injective.*

(ii) Let $(V_i)_i$ be as in (i). Then the rings $\mathcal{A}_{V_i} \# D_r(U_F^{(e)})$ exist for all i and the maps

$$\mathcal{A}_{V_i} \# D_r(U_F^{(e)}) \to \mathcal{A}_{V_{i+1}} \# D_r(U_F^{(e)})$$

induced by the restriction maps $\mathcal{A}_{V_i} \to \mathcal{A}_{V_{i+1}}$ are homomorphisms of K-algebras.

(iii) Let $(V_i)_i$ be as in (i). Then the maps

$$\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)}) \to \mathcal{O}_{\mathscr{B}}(C) \hat{\otimes}_L D_r(U_F^{(e)})$$

induced by the canonical maps $\mathcal{A}_{V_i} \to \mathcal{O}_{\mathscr{B}}(C)$ induce an isomorphism of topological vector spaces

$$\varinjlim_i \left(\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)}) \right) \longrightarrow \mathcal{O}_{\mathscr{B}}(C) \hat{\otimes}_L D_r(U_F^{(e)}) \ . \tag{6.4.16}$$

(iv) The left hand side of 6.4.16 carries a unique structure of a K-algebra, such that the canonical maps

$$\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)}) \longrightarrow \varinjlim_i \left(\mathcal{A}_{V_i} \hat{\otimes}_L D_r(U_F^{(e)}) \right)$$

become K-algebra homomorphisms. Consequently, via transport of structure, we give the $\mathcal{O}_{\mathscr{B}}(C) \hat{\otimes}_L D_r(U_F^{(e)})$ the unique K-algebra structure, henceforth denoted by $\mathcal{O}_{\mathscr{B}}(C) \# D_r(U_F^{(e)})$, such that 6.4.16 becomes an isomorphism of K-algebras.

Proof. (i) By [**?**, 3.2.9] there is a countable fundamental system of open neighborhoods $W_1 \supset W_2 \supset \ldots$ of C in X^{an}. We are going to find inductively the special domain $V_i \subset W_i$. To begin, use 6.2.6 to find for each $x \in C$ an affinoid neighborhood $W_{1,x} \subset W_1$ on which $U_F^{(e)}$ acts analytically. Clearly, we may furthermore assume that every $W_{1,x}$ is connected. Denote by $\mathrm{Int}(W_{1,x})$ the topological interior of $W_{1,x}$. As C is compact it is contained in a finite union $\mathrm{Int}(W_{1,x_1}) \cup \ldots \cup \mathrm{Int}(W_{1,x_{m_1}})$. Put

$$V_1 = W_{1,x_1} \cup \ldots \cup W_{1,x_{m_1}} \ .$$

Now suppose we have found a special domain $V_i = W_{i,z_1} \cup \ldots \cup W_{i,z_{m_i}}$ contained in W_i with the property that for all $1 \le j \le m_i$

- W_{i,z_j} is a connected affinoid neighborhood of z_i on which $U_F^{(e)}$ acts analytically, for all $1 \le j \le m_i$,
- C is contained in the union of the $\mathrm{Int}(W_{i,z_j})$, for $1 \le j \le m_i$.

For a given $z' \in C$ choose $j(z') \in \{1, \ldots, m_i\}$ such that z' is contained in $\mathrm{Int}(W_{i,z_{j(z')}})$. Use again 6.2.6 to find a connected affinoid neighborhood $W_{i+1,z'}$ of z' contained in $W_{i+1} \cap \mathrm{Int}(W_{i,z_{j(z')}})$ on which $U_F^{(e)}$ acts analytically. Put $z = z_{j(z')}$. As $W_{i,z}$ is connected, it is irreducible, cf. [?, 3.1.8], and so is $\mathrm{Spec}(\mathcal{A}_{W_{i,z}})$. The ring $\mathcal{A}_{W_{i,z}}$ is hence an integral domain, and the restriction map

$$\mathcal{A}_{W_{i,z}} \longrightarrow \mathcal{A}_{W_{i+1,z'}} \tag{6.4.17}$$

is thus injective. Moreover, since $W_{i+1,z'}$ is contained in $\mathrm{Int}(W_{i,z})$, the map 6.4.17 is inner, by [?, 2.5.9]. The arguments in [?, 2.1.16] then show that 6.4.17 is a compact map of Banach spaces. Now choose $z'_1, \ldots, z'_{m_{i+1}}$ such that C is contained in the union of the $\mathrm{Int}(W_{i+1,z'_j})$, for $1 \le j \le m_{i+1}$, and let V_{i+1} be the union of the W_{i+1,z'_j}, for $1 \le j \le m_{i+1}$. Recall that

$$\mathcal{A}_{V_i} = \ker\left(\prod_{j=1}^{m_i} \mathcal{A}_{W_{i,z_j}} \rightrightarrows \prod_{j,j'} \mathcal{A}_{W_{i,z_j} \cap W_{i,z_{j'}}}\right) ,$$

cf. [?, 2.2.6, 3.3]. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{V_i} & \longrightarrow & \prod_{j=1}^{m_i} \mathcal{A}_{W_{i,z_j}} \\ \downarrow & & \downarrow \\ \mathcal{A}_{V_{i+1}} & \longrightarrow & \prod_{j=1}^{m_{i+1}} \mathcal{A}_{W_{i+1,z'_j}} \end{array}$$

The horizontal arrows are the obvious inclusions. The vertical arrow on the right is the one induced by the maps 6.4.17, and is thus injective and compact. The canonical vertical arrow on the left is thus injective and compact too, cf. [?, 16.7 (ii)]. This proves the first assertion.

(ii) This follows immediately from the second part of 6.4.4, together with 6.4.13.

(iii) Because $(V_i)_i$ is a fundamental system of neighborhoods of C we have, by 6.3.2, $\mathcal{O}_{\mathscr{B}}(C) = \varinjlim_i \mathcal{A}_{V_i}$. The assertion now follows from (i) and the main result of section 13.

(iv) Using (ii) we see that the right hand side of 6.4.16 has a canonical K-algebra structure. The remaining assertions are now clear. □

7. A sheaf of 'distribution operators' on the building

In this section we assume throughout $L = \mathbb{Q}_p$, $e > e_{st}$ (cf. 6.2.6), and $r \in [r_0, 1)$. Because $e - 1 \ge e_{st} \ge e_{uni}$, cf. 6.2.6, all groups $U_z^{(e-1)}$ are uniform pro-p groups, cf. 4.3.4. We will work from now on exclusively over the coefficient field K. To ease notation we will therefore drop this coefficient field from the notation when working with distribution algebras. We thus write $D(G) = D(G,K)$, $D_r(U_F^{(e)}) = D_r(U_F^{(e)},K)$ etc.

Recall that the sheaf of (twisted) differential operators $\mathcal{D}_\chi$ on X may be constructed from the skew tensor product $\mathcal{O}_X\#U(\mathfrak{g})$, cf. sec. 5. In a similar way we are going to construct a sheaf of 'distribution operators' on $\mathscr{B}$ starting from a twisted tensor product $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$. Here, $\underline{D}_r$ replaces the constant sheaf $\underline{U(\mathfrak{g})}$ and equals a sheaf of distribution algebras on $\mathscr{B}$. It will be a constructible sheaf with respect to the usual polysimplicial structure of $\mathscr{B}$. Recall from sec. 4.2 that a sheaf on a polysimplicial space is called *constructible* if its restriction to a given geometric polysimplex is a constant sheaf.

7.1. **A constructible sheaf of distribution algebras.** Given an open set $\Omega \subset \mathscr{B}$ we have the natural map

$$\iota_z : \mathcal{O}_{\mathscr{B}}(\Omega) \longrightarrow \mathcal{O}_{\mathscr{B},z},\ f \mapsto \text{germ of } f \text{ at } z$$

for any $z \in \Omega$.

Definition 7.1.1. For an open subset $\Omega \subseteq \mathscr{B}$ let

$$\underline{D}_r(\Omega) := K\text{−vector space of all maps } s : \Omega \to \dot{\bigcup}_{z\in\Omega} D_r(U_z^{(e)}) \text{ s.t.}$$

(1) $s(z) \in D_r(U_z^{(e)})$ for all $z \in \Omega$,
(2) for each facet $F \subseteq \mathscr{B}$ there exists a finite open covering $\Omega \cap St(F) = \cup_{i\in I}\Omega_i$ with the property: for each i with $\Omega_i \cap F \neq \emptyset$ there is an element $s_i \in D_r(U_F^{(e)})$ such that
  (2a) $s(z) = s_i$ for any $z \in \Omega_i \cap F$,
  (2b) $s(z') = \sigma_r^{FF'}(s_i)$ for any $z' \in \Omega_i$. Here, F' is the unique facet in $St(F)$ that contains z'.

¿From (2a) it is easy to see that the restriction of $\underline{D}_r$ to a facet F is the constant sheaf with value $D_r(U_F^{(e)})$. Hence $\underline{D}_r$ is constructible. Furthermore, if $\Omega' \subseteq \Omega$ is an open subset there is the obvious restriction map $\underline{D}_r(\Omega) \to \underline{D}_r(\Omega')$. The proof of the following result is implicitly contained in the proofs of 7.2.2 and 7.2.3 below.

Lemma 7.1.2. *With pointwise multiplication $\underline{D}_r$ is a sheaf of K-algebras. For $z \in \mathscr{B}$ one has $(\underline{D}_r)_z = D_r(U_z^{(e)})$.*

7.2. **Sheaves of completed skew group rings.**

Definition 7.2.1. For an open subset $\Omega \subseteq \mathscr{B}$ let

$$(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega) := K\text{−vector space of all maps } s : \Omega \to \dot{\bigcup}_{z\in\Omega} \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}) \text{ such that}$$

(1) $s(z) \in \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)})$ for all $z \in \Omega$,

(2) for each facet $F \subseteq \mathscr{B}$ there exists an open covering $\Omega \cap St(F) = \cup_{i\in I}\Omega_i$ with the property: for each i with $\Omega_i \cap F \neq \emptyset$ there is an element

$$s_i \in \mathcal{O}_{\mathscr{B}}(\Omega_i)\hat{\otimes}_L D_r(U_F^{(e)})$$

such that

(2a) $s(z) = (\iota_z \hat{\otimes} \mathrm{id})(s_i)$ for any $z \in \Omega_i \cap F$,

(2b) $s(z') = (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s_i)$ for any $z' \in \Omega_i$. Here, F' is the unique facet in $St(F)$ that contains z'.

Consider a map $s : \Omega \to \dot{\cup}_{z\in\Omega}\mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)})$ satisfying (1). It will be convenient to call an open covering $\Omega \cap St(F) = \cup_{i\in I}\Omega_i$ together with the elements s_i such that ($2a$) and ($2b$) hold a *datum* for s with respect to the facet F. Any open covering of $\Omega \cap St(F)$ which is a refinement of the covering $\{\Omega_i\}_{i\in I}$, together with the same set of elements s_i is again a datum for s with respect to F.

Suppose $\Omega' \subseteq \Omega$ is an open subset and let $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. Let $F \subseteq \mathscr{B}$ be a facet. Given a corresponding datum $\{\Omega_i\}_{i\in I}$ for s put $\Omega'_i := \Omega' \cap \Omega_i$. Together with the elements s_i , in case $\Omega'_i \cap F \neq \emptyset$, we obtain a datum for the function $s|_{\Omega'}$. It follows that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is a presheaf of K-vector spaces on $\mathscr{B}$.

In the following it will be convenient to define $\mathcal{F}(\Omega)$ as the K-vector space of all maps

$$s : \Omega \to \dot{\bigcup}_{z\in\Omega}\mathcal{O}_{\mathscr{B},z}\#_L D_r(U_z^{(e)})$$

satisfying condition (1) in the above definition. It is clear that pointwise multiplication makes $\mathcal{F}$ a sheaf of K-algebras on $\mathscr{B}$ such that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is a subpresheaf of K-vector spaces.

Lemma 7.2.2. *The induced multiplication makes $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r) \subseteq \mathcal{F}$ an inclusion of sheaves of K-algebras.*

Proof. Consider an open subset $\Omega \subseteq \mathscr{B}$. Let us first show that for $s, s' \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ we have $ss' \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$, i.e. that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ is a subalgebra of $\mathcal{F}(\Omega)$.
To do this let $F \subseteq \mathscr{B}$ be a facet. Let $\{\Omega_i\}_{i\in I}$ and $\{\Omega'_j\}_{j\in J}$ be corresponding data for s and s' respectively. Passing to $\{\Omega_{ij}\}_{ij}$ with $\Omega_{ij} = \Omega_i \cap \Omega'_j$ and refining the coverings if necessary, we may assume: there exists one datum $\{\Omega_i\}_{i\in I}$ for both s, s' and each Ω_i is contained in a compact subset of $St(F)$. We will produce a datum for ss' by passing to a suitable open covering of Ω_i whenever $\Omega_i \cap F \neq \emptyset$. To this end, let us fix such an $i \in I$. We choose connected compact subsets $C \subset St(F)$ whose open interiors C° form a covering of Ω_i. We have the K-algebra $\mathcal{O}_{\mathscr{B}}(C)\#D_r(U_F^{(e)})$ from 6.4.15. We apply the base change $(\cdot)\hat{\otimes}_L D_r(U_F^{(e)})$ to the restriction map $\mathcal{O}_{\mathscr{B}}(\Omega) \to \mathcal{O}_{\mathscr{B}}(C)$ and consider the image of s_i and s'_i in $\mathcal{O}_{\mathscr{B}}(C)\#D_r(U_F^{(e)})$. Let $s_i s'_i \in \mathcal{O}_{\mathscr{B}}(C)\#D_r(U_F^{(e)})$ be their product. We apply the base change $(\cdot)\hat{\otimes}_L D_r(U_F^{(e)})$ to the restriction map $\mathcal{O}_{\mathscr{B}}(C) \to \mathcal{O}_{\mathscr{B}}(C^\circ)$ and consider the

image of $s_i s'_i$ in $\mathcal{O}_{\mathscr{B}}(C^\circ)\hat{\otimes}_L D_r(U_F^{(e)})$. We denote this image again by $s_i s'_i$. According to the definition of the product on $\mathcal{O}_{\mathscr{B}}(C)\# D_r(U_F^{(e)})$ in 6.4.15 we find, for any $z \in C^\circ \cap F$, that

$$(ss')(z) = s(z)s'(z) = (\iota_z\hat{\otimes}\mathrm{id})(s_i)\cdot(\iota_z\hat{\otimes}\mathrm{id})(s'_i) = (\iota_z\hat{\otimes}\mathrm{id})(s_i s'_i)$$

using 6.4.10 and we find, for any $z' \in C^\circ$, that

$$(ss')(z') = s(z')s'(z') = (\iota_z\hat{\otimes}\sigma_r^{F'F})(s_i)\cdot(\iota_z\hat{\otimes}\sigma_r^{F'F})(s'_i) = (\iota_z\hat{\otimes}\sigma_r^{F'F})(s_i s'_i)$$

using 6.4.13 (F' is the unique facet in $St(F)$ that contains z'). This shows that, if we replace each such Ω_i by the open covering given by the corresponding C° and invoke the corresponding sections $s_i s'_i \in \mathcal{O}_{\mathscr{B}}(C^\circ)\hat{\otimes}_L D_r(U_F^{(e)})$, we will have a datum for ss' relative to F. Consequently, $ss' \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ and hence, $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ is a subalgebra of $\mathcal{F}(\Omega)$. If $\Omega' \subseteq \Omega$ is an open subset the restriction map $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega) \to (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega')$ is obviously multiplicative. Thus, $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is a presheaf of K-algebras.
Let us show that $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ is in fact a sheaf. Since $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r) \subseteq \mathcal{F}$ is a subpresheaf and $\mathcal{F}$ is a sheaf it suffices to prove the following: if

$$\Omega = \bigcup_{j\in J} U_j$$

is an open covering of an open subset $\Omega \subseteq \mathscr{B}$ and if $s_j \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(U_j)$ are local sections with $s_j|_{U_j\cap U_i} = s_i|_{U_i\cap U_j}$ for all $i,j \in J$ then the unique section $s \in \mathcal{F}(\Omega)$ with $s|_{U_j} = s_j$ for all $j \in J$ lies in $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. To do this let $F \subseteq \mathscr{B}$ be a facet. Consider for each $j \in J$ a datum $\{U_{ji}\}_{i\in I}$ for s_j. In particular, $U_j \cap St(F) = \cup_{i\in I} U_{ji}$ and there are distinguished elements

$$s_{ji} \in \mathcal{O}_{\mathscr{B}}(U_{ji})\hat{\otimes}_L D_r(U_F^{(e)})$$

whenever $U_{ji} \cap F \neq \emptyset$. Then $\Omega \cap St(F) = \cup_{ji} U_{ji}$ (together with the elements s_{ji} whenever $U_{ji}\cap F \neq \emptyset$) is a datum for s. Indeed, given $z \in U_{ji}\cap F$ one has $s(z) = s_j(z) = (\iota_z\hat{\otimes}\mathrm{id})(s_{ji})$ which shows $(2a)$. Moreover, if $z' \in U_{ji}$ one has $s(z') = s_j(z') = (\iota_z\hat{\otimes}\sigma_r^{F'F})(s_{ji})$ which shows $(2b)$. Together this means $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. □

The next lemma shows that the stalks of the sheaf $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$ are as expected.

Lemma 7.2.3. *One has* $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z = \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)})$ *for any* $z \in \mathscr{B}$.

Proof. There is the K-algebra homomorphism

$$(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z \longrightarrow \mathcal{O}_{\mathscr{B},z}\# D_r(U_z^{(e)}),\quad \text{germ of } s \text{ at } z \mapsto s(z)\ .$$

Let us show that this map is injective. Let $[s]$ be the germ of a local section $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ over some open subset $\Omega \subseteq \mathscr{B}$ with the property $s(z) = 0$. Let F be the unique facet of $\mathscr{B}$ that contains z and let $\{\Omega_i\}_{i\in I}$ be a corresponding datum for s. According to 6.2.4 we may write the stalk

$$\mathcal{O}_{\mathscr{B},z} = \varinjlim_{V} \mathcal{A}_V$$

as a compact inductive limit of integral affinoid algebras with injective transition maps. Let us abbreviate $\mathcal{E} := D_r(U_F^{(e)})$. If $W \subseteq V$ is an inclusion of affinoids occuring in the above inductive limit, then [**?**, Cor. 1.1.27] implies that the base changed map

$$\mathcal{A}_V \hat{\otimes}_L \mathcal{E} \to \mathcal{A}_W \hat{\otimes}_L \mathcal{E}$$

remains injective. Let $i_0 \in I$ such that $z \in \Omega_{i_0} \cap F$ and consider the map

$$\iota_z \hat{\otimes} \mathrm{id} : \mathcal{O}_{\mathscr{B}}(\Omega_{i_0}) \hat{\otimes}_L \mathcal{E} \longrightarrow \mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L \mathcal{E} = \varinjlim_{V} (\mathcal{A}_V \hat{\otimes}_L \mathcal{E}) \ .$$

Let V be an affinoid in the inductive limit on the right-hand side such that $\mathcal{A}_V \hat{\otimes}_L \mathcal{E}$ contains the image of s_{i_0} under $\iota_z \hat{\otimes} \mathrm{id}$. Choose an open subset $U \subseteq X^{an}$ in V containing z and replace Ω_{i_0} by the intersection $\mathscr{B} \cap U$. Then replace s_{i_0} by its restriction to this intersection, in other words, s_{i_0} lies now in the image of the map

$$\mathcal{A}_V \hat{\otimes}_L \mathcal{E} \longrightarrow \mathcal{O}_{\mathscr{B}}(\Omega_{i_0}) \hat{\otimes}_L \mathcal{E} \ .$$

By our discussion above, the natural map from $\mathcal{A}_V \hat{\otimes}_L \mathcal{E}$ into $\mathcal{O}_{\mathscr{B},z} \hat{\otimes}_L \mathcal{E}$ is injective and lifts the map $\iota_z \hat{\otimes} \mathrm{id}$. We may therefore deduce from

$$0 = s(z) = (\iota_z \hat{\otimes} \mathrm{id})(s_{i_0})$$

that $s_{i_0} = 0$. Given $z' \in \Omega_{i_0}$ let F' be the unique facet of $St(F)$ containing z'. Then $s(z') = (\iota_z \hat{\otimes} \sigma_r^{F'F})(s_{i_0}) = 0$ according to ($2b$) and, consequently, $s|_{\Omega_{i_0}} = 0$. Since Ω_{i_0} is an open neighborhood of z this shows $[s] = 0$ and proves injectivity.

Let us now show that our map in question is surjective. Let $t \in \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ be an element in the target. Since the stalk $\mathcal{O}_{\mathscr{B},z}$ is an inductive limit with compact and injective transition maps and since $D_r(U_z^{(e)})$ is a Banach space, the main result of section 13 implies that there is an open neighborhood Ω' of z and an element $\tilde{s} \in \mathcal{O}_{\mathscr{B}}(\Omega') \hat{\otimes}_L D_r(U_z^{(e)})$ such that $(\iota_z \hat{\otimes} \mathrm{id})(\tilde{s}) = t$. Let $F \subseteq \mathscr{B}$ be a facet containing z and define

$$\Omega := \Omega' \cap St(F), \ \ s := (res^{W}_{\Omega} \hat{\otimes} \mathrm{id})(\tilde{s}) \in \mathcal{O}_{\mathscr{B}}(\Omega) \hat{\otimes}_L D_r(U_z^{(e)}) \ .$$

Since $St(F)$ is an open neighborhood of z and contains only finitely many facets of $\mathscr{B}$ we may pass to a smaller Ω' (and hence Ω) and therefore assume: any $F' \in (\mathscr{B} \setminus St(F))$ satisfies $F' \cap \Omega = \emptyset$. For any $z' \in \Omega$ let $s(z') := (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s)$ where F' denotes the facet in $St(F)$ containing z'. This defines a function

$$s : \Omega \to \dot{\bigcup}_{z' \in \Omega} \mathcal{O}_{\mathscr{B},z} \# D_r(U_{z'}^{(e)})$$

satisfying condition (1) of definition 7.2.1. According to 4.3.5 one has $\sigma_r^{FF} = \mathrm{id}$ whence

$$s(z) = (\iota_z \hat{\otimes} \mathrm{id})(s) = (\iota_z \hat{\otimes} \mathrm{id})(\tilde{s}) = t\ .$$

Thus, the germ of s at z will be a preimage of t once we have shown that $s \in (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)(\Omega)$. To do this consider an arbitrary facet $F' \subset \mathscr{B}$ together with the covering of $\Omega \cap St(F')$ consisting of the single element

$$\Omega_0 := \Omega \cap St(F')\ .$$

Suppose $\Omega_0 \cap F' \neq \emptyset$. We have to exhibit an element $s_0 \in \mathcal{O}_{\mathscr{B}}(\Omega_0) \hat{\otimes}_L D_r(U_{F'}^{(e)})$ satisfying (2a) and (2b). Since $F' \in St(F)$ we may define $s_0 := (\mathrm{id} \hat{\otimes} \sigma_r^{FF'})(s)$. For any $z' \in \Omega \cap F'$ we compute

$$s(z') = (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s) = (\iota_{z'} \hat{\otimes} \mathrm{id})(\mathrm{id} \hat{\otimes} \sigma_r^{FF'})(s) = (\iota_{z'} \hat{\otimes} \mathrm{id})(s_0)$$

which shows (2a). Moreover, for any $z' \in \Omega_0$ we compute

$$s(z') = (\iota_{z'} \hat{\otimes} \sigma_r^{FF''})(s) = (\iota_{z'} \hat{\otimes} \sigma_r^{F'F''})\ (\iota_{z'} \hat{\otimes} \sigma_r^{FF'})(s) = (\iota_{z'} \hat{\otimes} \sigma_r^{F'F''})(s_0)$$

by 4.3.5. Here F'' denote the facet of $St(F')$ that contains z'. This shows (2b) and completes the proof. □

Corollary 7.2.4. *The $\mathcal{O}_{\mathscr{B},z}$-module structure on $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z$ for any $z \in \mathscr{B}$ sheafifies to a $\mathcal{O}_{\mathscr{B}}$-module structure on $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$ (compatible with scalar multiplication by L).*

Proof. As with any sheaf ([?, II.1.2]) we may regard $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$ as the sheaf of *continuous* sections of its étale space

$$\begin{array}{c} \dot{\bigcup}_{z \in \mathscr{B}} (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z \\ \big\downarrow \\ \mathscr{B} \end{array}$$

and the same applies to the sheaf $\mathcal{O}_{\mathscr{B}}$. By the preceding proposition we have $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z = \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})$ for any $z \in \mathscr{B}$. Let $\Omega \subseteq \mathscr{B}$ be an open subset, $s \in (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$, $f \in \mathcal{O}_{\mathscr{B}}(\Omega)$. For $z \in \Omega$ put $(f \cdot s)(z) := f(z) \cdot s(z)$. This visibly defines an element $f \cdot s \in \mathcal{F}(\Omega)$. The '$\mathcal{O}_{\mathscr{B}}$-linearity' in the conditions (2a) and (2b) proves $f \cdot s \in (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$. It follows that $(\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)$ is an $\mathcal{O}_{\mathscr{B}}$-module in the prescribed way. □

Proposition 7.2.5. *The natural map*

$$D_r(U_z^{(e)}) \to (\mathcal{O}_{\mathscr{B}} \# \underline{D}_r)_z,\ \ \delta \mapsto 1 \hat{\otimes} \delta\ ,$$

sheafifies to a morphism of sheaves of K-algebras $\underline{D}_r \longrightarrow \mathcal{O}_{\mathscr{B}} \# \underline{D}_r$.

Proof. This is easy to see. □

Recall (6.4.12) that we have for any $z \in \mathscr{B}$ a canonical K-algebra homomorphism

$$\mathcal{O}_{\mathscr{B},z}\#U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)}) \ .$$

Proposition 7.2.6. *The homomorphisms (6.4.12) sheafify into a morphism*

$$\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})_K \longrightarrow \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$$

of sheaves of K-algebras. This morphism is $\mathcal{O}_{\mathscr{B}}$-linear.

Proof. We view $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})_K$ as the sheaf of continuous sections of its étale space

$$\begin{array}{c} \dot{\bigcup}_{z\in\mathscr{B}}\mathcal{O}_{\mathscr{B},z}\#_L U(\mathfrak{g})_K \\ \downarrow \\ \mathscr{B} \end{array}$$

Composing such a section with (6.4.12) defines a morphism $i : \mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})_K \longrightarrow \mathcal{F}$ of sheaves of K-algebras and we will prove that its image lies in the subsheaf $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)$. To do this let $\Omega \subseteq \mathscr{B}$ be an open subset and $s \in \mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})_K(\Omega)$ a local section.
Let $F \subseteq \mathscr{B}$ be a facet. Consider the covering of $\Omega \cap St(F)$ consisting of the single element $\Omega_0 := \Omega \cap St(F)$. In case $\Omega_0 \cap F \neq \emptyset$ let s_0 be the image of $\tilde{s}$ under the map

$$\mathcal{O}_{\mathscr{B}}(\Omega) \otimes_L U(\mathfrak{g})_K \longrightarrow \mathcal{O}_{\mathscr{B}}(\Omega)\hat{\otimes}_L D_r(U_F^{(e)})$$

induced by $U(\mathfrak{g})_K \subseteq D_r(U_F^{(e)})$. For any $z \in \Omega_0 \cap F$ we obviously have $i(s)(z) = (\iota_z\hat{\otimes}\mathrm{id})(s_0)$ which shows $(2a)$. For any $z \in \Omega_0$ we find $i(s)(z) = (\iota_z\hat{\otimes}\mathrm{id})(s_0) = (\iota_z\hat{\otimes}\sigma_r^{FF'})(s_0)$ by the last statement of 4.3.5. Here F' denotes the facet containing z. This shows $(2b)$. In the light of the definitions it is clear that the resulting morphism $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g})_K \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ is $\mathcal{O}_{\mathscr{B}}$-linear. ☐

7.3. **Infinitesimal characters.** In the following we write $\mathfrak{g}_K := \mathfrak{g} \otimes_{\mathbb{Q}_p} K, \mathfrak{t}_K := \mathfrak{t} \otimes_{\mathbb{Q}_p} K$ etc.

7.3.1. According to [?, Prop. 3.7] the ring $Z(\mathfrak{g}_K)$ lies in the center of the ring $D(G)$. In the following we fix a central character

$$\theta : Z(\mathfrak{g}_K) \longrightarrow K$$

and we let

$$D(G)_\theta := D(G) \otimes_{Z(\mathfrak{g}_K),\theta} K$$

be the corresponding central reduction of $D(G)$. A (left) $D(G)_\theta$-module M is called *coadmissible* if it is coadmissible as $D(G)$-module via the natural map $D(G) \to D(G)_\theta, \delta \mapsto$

$\delta\hat{\otimes}1$. In the following we are going to study the abelian category of coadmissible $D(G)_\theta$-modules. As explained in the beginning this category is anti-equivalent to the category of admissible locally analytic G-representations over K which have infinitesimal character θ.

Example. Let $\lambda_0 : D(T) \longrightarrow K$ denote the character of $D(T)$ induced by the augmentation map $K[T] \to K$. The restriction of λ_0 to the Lie algebra $\mathfrak{t}_K \subset D(T)$ vanishes identically whence $\chi = \rho$. Let $\theta_0 : Z(\mathfrak{g}_K) \longrightarrow K$ be the infinitesimal character associated to ρ via the Harish-Chandra homomorphism. Then $\ker \theta_0 = Z(\mathfrak{g}_K) \cap U(\mathfrak{g}_K)\mathfrak{g}_K$.

Remark. In [**?**, §8] K. Ardakov and S. Wadsley establish a version of Quillen's lemma for p-adically completed universal enveloping algebras. It implies that any topologically irreducible admissible locally analytic G-representation admits, up to a finite extension of K, a central character and an infinitesimal character ([**?**]).

7.3.2. To investigate the local situation let F be a facet in $\mathscr{B}$. We have

$$Z(\mathfrak{g}_K) \subseteq D(U_F^{(e)}) \cap Z(D(G)) \subseteq Z(D(U_F^{(e)})) \;,$$

again according to [**?**, Prop. 3.7]. We let $D_r(U_F^{(e)})_\theta := D_r(U_F^{(e)}) \otimes_{Z(\mathfrak{g})_K,\theta} K$ be the corresponding central reduction of $D_r(U_F^{(e)})$.

Let F, F' be two facets in $\mathscr{B}$ such that $F' \subseteq \overline{F}$ and consider the homomorphism $\sigma_r^{F'F}$. According to the last statement of 4.3.5 it factors by continuity into a homomorphism

$$\sigma_r^{F'F} : D_r(U_{F'}^{(e)})_\theta \longrightarrow D_r(U_F^{(e)})_\theta \;.$$

We may therefore define a sheaf of K-algebras $\underline{D}_{r,\theta}$ in complete analogy with the sheaf $\underline{D}_r$ by replacing each $D_r(U_z^{(e)})$ and each $D_r(U_F^{(e)})$ by their central reductions. In particular, $(\underline{D}_{r,\theta})_z = D_r(U_z^{(e)})_\theta$ for any $z \in \mathscr{B}$ and there is an obvious quotient morphism

$$\underline{D}_r \longrightarrow \underline{D}_{r,\theta} \;.$$

7.4. **Twisting.** We now bring in a toral character

$$\chi : \mathfrak{t}_K \longrightarrow K$$

such that $\sigma(\chi) = \theta$. We consider the two-sided ideals $\mathcal{I}_{\mathscr{B},\mathfrak{t}}^{an}$ and $\mathcal{I}_{\mathscr{B},\chi}^{an}$ of $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K)$. Denote the right ideal in $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ generated by the image of the first resp. second under the morphism

$$\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K) \longrightarrow \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$$

by $\mathscr{I}_{\mathfrak{t}}^{an}$ resp. $\mathscr{I}_{\chi}^{an}$.

Proposition 7.4.1. *Let $z \in \mathscr{B}$ a point and $e > e_{uni}$, cf. 4.3.4. Let V be a strictly affinoid neighborhood of z on which $U_z^{(e-1)}$ acts analytically. Then the ring $\mathcal{A}_V\#D_{r_0}(U_z^{(e)})$ is noetherian.*

Proof. As $e > e_{uni}$ the group $U_z^{(e-1)}$ is a uniform pro-p group. Let

$$\mathfrak{h}_{\mathbb{Z}_p} = \mathcal{L}(U_z^{(e-1)}) \subset \mathfrak{g}$$

be the $\mathbb{Z}_p$-Lie algebra of $U_z^{(e-1)}$, cf. [?, sec. 9.4]. We consider the bijective exponential map $\exp : \mathfrak{h}_{\mathbb{Z}_p} \to U_z^{(e-1)}$ which is used to define the affinoid analytic subgroup $\mathbb{U}_z^{(e-1)} \subset \mathbf{G}^{an}$, cf. 6.2.5. This exponential map gives then rise to an exponential map of affinoid analytic spaces $\exp : \mathbb{B} \otimes_{\mathbb{Z}_p} \mathfrak{h}_{\mathbb{Z}_p} \to \mathbb{U}_z^{(e-1)}$, where $\mathbb{B}$ is the closed unit disc over $\mathbb{Q}_p$ and $\mathbb{B} \otimes_{\mathbb{Z}_p} \mathfrak{h}_{\mathbb{Z}_p}$ is the strictly $\mathbb{Q}_p$-analytic space whose affinoid algebra is

$$\mathrm{Sym}_{\mathbb{Z}_p}(\mathfrak{h}_{\mathbb{Z}_p}^\vee)^\wedge \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \ .$$

Here, $\mathfrak{h}_{\mathbb{Z}_p}^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(\mathfrak{h}_{\mathbb{Z}_p}, \mathbb{Z}_p)$ and $(\cdot)^\wedge$ means the p-adic completion.

The affinoid algebra $\mathcal{A}_V$ is a $\mathfrak{g}$-module. As a first step we want to show that the subring $A \subset \mathcal{A}_V$ of power-bounded elements is stable under the action of $\mathfrak{h}_{\mathbb{Z}_p}$. Because $U_z^{(e-1)}$ acts analytically on V we have for any $\mathfrak{x} \in \mathfrak{h}_{\mathbb{Z}_p}$ and $f \in A$

$$\exp(t\mathfrak{x}).f = \sum_{n \geq 0} \left(\frac{\mathfrak{x}^n}{n!}.f\right) t^n \ ,$$

where the right hand side is a convergent power series in $t \in \mathbb{B}$. If we evaluate this identity at a point $z' \in V$ we get

$$f\left(\exp(-t\mathfrak{x}).z'\right) = \sum_n \left(\frac{\mathfrak{x}^n}{n!}.f\right)(z')t^n \ , \tag{7.4.2}$$

which holds for all $t \in \mathbb{B}$. The left hand side of 7.4.2 is bounded by 1 in absolute value for all $t \in \mathbb{B}$, and so is the right hand side. But this means that all coefficients $(\frac{1}{n!}\mathfrak{x}^n.f)(z') \in \mathcal{A}_V$ on the right hand side of 7.4.2 must be bounded by one in absolute value, and, in particular, the coefficient $(\mathfrak{x}.f)(z')$. This shows that the supremum norm of $\mathfrak{x}.f$ on V is bounded by 1, i.e., that we have $\mathfrak{x}.f \in A$.

We let $U(\mathfrak{h}_{\mathbb{Z}_p})$ be the universal enveloping algebra over $\mathbb{Z}_p$ of $\mathfrak{h}_{\mathbb{Z}_p}$. As we have seen above, the ring A is a $U(\mathfrak{h}_{\mathbb{Z}_p})$-module, and we can consider the skew enveloping algebra $A\#U(\mathfrak{h}_{\mathbb{Z}_p}) := A \otimes_{\mathbb{Z}_p} U(\mathfrak{h}_{\mathbb{Z}_p})$. We denote its p-adic completion by

$$R_A := A\#\hat{U}(\mathfrak{h}_{\mathbb{Z}_p}) \ .$$

In a manner completely analogous to sec. 3, this becomes a p-adically complete topological $\mathbb{Z}_p$-algebra. Its mod p-reduction is equal to

$$gr_0(R_A) := \bar{A}\#U(\mathfrak{h}_{\mathbb{F}_p})$$

where $\bar{A} = A/pA$ and $\mathfrak{h}_{\mathbb{F}_p} := \mathfrak{h}_{\mathbb{Z}_p} \otimes \mathbb{F}_p$. The vector space underlying $gr_0(R_A)$ equals $\bar{A} \otimes_{\mathbb{F}_p} U(\mathfrak{h}_{\mathbb{F}_p})$. The second factor in this tensor product has its PBW-filtration. It induces a positive $\mathbb{Z}$-filtration on $gr_0(R_A)$ with $\bar{A}$ concentrated in degree zero. Let deg be the degree function of this filtration. If $f \in \bar{A}, \mathfrak{x} \in \mathfrak{h}_{\mathbb{F}_p}$ we have $[f, \mathfrak{x}] = \mathfrak{x}(f)$ from which it follows that $gr_0(R_A)$ is a $\mathbb{Z}$-filtered ring. Moreover,

$$\deg\, [f, \mathfrak{x}] < \deg\, \mathfrak{x}$$

which means that the associated graded ring

$$\mathrm{Gr}(R_A) := gr\ gr_0(R_A) = \bar{A} \otimes_{\mathbb{F}_p} S(\mathfrak{h}_{\mathbb{F}_p})$$

is commutative and therefore a polynomial ring over $\bar{A}$. Since $\bar{A}$ is noetherian, so is $\mathrm{Gr}(R_A)$. By [?, Prop. 1.1] the ring $gr_0(R_A)$ is noetherian. Now R_A is complete with respect to the p-adic topology and the graded ring associated with the p-adic filtration equals

$$gr(R_A) = (gr_0 R_A)[Z, Z^{-1}]\ ,$$

the Laurent polynomials over $gr_0(R_A)$ in one variable Z (e.g. [?, Lem. 3.1]). It is noetherian, since $gr_0(R_A)$ is noetherian. Another application of [?, Prop. 1.2] now yields that R_A is noetherian. The embedding $\mathfrak{h}_{\mathbb{Z}_p} \subset \mathfrak{g} \subset D(U_z^{(e)})$ induces a ring isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{U}(\mathfrak{h}_{\mathbb{Z}_p}) \xrightarrow{\cong} D_{r_0}(U_z^{(e)}, \mathbb{Q}_p)\ ,$$

cf. [?, Prop. 6.3]. We have $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A = \mathcal{A}_V$ and thus a ring isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} R_A \simeq \mathcal{A}_V \# D_{r_0}(U_z^{(e)}, \mathbb{Q}_p)\ .$$

Therefore, the right hand side is noetherian. Base change from $\mathbb{Q}_p$ to K finally yields the assertion of the lemma. □

Lemma 7.4.3. *For all $m \geq 0$ the inclusion $U_z^{(e+m)} \subseteq U_z^{(e)}$ induces a finite free ring homomorphism*

$$D_{r_0}(U_z^{(e+m)}) \longrightarrow D_{r_m}(U_z^{(e)}) \tag{7.4.4}$$

which is an isometry between Banach algebras. A basis for this free extension is given by any choice of system of coset representatives for the finite group $U_z^{(e)}/U_z^{(e+m)}$.

Proof. Since $e \geq e_{uni}$ each group $U_z^{(e)}$ is a uniform pro-p group with lower p-series given by the subgroups $U_z^{(e+m)}$ for $m \geq 0$. The claim follows therefore from the discussion at the end of subsection 2.2. □

Keep the assumptions of the preceding proposition and lemma.

Lemma 7.4.5. *The rings* $\mathcal{A}_V\#D_{r_0}(U_z^{(e+m)})$ *and* $\mathcal{A}_V\#D_{r_m}(U_z^{(e)})$ *are noetherian.*

Proof. We abbreviate

$$\mathcal{E}_V^m := \mathcal{A}_V\#D_{r_0}(U_z^{(e+m)}) \text{ and } \mathcal{E}_{m,V} := \mathcal{A}_V\#D_{r_m}(U_z^{(e)}).$$

The ring homomorphism (7.4.4) induces a ring homomorphism $\phi : \mathcal{E}_V^m \to \mathcal{E}_{m,V}$.
The bijectivity of the natural map

$$\mathcal{E}_V^m \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)}) \longrightarrow \mathcal{E}_{m,V}, f\otimes h \mapsto \phi(f)\cdot h$$

of $(\mathcal{E}_V^m, D_{r_m}(U_z^{(e)})$-bimodules can be checked on the level of vector spaces. It follows there from functoriality of $\mathcal{A}_V\hat{\otimes}_L(.)$ applied to the obvious bijective linear map

$$D_{r_0}(U_z^{(e+m)}) \otimes_{D_{r_0}(U_z^{(e+m)})} D_{r_m}(U_z^{(e)}) \xrightarrow{\cong} D_{r_m}(U_z^{(e)}) \ .$$

Using 7.4.3 we conclude that $\mathcal{E}_{m,V}$ is a finite free left $\mathcal{E}_V^m$-module. By 7.4.1, the ring $\mathcal{E}_V^m$ is noetherian and thus, $\mathcal{E}_{m,V}$ is left noetherian. A similar argument proves that $\mathcal{E}_{m,V}$ is right noetherian.

□

Corollary 7.4.6. *Let* $r = r_m$ *for some* $m \geq 0$. *The ideals* $\mathscr{I}_{\mathfrak{t},z}^{an}$ *and* $\mathscr{I}_{\chi,z}^{an}$ *are closed in* $\mathcal{O}_{\mathscr{B},z}\#D_r(U_z^{(e)})$.

Proof. Let V be a strictly affinoid neighborhood of z in X^{an} on which $U_z^{(e-1)}$ acts analytically, and let $\mathcal{A}_V$ be its affinoid algebra.
Consider the subsheaves $\mathfrak{n}^{\circ,an}$ and $\ker\lambda^{\circ,an}$ of the sheaf $\mathcal{O}_{X^{an}}\#U(\mathfrak{g})$ on X^{an}. Let $\mathcal{K}_V$ reps. $\mathcal{K}_z$ be the vector space of sections[7] over the affinoid $V \subset X^{an}$ resp. the stalk at $z \in V$ of one of these subsheaves respectively. Abbreviating $\mathcal{E}_V := \mathcal{A}_V\#D_{r_m}(U_z^{(e)})$ and $\mathcal{E}_z := \mathcal{O}_{\mathscr{B},z}\#D_{r_m}(U_z^{(e)})$, the assertion amounts to show that $\mathcal{K}_z\mathcal{E}_z \subseteq \mathcal{E}_z$ is closed. Consider the diagram of continuous K-linear maps

$$0 \longrightarrow \varinjlim_V \mathcal{K}_V\mathcal{E}_V \xrightarrow{\iota} \varinjlim_V \mathcal{E}_V \longrightarrow \varinjlim_V \mathcal{E}_V/\mathcal{K}_V\mathcal{E}_V \longrightarrow 0$$

which is short exact as a diagram of abstract K-vector spaces. The right-hand term is Hausdorff (why?) and the injection ι has therefore closed image. The main result of section 13 provides a (topological) isomorphism $\varinjlim_V \mathcal{E}_V \simeq \mathcal{E}_z$ which identifies the image of ι with $\mathcal{K}_z\mathcal{E}_z$. □

Lemma 7.4.7. *Let* $r = r_m$ *for some* $m \geq 0$. *The right ideals* $\mathscr{I}_{\mathfrak{t}}^{an}$ *and* $\mathscr{I}_{\chi}^{an}$ *are two-sided ideals.*

[7]Of course, V is not an open subset of the topological space X^{an}. However, all sheaves in fact extend to sheaves with respect to the Grothendieck topology on the analytic space X^{an}. This technical point is of minor importance.

Proof. According to section 8.3 the sheaves $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K)$ and $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ have a natural G-equivariant structure such that the morphism $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ is equivariant. Moreover, the ideals $\mathcal{I}^{an}_{\mathscr{B},\mathfrak{t}}$ and $\mathcal{I}^{an}_{\mathscr{B},\chi}$ are G-stable. Hence, so are the right ideals $\mathscr{I}^{an}_{\mathfrak{t}}$ and $\mathscr{I}^{an}_{\chi}$. That these ideals are two-sided can be checked stalkwise. We give the argument in the case $\mathscr{I}^{an}_{\mathfrak{t}}$. The other case follows in the same way. By [**?**, Lem. 3.1] and the preceding corollary 7.4.6, it suffices to prove that the product $\delta_g \cdot \partial \in \mathcal{O}_{\mathscr{B},z}\#D_r(U^{(e)}_z)$ lies in the subspace $\mathscr{I}^{an}_{\mathfrak{t},z}$ for $g \in U^{(e)}_z$ and $\partial \in \mathscr{I}^{an}_{\mathfrak{t},z}$. Using the power series expansions for elements of completed distribution algebras (2.2.3) together with the main result of section 13 we may write ∂ as an infinite convergent sum $\partial = \sum_{\alpha\in\mathbb{N}_0^d} f_\alpha \hat{\otimes}\mathbf{b}^\alpha$ with $f_\alpha \in \mathcal{O}_{\mathscr{B},z}$. By definition of the skew multiplication (3.2.2) we have

$$\delta_g \cdot \partial = \sum_{\alpha\in\mathbb{N}_0^d} (g.f_\alpha)\hat{\otimes}\delta_g\mathbf{b}^\alpha = \sum_{\alpha\in\mathbb{N}_0^d} (g.f_\alpha)\hat{\otimes}(\delta_g\mathbf{b}^\alpha\delta_g^{-1})\delta_g = \sum_{\alpha\in\mathbb{N}_0^d} g^*(f_\alpha\hat{\otimes}\mathbf{b}^\alpha)\delta_g = g^*(\partial)\delta_g \in \mathscr{I}^{an}_{\mathfrak{t},z}$$

where $g^* : \mathscr{I}^{an}_{\mathfrak{t},z} \xrightarrow{\cong} \mathscr{I}^{an}_{\mathfrak{t},z}$ is induced bythe equivariant structure on the sheaf $\mathscr{I}^{an}_{\mathfrak{t}}$ (note that $U^{(e)}_z$ stabilizes the point z). □

In the following we tacitly restrict to numbers r of the form $r = r_m$ for some $m \geq 0$. By the preceding lemma we may form the quotient sheaves

$$\mathscr{D}_{r,\mathfrak{t}} := (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)/\mathscr{I}^{an}_{\mathfrak{t}}, \qquad \mathscr{D}_{r,\chi} := (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)/\mathscr{I}^{an}_{\chi}.$$

These are sheaves of (noncommutative) K-algebras on $\mathscr{B}$ and, at the same time, $\mathcal{O}_{\mathscr{B}}$-modules. We have a commutative diagram of morphisms

$$\text{(7.4.8)} \qquad \begin{array}{ccc} \mathcal{D}^{an}_{\mathscr{B},\mathfrak{t}} & \longrightarrow & \mathcal{D}^{an}_{\mathscr{B},\chi} \\ \downarrow & & \downarrow \\ \mathscr{D}_{r,\mathfrak{t}} & \longrightarrow & \mathscr{D}_{r,\chi} \end{array}$$

with surjective horizontal arrows. Moreover, it follows from (6.1.4) and the preceding lemma that the lower horizontal arrow induces an isomorphism

$$\text{(7.4.9)} \qquad \mathscr{D}_{r,\mathfrak{t}}/(\ker\lambda)\mathscr{D}_{r,\mathfrak{t}} \xrightarrow{\cong} \mathscr{D}_{r,\chi}\ .$$

We have the following extension of the property 2. in [**?**, §2, Lemme].

Proposition 7.4.10. *The morphism $\underline{D}_r \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r \to \mathscr{D}_{r,\chi}$ factors through $\underline{D}_r \to \underline{D}_{r,\theta}$.*

Proof. Letting $\mathscr{K}$ be the kernel of the morphism $\underline{D}_r \to \underline{D}_{r,\theta}$ the claim amounts to

$$\mathscr{K} \subseteq \ker(\underline{D}_r \to \mathscr{D}_{r,\chi})\ .$$

This can be checked stalkwise, i.e. we are reduced to show that, for each $z \in \mathscr{B}$ the natural map $D_r(U_z^{(e)}) \to \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)})/(\mathscr{I}_\chi)_z$ factors through $D_r(U_z^{(e)})_\theta$. The kernel of the map $D_r(U_z^{(e)}) \to D_r(U_z^{(e)})_\theta$ is generated by

$$I_\theta := \ker(U(\mathfrak{g}_K) \to U(\mathfrak{g}_K)_\theta)$$

and the ideal $(\mathscr{I}_\chi)_z$ is generated by the image of $\mathcal{I}^{an}_{\mathscr{B},\chi,z}$. It therefore suffices to show that the natural map $U(\mathfrak{g}_K) \to \mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g}_K)$ maps I_θ into $\mathcal{I}^{an}_{\mathscr{B},\chi,z}$. This follows from loc.cit. $\square$

7.4.11. Let us finally make the structure of the stalks of the sheaves $\mathscr{D}_{r,\mathfrak{t}}$ and $\mathscr{D}_{r,\chi}$ at a point $z \in \mathscr{B}$ more explicit. According to 6.2.2 the local ring $\mathcal{O}_{\mathscr{B},z}$ is a field. For simplicity we put $\kappa(z) := \mathcal{O}_{\mathscr{B},z}$ and view this as a topological field of compact type. Note that the Berkovich point $z \in \mathscr{B} \subset X^{an}$ canonically induces a norm topology on $\kappa(z)$ which is weaker than our topology. We shall not make use of this norm topology in the following.

By loc.cit. together with (5.1.1) we furthermore have $(\mathfrak{n}^\circ)_{\pi(z)} = \mathfrak{n}_{\pi(z)}$ and $(\mathfrak{b}^\circ)_{\pi(z)} = \mathfrak{b}_{\pi(z)}$ for the stalks of the sheaves $\mathfrak{n}^\circ$ and $\mathfrak{b}^\circ$ at $\pi(z) = \eta$ (the generic point of X). Since passage to the stalk is exact, this proves the following lemma.

Lemma 7.4.12. *Assume $r = r_m$ for some $m \geq 0$. Let $z \in \mathscr{B}$. There is a canonical isomorphism*

$$\mathscr{D}_{r,\mathfrak{t},z} \xrightarrow{\cong} (\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))\ .$$

This isomorphism induces a canonical isomorphism between $\mathscr{D}_{r,\chi,z}$ and the λ-coinvariants of the $\mathfrak{t}_K$-module $(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))$. In particular,

$$\mathscr{D}_{r,\rho,z} \xrightarrow{\cong} (\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{b}_{\pi(z)}(\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))\ .$$

8. From representations to sheaves

In this section, as well as in sections 10 and 11, we assume that

$$L = \mathbb{Q}_p \ \text{ and } \ e > e_{st} \ \text{ and } \ r = r_m = \sqrt[p^m]{1/p} \ \text{ for some } \ m \geq 0\ . \tag{8.0.1}$$

Our proposed 'localization functor' from representations to sheaves associated to the pair $\sigma(\chi) = \theta$ will be a functor

$$\mathscr{L}_{r,\chi} : M \mapsto \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

from (coadmissible) left $D(G)_\theta$-modules M to left $\mathscr{D}_{r,\chi}$-modules satisfying additional properties. Here $\underline{M}_r$ is a constructible sheaf replacing the constant sheaf $\underline{M}$ appearing in the Beilinson-Bernstein construction, cf. 5.2.2. It is a modest generalization of the sheaf $\underline{D}_r$ as follows.

8.1. **A constructible sheaf of modules.** Suppose we are given any (left) $D(G)$-module M. Let $F \subseteq \mathscr{B}$ be a facet. We may regard M as a $D(U_F^{(e)})$-module via the natural map $D(U_F^{(e)}) \to D(G)$. We put

$$M_r(U_F^{(e)}) := D_r(U_F^{(e)}) \otimes_{D(U_F^{(e)})} M \ ,$$

a (left) $D_r(U_F^{(e)})$-module. If $F' \subseteq \mathscr{B}$ is another facet such that $F' \subset \overline{F}$ the map

$$\sigma_r^{F'F} \otimes \mathrm{id} : M_r(U_{F'}^{(e)}) \longrightarrow M_r(U_F^{(e)}),\ \delta \otimes m \mapsto \sigma_r^{F'F}(\delta) \otimes m$$

is a module homomorphism relative to $\sigma_r^{F'F}$ and inherits the homomorphic properties from $\sigma_r^{F'F}$ (cf. 4.3.5). Again, we may define a sheaf of K-vector spaces $\underline{M}_r$ on $\mathscr{B}$ in a completely analogous way as the sheaf $\underline{D}_r$ by replacing each $D_r(U_F^{(e)})$ and each $D_r(U_z^{(e)})$ by $M_r(U_F^{(e)})$ and $M_r(U_z^{(e)})$ respectively. In particular, $\underline{M}_r$ restricted to a facet F is the constant sheaf with value $M_r(U_F^{(e)})$ and therefore $\underline{M}_r$ is a constructible sheaf. If $s \in D_r(U_z^{(e)}), m \in M_r(U_z^{(e)})$ the 'pointwise multiplication' $(s \cdot m)(z) := s(z)m(z)$ makes $\underline{M}_r$ a $\underline{D}_r$-module.

Lemma 8.1.1. *If M is a $D(G)_\theta$-module then $\underline{M}_r$ is a $\underline{D}_{r,\theta}$-module via the morphism $\underline{D}_r \to \underline{D}_{r,\theta}$.*

Proof. This is easy to see. □

8.2. **A localization functor.**

8.2.1. As usual, $\mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$ denotes the sheaf associated to the presheaf $V \mapsto \mathscr{D}_{r,\chi}(V) \otimes_{\underline{D}_{r,\theta}(V)} \underline{M}_r(V)$ on $\mathscr{B}$. The construction $M \mapsto \underline{M}_r$ is functorial in M and commutes with arbitrary direct sums. Thus the correspondence

$$\mathscr{L}_{r,\chi} : M \mapsto \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

is a covariant functor from (left) $D(G)_\theta$-modules to (left) $\mathscr{D}_{r,\chi}$-modules. It commutes with arbitrary direct sums. We call it tentatively a *localization functor* associated to χ.
We emphasize that the functor $\mathscr{L}_{r,\chi}$ depends on the choice of the level e. As we did before we suppress this dependence in the notation. As a second remark, let $\mathcal{M}$ be an arbitrary $\mathscr{D}_{r,\chi}$-module and $f : \mathscr{L}_{r,\chi}(M) \to \mathcal{M}$ a morphism. The composite

$$M \to \Gamma(\mathscr{B}, \underline{M}_r) \to \Gamma(\mathscr{B}, \mathscr{L}_{r,\chi}(M)) \xrightarrow{f} \Gamma(\mathscr{B}, \mathcal{M})$$

is a K-linear map. We therefore have a natural transformation of functors

$$\mathrm{Hom}_{\mathscr{D}_{r,\chi}}(\mathscr{L}_{r,\chi}(\cdot), .) \longrightarrow \mathrm{Hom}_K(\cdot, \Gamma(\mathscr{B}, \cdot)) \ .$$

Generally, it is far from being an equivalence.

We compute the stalks of the localization $\mathscr{L}_{r,\chi}(M)$ for a *coadmissible* module M. In this case $(\underline{M}_r)_z$ is finitely generated over the Banach algebra $D_r(U_z^{(e)})_\theta$ and therefore has a unique structure as a Banach module over $D_r(U_z^{(e)})_\theta$. Let $z \in \mathscr{B} \subset X^{an}$ with residue field $\kappa(z)$. Recall that $\pi(z)$ equals the generic point of X.

Proposition 8.2.2. *Let M be a coadmissible left $D(G)_\theta$-module and let $z \in \mathscr{B}$. The morphism $\underline{M}_r \to \mathscr{L}_{r,\chi}(M)$ induces an isomorphism between the λ-coinvariants of the $\mathfrak{t}_K$-module*

$$(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)$$

and the stalk $\mathscr{L}_{r,\chi}(M)_z$. In particular, if $\theta = \theta_0$ we have

$$(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z)/\mathfrak{b}_{\pi(z)}(\kappa(z)\hat{\otimes}_L(\underline{M}_r)_z) \xrightarrow{\cong} \mathscr{L}_{r,\rho}(M)_z \ .$$

Proof. Let N be an arbitrary finitely generated $D_r(U_z^{(e)})$-module. According to 7.4.12 the space $\mathscr{D}_{r,\mathfrak{t},z} \otimes_{D_r(U_z^{(e)})} N$ may be written as

$$\Big[\big((\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)}))/\mathfrak{n}_{\pi(z)}(\kappa(z))\hat{\otimes}_L D_r(U_z^{(e)})\big) \otimes_{\kappa(z)\hat{\otimes}_L D_r(U_z^{(e)})} \kappa(z)\hat{\otimes}_L D_r(U_z^{(e)})\Big] \otimes_{D_r(U_z^{(e)})} N \ .$$

Since N is a complete Banach module this may be identified with

$$(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N)$$

by associativity of the completed tensor product. The resulting isomorphism

$$(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N) \xrightarrow{\cong} \mathscr{D}_{r,\mathfrak{t},z} \otimes_{D_r(U_z^{(e)})} N$$

is functorial in N. According to the second part of loc.cit. we obtain a functorial homomorphism

$$(\lambda - \text{coinvariants of } (\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N)) \longrightarrow \mathscr{D}_{r,\chi,z} \otimes_{D_r(U_z^{(e)})} N$$

which is an isomorphism in the case $N = D_r(U_z^{(e)})$. Note that the target is a right exact functor in N. Similarly, the source is also a right exact functor in N. To see this, it suffices to note that the functor which sends N to $\kappa(z)\hat{\otimes}_L N$ is exact. Indeed, any short exact sequence of finitely generated $D_r(U_z^{(e)})$-modules is a strict exact sequence relative to the unique Banach topology on such modules (cf. [**?**, Prop. 2.1.iii]) and so the claim follows from a well-known result of L. Gruson ([**?**, 3.2 Corollaire 1]). Since the source and the target are both right exact functors in N commuting with finite direct sums, we may use a finite free presentation of N to obtain that it is an isomorphism in general. The assertion of the proposition follows by taking $N = (\underline{M}_r)_z$. ☐

Corollary 8.2.3. *Let χ be dominant and regular. The functor $\mathscr{L}_{r,\chi}$, restricted to coadmissible modules, is exact.*

Proof. Exactness can be checked at a point $z \in \mathscr{B}$ where the functor in question equals the composite of three functors. The first functor equals $N \mapsto D_r(U_z^{(e)})_\theta \otimes_{D(U_z^{(e)})_\theta} N$ on the category of coadmissible $D(U_z^{(e)})_\theta$-modules. It is exact by [?, Rem. 3.2]. The second functor equals $N \mapsto \kappa(z)\hat{\otimes}_L N$ on the category of finitely generated $D_r(U_z^{(e)})_\theta$-modules. It is exact as we have explained in the proof of the preceding proposition. The natural inclusion $U(\mathfrak{g})_\theta \to D_r(U_z^{(e)})_\theta$ allows one to consider $\kappa(z)\hat{\otimes}_L N$ as a $U(\kappa(z) \otimes_L \mathfrak{g})_\theta$-module. The Beilinson-Bernstein stalk functor at $\pi(z)$ of the corresponding localization on the flag variety $X_{\kappa(z)}$ (note that the natural embedding $k(\pi(z)) \to \kappa(z)$ gives a canonical lift of $\pi(z)$ to a $\kappa(z)$-rational point of $X_{\kappa(z)}$) is given by the λ-coinvariants of the $\kappa(z) \otimes_L \mathfrak{h}$-module

$$(\kappa(z)\hat{\otimes}_L N)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L N)$$

according to 5.2.2 (iii). By loc.cit., part (ii), this functor is exact if χ is dominant and regular. □

Lemma 8.2.4. *Let $z \in \mathscr{B}$. If N is a finitely generated $D_r(U_z^{(e)})_\theta$-module which is finite dimensional over K, then the natural map*

$$\mathcal{D}^{an}_{\mathscr{B},\chi,z} \otimes_{U(\mathfrak{g}_K)_\theta} N \xrightarrow{\cong} \mathscr{D}_{r,\chi,z} \otimes_{D_r(U_z^{(e)})_\theta} N$$

is an isomorphism which is functorial in modules of this kind.

Proof. We adopt the notation of 6.4.8 and write

$$\varinjlim_V (\mathcal{A}_V \# D_r(U_z^{(e)})) \xrightarrow{\cong} \mathcal{O}_{\mathscr{B},z} \# D_r(U_z^{(e)}) ,$$

an isomorphism of topological K-algebras according to the main result of section 13. By [?, Prop. 2.1] the finitely generated module $(\mathcal{A}_V \# D_r(U_z^{(e)})) \otimes_{D_r(U_z^{(e)})} N$ has a unique Banach topology. We therefore have canonical $\mathcal{A}_V$-linear isomorphisms

$$(\mathcal{A}_V \hat{\otimes}_L D_r(U_z^{(e)})) \otimes_{D_r(U_z^{(e)})} N \simeq \mathcal{A}_V \hat{\otimes}_L N = \mathcal{A}_V \otimes_L N .$$

Passage to the inductive limit yields, by the main result of section 13, the $\mathcal{O}_{\mathscr{B},z}$-linear map

$$(\mathcal{O}_{\mathscr{B},z} \hat{\otimes} D_r(U_z^{(e)})) \otimes_{D_r(U_z^{(e)})} N \simeq \mathcal{O}_{\mathscr{B},z} \otimes_L N = (\mathcal{O}_{\mathscr{B},z} \# U(\mathfrak{g}_K)) \otimes_{U(\mathfrak{g}_K)} N .$$

The target maps canonically to $\mathcal{D}^{an}_{\mathscr{B},\chi,z} \otimes_{U(\mathfrak{g}_K)_\theta} N$ and the composed map annihilates all elements of the form $\xi \hat{\otimes} n$ with $n \in N$ and $\xi \in \mathcal{I}^{an}_{\mathscr{B},\chi,z}$. Since such ξ generate $\mathscr{I}^{an}_{\chi,z}$ the composed map factors therefore into a map

$$\mathscr{D}_{r,\chi,z} \otimes_{D_r(U_z^{(e)})_\theta} N \to \mathcal{D}^{an}_{\mathscr{B},\chi,z} \otimes_{U(\mathfrak{g}_K)_\theta} N .$$

This gives the required inverse map. □

Corollary 8.2.5. *Let M be a left $D(G)_\theta$-module such that* $\dim_K M_r(U_z^{(e)}) < \infty$ *for all* $z \in \mathscr{B}$. *The natural morphism of sheaves*

$$\mathcal{D}^{an}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} \underline{M}_r \xrightarrow{\cong} \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r = \mathscr{L}_{r,\chi}(M)$$

induced from (7.4.8) is an isomorphism.

Proof. Let $z \in \mathscr{B}$. Applying the preceding lemma to $N := M_r(U_z^{(e)})$ we see that the morphism is an isomorphism at the point z. This proves the claim. □

8.3. **Equivariance.**

8.3.1. Consider for a moment an arbitrary ringed space $(Y, \mathcal{A})$ where $\mathcal{A}$ is a sheaf of (not necessarily commutative) K-algebras on Y. Let Γ be an abstract group acting (from the right) on $(Y, \mathcal{A})$. In other words, for every $g, h \in \Gamma$ and every open subset $U \subseteq Y$ there is an isomorphism of K-algebras $g^* : \mathcal{A}(U) \xrightarrow{\cong} \mathcal{A}(g^{-1}U)$ compatible in an obvious sense with restriction maps and satisfying $(gh)^* = h^* g^*$.

A Γ-*equivariant* $\mathcal{A}$-module (cf. [**?**, II.F.5]) is a (left) $\mathcal{A}$-module $\mathcal{M}$ equipped, for any open subset $U \subseteq Y$ and for $g \in G$, with K-linear isomorphisms $g^* : \mathcal{M}(U) \xrightarrow{\cong} \mathcal{M}(g^{-1}U)$ compatible with restriction maps and such that $g^*(am) = g^*(a)g^*(m)$ for $a \in \mathcal{A}(U), m \in \mathcal{M}(U)$. If $g, h \in G$ we require $(gh)^* = h^* g^*$.

An obvious example is $\mathcal{M} = \mathcal{A}$. If $\mathcal{M}$ is equivariant we have a K-linear isomorphism $\mathcal{M}_z \xrightarrow{\cong} \mathcal{M}_{g^{-1}z}$ between the stalks of the sheaf $\mathcal{M}$ at z and $g^{-1}z$ for any $g \in G$. Finally, a morphism of equivariant modules is an $\mathcal{A}$-linear map compatible with the Γ-actions. The equivariant modules form an abelian category.

8.3.2. After these preliminaries we go back to the situation discussed in the previous section. We keep all the assumptions from this section. The group G naturally acts on the ringed space $(X^{an}, \mathcal{O}_{X^{an}})$. Moreover, G acts on $\mathfrak{g}$ and $U(\mathfrak{g})$ via the adjoint action as usual. It follows from the classical argument ([**?**, §3]) that the sheaves

$$\mathcal{O}_{X^{an}} \# U(\mathfrak{g}), \mathcal{I}^{an}_\chi \text{ and } \mathcal{D}^{an}_\chi := (\mathcal{O}_{X^{an}} \# U(\mathfrak{g}))/\mathcal{I}^{an}_\chi$$

(as defined in section 6) are equivariant $\mathcal{O}_{X^{an}}$-modules. Of course, here $g^* : \mathcal{D}^{an}_\chi(U) \xrightarrow{\cong} \mathcal{D}^{an}_\chi(g^{-1}U)$ is even a K-algebra isomorphism for all $g \in G$ and open subsets $U \subseteq X^{an}$.

On the other hand, the group G acts on the ringed space $(\mathscr{B}, \mathcal{O}_{\mathscr{B}})$ and the natural map $\vartheta_{\mathbf{B}} : \mathscr{B} \longrightarrow X^{an}$ is G-equivariant, cf. 6.2.1. Since our functor $\vartheta_{\mathbf{B}}^{-1}$ preserves G-equivariance the $\mathcal{O}_{\mathscr{B}}$-modules

$$\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}_K), \mathcal{I}^{an}_{\mathscr{B},\chi} \quad \text{and} \quad \mathcal{D}^{an}_{\mathscr{B},\chi} = (\mathcal{O}_{\mathscr{B}} \# U(\mathfrak{g}_K))/\mathcal{I}^{an}_{\mathscr{B},\chi}$$

are G-equivariant. Again, here $g^* : \mathcal{D}^{an}_{\mathscr{B},\chi}(U) \xrightarrow{\cong} \mathcal{D}^{an}_{\mathscr{B},\chi}(g^{-1}U)$ is a K-algebra isomorphism for all $g \in G$ and open subsets $U \subseteq \mathscr{B}$. Recall (7.2.1) the sheaf of K-algebras $\mathcal{O}_{\mathscr{B}} \# \underline{D}_r$.

Proposition 8.3.3. *The $\mathcal{O}_{\mathscr{B}}$-module $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ is G-equivariant. For $g \in G$ the map g^* is a K-algebra isomorphism.*

Proof. Given $g \in G$ and $z \in \mathscr{B}$ we have the group isomorphism $g^{-1}(\cdot)g : U_z^{(e)} \xrightarrow{\cong} U_{g^{-1}z}^{(e)}$ by (4.1.6). Since it is compatible with variation of the level e it is compatible with the p-valuations $\mathring{\omega}_z$ and $\mathring{\omega}_{g^{-1}z}$. It induces therefore an isometric isomorphism of Banach algebras

$$g^{-1}(\cdot)g : D_r(U_z^{(e)}) \xrightarrow{\cong} D_r(U_{g^{-1}z}^{(e)}). \tag{8.3.4}$$

The induced map

$$\mathcal{O}_{\mathscr{B},z}\hat{\otimes}_L D_r(U_z^{(e)}) \xrightarrow{\cong} \mathcal{O}_{\mathscr{B},z}\hat{\otimes}_L D_r(U_{g^{-1}z}^{(e)})$$

is multiplicative with respect to the skew multiplication and we obtain an isomorphism of topological K-algebras

$$g^* : (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z \xrightarrow{\cong} (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_{g^{-1}z} \tag{8.3.5}$$

according to 7.2.3. Since we have the identity $g\mathfrak{x}g^{-1} = \mathrm{Ad}(g)(\mathfrak{x})$ in $D(G)$ this isomorphism fits into the commutative diagram

$$\begin{array}{ccc}
(\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))_z & \xrightarrow{\simeq} & (\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}))_{g^{-1}z} \\
\downarrow & & \downarrow \\
(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_z & \xrightarrow{\simeq} & (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)_{g^{-1}z}
\end{array}$$

where the vertical arrows are the inclusions from (6.4.12). Recall the sheaf $\mathcal{F}$ appearing in 7.2.2. Let $\Omega \subseteq \mathscr{B}$ be an open subset. The isomorphisms (8.3.5) for $z \in \Omega$ assemble to a K-algebra isomorphism

$$g^* : \mathcal{F}(\Omega) \xrightarrow{\cong} \mathcal{F}(g^{-1}\Omega), s \mapsto [z \mapsto (g^*)^{-1}(s(gz))]$$

compatible with restriction maps and satisfying $(gh)^* = h^*g^*$ for $g, h \in G$. It now suffices to see that g^* maps the subspace $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$ into $(\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(g^{-1}\Omega)$. Let $s \in (\mathcal{O}_{\mathscr{B}}\#\underline{D}_r)(\Omega)$. If F is a facet in $\mathscr{B}$ we let $\Omega = \cup_{i\in I}\Omega_i$ be a datum for s with respect to F. If $F \cap \Omega_i \neq \emptyset$ we consider $g^{-1}V_i$ and $(g^*)^{-1}(s_i)$ and obtain a datum $g^{-1}\Omega = \cup_{i\in I} g^{-1}\Omega_i$ for the section $(g^*)^{-1}sg \in \mathcal{F}(g^{-1}\Omega)$ with respect to the facet $g^{-1}F$. Indeed, the axiom (2a) for the section$(g^*)^{-1}sg$ follows from the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathscr{B}}(U) & \xrightarrow{\iota_z} & \mathcal{O}_{\mathscr{B},z} \\
\downarrow{\scriptstyle (g^*)^{-1}} & & \downarrow{\scriptstyle (g^*)^{-1}} \\
\mathcal{O}_{\mathscr{B}}(gU) & \xrightarrow{\iota_{gz}} & \mathcal{O}_{\mathscr{B},gz}
\end{array}$$

valid for any open subset $U \subseteq \mathscr{B}$ containing z. Moreover, we have a commutative diagram

$$\begin{array}{ccc} D_r(U_{F'}^{(e)}) & \xrightarrow{\sigma_r^{F'F}} & D_r(U_F^{(e)}) \\ \downarrow{\scriptstyle (g^*)^{-1}} & & \downarrow{\scriptstyle (g^*)^{-1}} \\ D_r(U_{gF'}^{(e)}) & \xrightarrow{\sigma_r^{gF'gF}} & D_r(U_{gF}^{(e)}) \end{array}$$

whenever F', F are two facets in $\mathscr{B}$ with $F' \subseteq \overline{F}$. From this the axiom (2b) for the section $(g^*)^{-1}sg$ follows easily. ☐

It follows from the preceding proof that the morphism $\mathcal{O}_{\mathscr{B}}\#U(\mathfrak{g}) \to \mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ from 7.2.6 is equivariant. The equivariant structure of $\mathcal{I}^{an}_{\mathscr{B},\chi}$ therefore implies that the ideal sheaf $\mathscr{I}^{an}_{\chi}$ of $\mathcal{O}_{\mathscr{B}}\#\underline{D}_r$ is naturally equivariant. This yields the following corollary.

Corollary 8.3.6. *The $\mathcal{O}_{\mathscr{B}}$-module $\mathscr{D}_{r,\chi}$ is equivariant. The map g^* is a K-algebra isomorphism for any $g \in G$. The morphism $\mathcal{D}^{an}_{\mathscr{B},\chi} \to \mathscr{D}_{r,\chi}$ from (7.4.8) is equivariant.*

The above discussion shows that there is a natural right action of G on the ringed space $(\mathscr{B}, \mathscr{D}_{r,\chi})$. We let $\mathrm{Mod}_G(\mathscr{D}_{r,\chi})$ be the abelian category of G-equivariant (left) $\mathscr{D}_{r,\chi}$-modules.

8.3.7. Using very similar arguments we may use the isomorphisms (8.3.4) appearing in the above proof to define an equivariant structure on the sheaves $\underline{D}_r$ and $\underline{D}_{r,\theta}$. As before we suppose $\sigma(\chi) = \theta$. If M is a $D(G)$-module (resp. $D(G)_\theta$-module) with $m \in M$ and $g \in G$ we put $g.m := \delta_{g^{-1}}m$. This defines a K-linear isomorphism

$$g^* : M_r(U_z^{(e)}) \xrightarrow{\cong} M_r(U_{g^{-1}z}^{(e)})$$

via $g^*(\delta \otimes m) := g^*(\delta) \otimes gm$ for any $\delta \in D_r(U_z^{(e)})$. As in the case of $\underline{D}_r$ these isomorphisms lift to an equivariant structure on the sheaf $\underline{M}_r$. Since these isomorphisms are compatible with the isomorphisms (8.3.4) we obtain that $\underline{M}_r$ is an equivariant $\underline{D}_r$-module (resp. $\underline{D}_{r,\theta}$-module). We now define $g^*(\partial \otimes m) := g^*(\partial) \otimes g^*(m)$ for local sections ∂ and m of $\mathscr{D}_{r,\chi}$ and $\underline{M}_r$ respectively. Since the morphism $\underline{D}_{r,\theta} \to \mathscr{D}_{r,\chi}$ induced by 7.2.5 is equivariant this yields an equivariant structure on $\mathscr{L}_{r,\chi}(M)$. If $M \to N$ is a $D(G)_\theta$-linear map the resulting morphism $\mathscr{L}_{r,\chi}(M) \to \mathscr{L}_{r,\chi}(N)$ is easily seen to be equivariant. This shows

Corollary 8.3.8. *The functor $\mathscr{L}_{r,\chi}$ takes values in $\mathrm{Mod}_G(\mathscr{D}_{r,\chi})$.*

9. Comparison with the Schneider-Stuhler construction

In this section we assume $L = \mathbb{Q}_p$, $e > e_{st}$ and $r \in [r_0, 1)$. We will work in this section with the trivial infinitesimal character, i.e., $\lambda := \lambda_0$ and $\theta := \theta_0$.

9.1. **Preliminaries on smooth distributions.**

9.1.1. Let M be a co-admissible $D(G)$-module such the the associated locally analytic representation $V = M'_b$ is smooth. In the previous section, we have associated to M a sheaf $\underline{M}_r$ on the Bruhat-Tits building $\mathscr{B}$. On the other hand, we also have the sheaf $\underset{\approx}{V}$ on $\mathscr{B}$ constructed by Schneider and Stuhler (cf. 4.6). We now show that for $r < p^{-\frac{1}{p-1}}$, the two sheaves $\underset{\approx}{\check{V}}$ and $\underline{M}_r$ are canonically isomorphic. Here, $\check{V}$ denotes the smooth dual. We remark straightaway that V is admissible-smooth ([**?**, Thm. 6.6]) and hence, so is $\check{V}$ ([**?**, 1.5 (c)]).

Suppose H is a uniform locally $\mathbb{Q}_p$-analytic group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{h}$. Let $D^\infty(H)$ denote the quotient of $D(H)$ by the ideal generated by $\mathfrak{h}$. Let $\mathcal{C}_H^\infty$ denote the category of coadmissible $D^\infty(H)$-modules. If $U_r(\mathfrak{h})$ denotes the closure of $U(\mathfrak{h})$ inside $D_r(H)$ we put

$$H_{(r)} := H \cap U_r(\mathfrak{h}) \ .$$

Lemma 9.1.2. *The set $H_{(r)}$ is an open normal subgroup of H constituting, for $r \uparrow 1$, a neighborhood basis of $1 \in H$.*

Proof. As the norm $\|\cdot\|_r$ on $D_r(H)$ does not depend on the choice of ordered basis the inversion map $h \mapsto h^{-1}$ induces an automorphism of $D_r(H)$. It induces an automorphism of $U_r(\mathfrak{h})$ which implies that $H_{(r)}$ is a subgroup of H. A similar argument with the conjugation automorphism $h \mapsto ghg^{-1}$ for a $g \in H$ implies that this subgroup is normal in H. For the remaining assertions we choose $m \geq 0$ such that $r_m = \sqrt[p^m]{r_0} \geq r$ and consider $D(P_{m+1}(H))$. The inclusion $D(P_{m+1}(H)) \subseteq D(H)$ gives rise to an isometric embedding

$$D_{r_0}(P_{m+1}(H)) \hookrightarrow D_{r_m}(H)$$

(final remark in 2.2.3). Since $U(\mathfrak{h})$ is norm-dense inside $D_{r_0}(P_{m+1}(H))$ it follows that

$$P_{m+1}(H) \subset U_{r_m}(\mathfrak{h}) \subseteq U_r(\mathfrak{h})$$

which implies $P_{m+1}(H) \subseteq H_{(r)}$ and therefore $H_{(r)}$ is open. Finally, if $r \uparrow 1$ then $r_m \uparrow 1$ whence $m \uparrow \infty$. Since the lower p-series $\{P_m(H)\}_m$ constitutes a neighborhood basis of $1 \in H$ the last assertion of the lemma follows. $\square$

The lemma implies (cf. [**?**, pf. of Thm. 6.6]) a canonical K-algebra isomorphism $D^\infty(H) \simeq \varprojlim_r K[H/H_{(r)}]$ coming from restricting distributions to the subspace of K-valued locally constant functions on H.

Proposition 9.1.3. *(i) We have $D_r(H) \otimes_{D(H)} D^\infty(H) \simeq K[H/H_{(r)}]$ as right $D^\infty(H)$-modules;*
(ii) If $M \in \mathcal{C}_H^\infty$ and $V = M'_b$ denotes the corresponding smooth representation then $D_r(H) \otimes_{D(H)} M \simeq (\check{V})_{H_{(r)}}$ as K-vector spaces. Here, $(\cdot)_{H_{(r)}}$ denotes $H_{(r)}$-coinvariants and $\check{(\cdot)}$ denotes the smooth dual.

Proof. The first statement follows from $D_r(H) = \oplus_{h \in H/H_{(r)}} \delta_h U_r(\mathfrak{h})$ as right $U_r(\mathfrak{h})$-modules by passing to quotients modulo the ideals generated by $\mathfrak{h}$. The second statement follows from (i) by observing the general identities $K[H/N] \otimes_{D^\infty(H)} M = \mathrm{Hom}_K(V^N, K) = (\check{V})_N$ valid for any normal open subgroup N of H. □

Corollary 9.1.4. *If $M \in \mathcal{C}_H^\infty$ and $r_0 \leq r < p^{-1/p-1}$ then $D_r(H) \otimes_{D(H)} M \simeq (\check{V})_H$.*

Proof. We have $U_r(\mathfrak{h}) = D_r(H)$ for such an r and therefore $H_{(r)} = H$. □

9.2. **The comparison isomorphism.**

9.2.1. Let us return to our sheaf $M \mapsto \underline{M}_r$. We assume in the following

$$r_0 \leq r < p^{-1/p-1} .$$

Let F be a facet in X. If we apply the above corollary to the uniform group $U_F^{(e)}$ we obtain a canonical linear isomorphism

$$f_r^F : M(U_F^{(e)}) = D_r(U_F^{(e)}) \otimes_{D(U_F^{(e)})} M \xrightarrow{\cong} (\check{V})_{U_F^{(e)}} .$$

If $F \subseteq \overline{F'}$ for two facets F, F' in X it follows that

$$f_r^{F'} \circ \sigma_r^{FF'} = pr^{FF'} \circ f_r^F \tag{9.2.1}$$

where $pr^{FF'} : (\check{V})_{U_F^{(e)}} \to (\check{V})_{U_{F'}^{(e)}}$ denotes the natural projection.

Proposition 9.2.2. *Given an open subset $\Omega \subseteq X$ the collection of maps f_r^z for $z \in \Omega$ induces a K-linear isomorphism $\underline{M}_r(\Omega) \simeq \underset{\approx}{\check{V}}(\Omega)$ compatible with restriction maps whence a canonical isomorphism of sheaves*

$$\underline{M}_r \xrightarrow{\cong} \underset{\approx}{\check{V}}$$

which is natural in admissible V.

Proof. Given $z \in \mathscr{B}$ we have the isomorphism $f_r^z : M_r(U_z^{(e)}) \xrightarrow{\cong} (\check{V})_{U_z^{(e)}}$ as explained above. These maps assemble to a K-linear isomorphism, say f_r^Ω, between the space of maps

$$s : \Omega \to \dot{\bigcup}_{z \in \Omega} M_r(U_z^{(e)})$$

such that $s(z) \in M_r(U_z^{(e)})$ for all $z \in \mathscr{B}$ and the space of maps

$$s : \Omega \to \dot{\bigcup}_{z \in \Omega} (\check{V})_{U_z^{(e)}}$$

such that $s(z) \in (\check{V})_{U_z^{(e)}}$ for all $z \in \mathscr{B}$. It is clearly compatible with restriction. It therefore suffices to show that it descends to an isomorphism between the subspaces $\underline{M}_r(\Omega)$ and

$\underset{\approx}{\check{V}}(\Omega)$ respectively. Since $\underline{M}_r$ and $\underset{\approx}{\check{V}}$ are sheaves it suffices to verify this over the open sets $\Omega \cap St(F)$ for facets $F \subset \mathscr{B}$. We may therefore fix a facet $F \subset \mathscr{B}$ and assume that $\Omega \subseteq St(F)$. Restricting to members Ω_i with $\Omega_i \cap F \neq \emptyset$ of a datum for s with respect to F and using the sheaf property a second time we may assume that the covering $\{\Omega\}$ of $\Omega = \Omega \cap St(F)$ is a datum for s with respect to F satisfying $\Omega \cap F \neq \emptyset$. Let $s \in M_r(U_F^{(e)})$ be the corresponding element of the datum. We let $\check{v}$ be any preimage in $\check{V}$ of $f_r^F(s) \in (\check{V})_{U_F^{(e)}}$. The value of the function $f_r^\Omega(s)$ at $z \in \Omega$ is then given by

$$f_r^z(s(z)) = f_r^{F'}(\sigma_r^{FF'}(s)) \overset{(9.2.1)}{=} pr^{FF'}(f_r^F(s)) = \text{class of } \check{v} \in (\check{V})_{U_{F'}^{(e)}}$$

where $F' \in St(F)$ is the unique open facet containing z. This means $f_r^\Omega(s) \in \underset{\approx}{\check{V}}(\Omega)$.

Conversely, let $\check{s} \in \underset{\approx}{\check{V}}(\Omega)$ and consider $s := (f_r^\Omega)^{-1}(\check{s})$. Let $F \subset \mathscr{B}$ be a facet. Any defining open covering $\Omega = \cup_{i \in I} \Omega_i$ with vectors $\check{v}_i \in \check{V}$ for the section $\check{s}$ induces an open covering $\Omega \cap St(F) = \cup_{i \in I} \Omega_{i,F}$ where $\Omega_{i,F} := \Omega_i \cap St(F)$. If $F \cap \Omega_{i,F} \neq \emptyset$ we let $s_i \in M_r(U_F^{(e)})$ be the inverse image of the class of $\check{v}_i$ under $(f_r^F)^{-1}$. We claim that this gives a datum for s with respect to F. Indeed, for any $z \in \Omega_{i,F} \cap F$ we compute

$$s(z) = (f_r^z)^{-1}(\check{s}(z)) = (f_r^z)^{-1}(\text{class of } \check{v}_i) = s_i$$

which settles the axiom $(2a)$ for s. Similarly, for any $z' \in \Omega_{i,F}$ the value of $s(z')$ equals

$$\begin{aligned}(f_r^{z'})^{-1}(\check{s}(z')) = (f_r^{F'})^{-1}(\text{class of } \check{v}_i) = \; & (f_r^{F'})^{-1}(pr^{FF'}(\check{v}_i)) \overset{(9.2.1)}{=} \sigma_r^{FF'}((f_r^F)^{-1}(\check{v}_i)) \\ & = \sigma_r^{FF'}(s_i)\end{aligned}$$

where F' denotes the unique open facet of $St(F)$ containing z'. This proves (2b) for s. All in all $s \in \underline{M}_r(\Omega)$. This proves the proposition. □

Lemma 9.2.3. *Let M be a coadmissible $D^\infty(G)$-module. Then M is a $D(G)_{\theta_0}$-module.*

Proof. We have to show that the canonical map $D(G) \to D^\infty(G)$ factors through $D(G)_{\theta_0}$. The kernel of $D(G) \to D^\infty(G)$ is the two sided ideal generated by $\mathfrak{g}$. The intersection of this latter ideal with $Z(\mathfrak{g}_K)$ equals $\ker \theta_0$ (cf. example 8.1.1). It follows that the map $Z(\mathfrak{g}_K) \to D^\infty(G)$ factors through θ_0. □

Theorem 9.2.4. *Let $r = r_0$. Suppose M is a coadmissible $D^\infty(G)$-module. Then there is a canonical isomorphism of $\mathcal{O}_{\mathscr{B}}$-modules*

$$C^{SS} : \mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx}{\check{V}} \xrightarrow{\cong} \mathscr{L}_{r_0,\rho}(M)$$

which is natural in such M. Here, as above, $V = M'_b$.

Proof. Since $\mathfrak{g}M = 0$ there is a canonical isomorphism

$$\mathcal{O}_{\mathscr{B}} \otimes_L \underline{M}_{r_0} \xrightarrow{\cong} \mathcal{D}^{an}_{\mathscr{B}\chi} \otimes_{U(\mathfrak{g}_K)_\theta} \underline{M}_{r_0} \ .$$

Arguing stalkwise the assertion is a combination of 8.2.5 and 9.2.2. □

10. Compatibility with the Beilinson-Bernstein localization

Throughout this section we suppose that the conditions (8.0.1) are fulfilled.

Let V denote a finite dimensional algebraic representation of $\mathbf{G}$. Then V gives rise to a $U(\mathfrak{g})$-module. Let $M = V'$ denote the dual of V. It is a coadmissible $D(G)$-module. Suppose the $U(\mathfrak{g}_K)$-module underlying M is a $U(\mathfrak{g}_K)_\theta$-module.

Recall that to any $U(\mathfrak{g}_K)_\theta$-module M, Beilinson and Bernstein associate a $\mathcal{D}_\chi$-module which will be denoted $\Delta(M)$ (cf. §5). We can pull this back under the natural map $\pi : X^{an} \to X$ to get a $\mathcal{D}^{an}_\chi$-module $\Delta(M)^{an}$. Finally, we may apply the functor $\vartheta_{\mathbf{B}}^{-1}$ to this module. Denote the latter $\mathcal{O}_{\mathscr{B}}$-module by $\Delta(M)^{an}_{\mathscr{B}}$. One has the following description of $\Delta(M)^{an}$ and $\Delta(M)^{an}_{\mathscr{B}}$:

$$\Delta(M)^{an} = \mathcal{D}^{an}_\chi \otimes_{U(\mathfrak{g}_K)_\theta} M$$

$$\Delta(M)^{an}_{\mathscr{B}} = \mathcal{D}^{an}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} M.$$

Here, the second identity follows from the compatibility between tensor products with restriction functors ([**?**, Prop. 2.3.5]). On the other hand, any finite dimensional algebraic representation V gives rise to a $D(G)$-module M, where $M = V'$. If V is a $U(\mathfrak{g}_K)_\theta$-module, then M is a $D(G)_\theta$-module. In particular, the results of section 8 allow us to associate to M the $\mathscr{D}_{r,\chi}$-module $\mathscr{L}_{r,\chi}(M)$. Recall, this module is given by:

$$\mathscr{L}_{r,\chi}(M) = \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r$$

Now the canonical morphism $\mathcal{D}^{an}_{\mathscr{B},\chi} \to \mathscr{D}_{r,\chi}$ induces a morphism

$$C^{BB} : \mathcal{D}^{an}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} M \longrightarrow \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r \ .$$

Recall that $r = r_m$ for some m.

Theorem 10.1.1. *There is $r(M) \in [r_0, 1)$ such that for $r \geq r(M)$ (i.e. $m >> 0$ sufficiently large) the canonical morphism*

$$C^{BB} : \Delta(M)^{an}_{\mathscr{B}} \xrightarrow{\cong} \mathscr{L}_{r,\chi}(M)$$

is an isomorphism of $\mathcal{D}^{an}_{\mathscr{B},\chi}$-modules.

Proof. Let F be a facet in $\mathscr{B}$ such that $F \subseteq \overline{\mathscr{C}}$. By [**?**, Prop. 4.2.10] the $D(U_F^{(e)})$-module M decomposes into a *finite* direct sum of irreducible $D(U_F^{(e)})$-modules M_i. Since all M_i are coadmissible $D(U_F^{(e)})$-modules there exists $r(F) \in [r_0, 1)$ such that

$$M_{i,r} := D_r(U_F^{(e)}) \otimes_{D(U_F^{(e)})} M_i \neq 0$$

for all $r \geq r(F)$ and all i. By Theorem A ([**?**, §3]) the $D(U_F^{(e)})$-equivariant map $M_i \to M_{i,r}, m \mapsto 1 \otimes m$ has dense image and is therefore surjective. Since M_i is irreducible the map is therefore bijective whenever $r \geq r(F)$. It follows $M \xrightarrow{\cong} M_r(U_F^{(e)})$ for $r \geq r(F)$. Given $g \in G$ we can use the G-equivariance of the sheaf $\underline{M}_r$ to express the canonical map $M \to M_r(U_{g^{-1}F}^{(e)})$ as the composite

$$M \xrightarrow{g\cdot} M \xrightarrow{\cong} M_r(U_F^{(e)}) \xrightarrow{g^*} M_r(U_{g^{-1}F}^{(e)}) \ .$$

It is therefore bijective. Put $r(M) := \max_{F \subseteq \overline{\mathscr{C}}} r(F)$. Then $M \xrightarrow{\cong} M_r(U_F^{(e)})$ for all $F \subset \mathscr{B}$ and all $r \geq r(M)$. Identifying M with its constant sheaf on $\mathscr{B}$ the natural morphism $M \xrightarrow{\cong} \underline{M}_r$ is therefore an isomorphism for all $r \geq r(M)$. On the other hand, arguing stalkwise gives, by 8.2.4, a canonical isomorphism

$$\mathcal{D}^{an}_{\mathscr{B},\chi} \otimes_{U(\mathfrak{g}_K)_\theta} \underline{M}_r \xrightarrow{\cong} \mathscr{D}_{r,\chi} \otimes_{\underline{D}_{r,\theta}} \underline{M}_r \ .$$

□

11. A class of examples

Throughout this section we suppose that the conditions (8.0.1) are fulfilled.

11.1.2. Let $\mathcal{O}$ be the classical BGG-category for the reductive Lie algebra $\mathfrak{g}_K$ relative to the choice of Borel subalgebra $\mathfrak{b}_K$ ([?]). Since this category was originally defined for complex semisimple Lie algebras only we briefly repeat what we mean by it here. The category $\mathcal{O}$ equals the full subcategory of all (left) $U(\mathfrak{g}_K)$-modules consisting of modules M such that

(i) M is finitely generated as $U(\mathfrak{g}_K)$-module;

(ii) the action of $\mathfrak{t}_K$ on M is semisimple and locally finite;

(iii) the action of $\mathfrak{n}_K$ on M is locally finite.

Recall here that $\mathfrak{t}_K$ acts locally finitely on some module M if $U(\mathfrak{t}_K).m$ is finite dimensional for all $m \in M$ (similar for $\mathfrak{n}_K$).

Let $\mathcal{O}_{\rm alg}$ be the full abelian subcategory of $\mathcal{O}$ consisting of those $U(\mathfrak{g}_K)$-modules whose $\mathfrak{t}_K$-weights are integral, i.e., are contained in the lattice $X^*(\mathbf{T}) \subset \mathfrak{t}_K^*$.

11.1.3. In [?] the authors study an exact functor

$$M \mapsto \mathcal{F}_B^G(M)$$

from $\mathcal{O}_{\rm alg}$ to admissible locally analytic G-representations. It maps irreducible modules to (topologically) irreducible representations. The image of $\mathcal{F}_B^G$ comprises a wide class of interesting representations containing all principal series representations and many representations arising from homogeneous vector bundles on p-adic symmetric spaces. In this final section we wish to study the localizations of representations in this class. We restrict our attention to modules $M \in \mathcal{O}_{\mathrm{alg},\theta}$ having fixed central character θ. Let $\chi \in \mathfrak{t}_K^*$ be such that $\sigma(\chi) = \theta$.

11.1.4. To start with let $U(\mathfrak{g}_K, B)$ be the smallest subring of $D(G)$ containing $U(\mathfrak{g}_K)$ and $D(B)$. The $\mathfrak{b}$-action on any $M \in \mathcal{O}_{\mathrm{alg}}$ integrates to an algebraic, and hence, locally analytic B-action on M and one has a canonical $D(G)$-module isomorphism

$$\mathcal{F}_B^G(M)'_b \xrightarrow{\cong} D(G) \otimes_{U(\mathfrak{g}_K,B)} M =: N$$

(loc.cit., Prop. 3.6). Of course, N is a $D(G)_\theta$-module. We may therefore consider its localization $\mathscr{L}_{r,\chi}(N)$ on $\mathscr{B}$. We recall that the stalk $\mathscr{L}_{r,\chi}(N)_z$ at a point z is a quotient of $\kappa(z)\hat{\otimes}(\underline{N}_r)_z$ (cf. 8.2.2) and therefore has its quotient topology. We finally say a morphism of sheaves to $\mathscr{L}_{r,\chi}(N)$ has *dense image* if this holds stalkwise at all points.

On the other hand, we may form

$$GM := K[G] \otimes_{K[B]} M \ .$$

It may be viewed as a $U(\mathfrak{g}_K)$-module via $x.(g \otimes m) := g \otimes \mathrm{Ad}(g^{-1})(x).m$ for $g \in G, m \in M, x \in \mathfrak{g}_K$. Since $K[G]$ is a free right $K[B]$-module, GM equals the direct sum of $U(\mathfrak{g}_K)$-submodules $gM := g \otimes M$ indexed by a system of coset representatives g for G/B. Since the group $\mathbf{G}$ is connected, the adjoint action of $G = \mathbf{G}(L)$ fixes the center $Z(\mathfrak{g}_K) \subset U(\mathfrak{g}_K)$ ([**?**, II §6.1.5]) and therefore GM still has central character θ. Let us consider its Beilinson-Bernstein module $\Delta(GM)$ over X. The linear map $gM \xrightarrow{\cong} M, g \otimes m \mapsto m$ is an isomorphism and equivariant with respect to the automorphism $\mathrm{Ad}(g^{-1})$ of $U(\mathfrak{g}_K)$. It follows that, given an open subset $V \subseteq X$, we have a linear isomorphism $\Delta(gM)(V) \xrightarrow{\cong} \Delta(M)(g^{-1}V)$ given by $\delta \otimes (g \otimes m) \mapsto g^*(\delta) \otimes m$ for a local section δ of $\mathcal{D}_\chi$ and $m \in M$. Here g^* r efers to the G-equivariant structure on $\mathcal{D}_\chi$ (8.3.2). The same argument works for the analytifications $\Delta^{an}(gM)$ and $\Delta^{an}(M)$. In particular, the stalks $\Delta^{an}(gM)_z \simeq \Delta^{an}(M)_{g^{-1}z}$ are isomorphic vector spaces for any $z \in \mathscr{B}$ and any $g \in G$.

Lemma 11.1.5. *We have*

$$\Delta^{an}(M)|_{\mathscr{B}} = 0 \Longleftrightarrow \Delta^{an}(GM)|_{\mathscr{B}} = 0 \ .$$

Proof. Suppose $\Delta^{an}(M)|_{\mathscr{B}} = 0$. Let $g \in G$. For any $z \in \mathscr{B}$ we compute $\Delta^{an}(gM)_z \simeq \Delta^{an}(M)_{g^{-1}z} = 0$ whence $\Delta^{an}(gM)|_{\mathscr{B}} = 0$. This yields $\Delta^{an}(GM)|_{\mathscr{B}} = 0$, since $\Delta^{an}(\cdot)_{\mathscr{B}}$ commutes with arbitrary direct sums. The converse is clear. □

Lemma 11.1.6. *There is a canonical morphism of $\mathscr{D}_{r,\chi}$-modules*

$$\mathscr{D}_{r,\chi} \otimes_{\mathcal{D}^{an}_{\mathscr{B},\chi}} \Delta(GM)^{an}_{\mathscr{B}} \longrightarrow \mathscr{L}_{r,\chi}(\mathcal{F}^G_B(M)')$$

functorial in M and with dense image.

Proof. The morphism is induced from the functorial map

$$P : GM \to D(G) \otimes_{U(\mathfrak{g}_K,B)} M = N$$

via the inclusions $K[B] \subset D(B)$ and $K[G] \subset D(G)$. Let us show that the morphism has dense image. We claim first that the map P has dense image with respect to the canonical topology on the coadmissible module N. Let G_0 be the (hyper-)special maximal compact open subgroup of G equal to the stabilizer of the origin $x_0 \in A$. Let $B_0 := B \cap G_0$. The Iwasawa decomposition $G = G_0 \cdot B$ implies $K[G] = K[G_0] \otimes_{K[B_0]} K[B]$ and similarly for distributions $D(\cdot)$. Let $G_0M := K[G_0] \otimes_{K[B_0]} M$ and $N_0 := D(G_0) \otimes_{U(\mathfrak{g}_K,B_0)} M$. Then $G_0M \simeq GM$ as $K[G_0]$-modules and $N \simeq N_0$ as $D(G_0)$-modules via the obvious maps. Write $D(G_0) = \varprojlim_r D_r(G_0)$ with some Banach algebra completions $D_r(G_0)$. The map P induces maps $P_r : G_0M \to D_r(G_0) \otimes_{U(\mathfrak{g}_K,B_0)} M$. Since $K[G_0] \subset D_r(G_0)$ is dense, the definition of the Banachtopology on the target implies that P_r has dense image. Passing to the limit over r shows that P has dense image. Let $z \in \mathscr{B}$. Then the map P composed with the map $N \to \underline{N}_{r,z}$ has dense image ([**?**, §3 Thm. A]). Now we are done: the map

$$\mathscr{D}_{r,\chi,z} \otimes_{\mathcal{D}^{an}_{\mathscr{B},\chi,z}} \Delta(GM)^{an}_{\mathscr{B},z} \to \mathscr{L}_{r,\chi}(N)_z \ ,$$

pulled back to $\Delta(GM)^{an}_{\mathscr{B},z}$, may be written as

$$((\kappa(z)\hat{\otimes}_L GM)/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L GM))_{\lambda-coinv.} \longrightarrow ((\kappa(z)\hat{\otimes}_L \underline{N}_{r,z})/\mathfrak{n}_{\pi(z)}(\kappa(z)\hat{\otimes}_L \underline{N}_{r,z}))_{\lambda-coinv.}$$

by 5.2.2 and 8.2.2. Consequently, it has dense image. □

11.1.7. We now look closer at the case $\theta = \theta_0$ and $\chi = \rho$. Let $V := ind^G_B(1)$ be the smooth induction of the trivial character of B. Its smooth dual $\check{V}$ equals the smooth induction $ind^G_B(\delta_B^{-1})$ where $\delta_B : B \to \mathbb{Q}^\times \subseteq K^\times$ is the modulus character of the locally compact group B. We choose e large enough so that the Schneider-Stuhler sheaf $\underset{\approx}{\check{V}}$ of $\check{V}$ is non-zero ([**?**, Thm. IV.4.1]).

The finitely many irreducible modules in $\mathcal{O}_{\mathrm{alg},\theta_0}$ are given by the irreducible quotients L_w of the Verma modules M_w of highest weight $-w(\rho) - \rho$ for $w \in W$. The cardinality of the latter set of weights is $|W|$. As usual, w_0 denotes the longest element in W. Let $w \in W$. Let $\mathcal{M}_w$ and $\mathcal{L}_w$ be the Beilinson-Bernstein localizations over X of M_w and L_w respectively. Let $\iota_w : X_w \hookrightarrow X$ be the inclusion of the Bruhat cell $\mathbf{B}w\mathbf{B}/\mathbf{B}$ into X and let $\mathcal{O}_{X_w}$ be its structure sheaf with its natural (left) D_{X_w}-module structure. Let $\mathcal{N}_w = \iota_{w*}\mathcal{O}_w$ be its D-module push-forward to X. Since $\mathcal{O}_{X_w}$ is a holonomic module and ι_w is an affine morphism, $\mathcal{N}_w$ may be viewed as an D_X-module (rather than just a complex of such), cf. [**?**, 3.4].

Proposition 11.1.8. *Let $w \in W$ and $\mathcal{L}_w^{an}$ be the analytification of $\mathcal{L}_w$. Then $\mathcal{L}_{w_0}^{an}|_{\mathscr{B}} = \mathcal{O}_{\mathscr{B}}$ and $\mathcal{L}_w^{an}|_{\mathscr{B}} = 0$ for $w \neq w_0$.*

Proof. By loc.cit., Lem. 12.3.1 the sheaf $\mathcal{N}_w$ has support contained in X_w. By loc.cit., Prop. 12.3.2 (i) the module $\mathcal{L}_w$ injects into $\mathcal{N}_w$. Now let $w \neq w_0$. Let $\eta \in X$ be the generic point of X and ${X^{an}}_\eta$ the fibre of $\pi : X^{an} \to X$ over η. Since $\eta \notin X_w$ one has $(\mathcal{N}_w)_\eta = 0$ and therefore $\mathcal{N}_w^{an}|_{{X^{an}}_\eta} = 0$. Lemma 6.2.2 states that $\mathscr{B} \subset {X^{an}}_\eta$ whence $\mathcal{L}_w^{an}|_{\mathscr{B}} = 0$. The converse is clear: the module L_{w_0} equals the trivial one-dimensional $U(\mathfrak{g})$-module having localization $\mathcal{L}_{w_0} = \mathcal{O}_X$ (e.g. by the Borel-Weil theorem). Hence, $\mathcal{L}_{w_0}^{an}|_{\mathscr{B}} = \mathcal{O}_{\mathscr{B}}$. □

Corollary 11.1.9. *Let $w \in W$. Then $\mathscr{L}_{r,\rho}(\mathcal{F}_B^G(L_w)') \neq 0$ if and only if $w = w_0$.*

Proof. Let $w \neq w_0$. The preceding proposition together with the first lemma above yields $\Delta^{an}(GL_w)|_{\mathscr{B}} = 0$. The second lemma then yields $\mathscr{L}_{r,\rho}(\mathcal{F}_B^G(L_w)') = 0$. Conversely, let $w = w_0$. We have $\mathcal{F}_B^G(L_{w_0}) = ind_B^G(1) = V$, the smooth induction of the trivial B-representation ([?]). By choice of e we have $\underset{\approx}{\check{V}} \neq 0$. Let $z \in \mathscr{B}$ be a point such that $\check{V}_{U_z^{(e)}} \neq 0$. With $N := V'$ and $(U_z^{(e)})_{(r)} := U_z^{(e)} \cap U_r(U_z^{(e)})$, Prop. 9.1.3 yields a surjection

$$(\underline{N}_r)_z = D_r(U_z^{(e)}) \otimes_{D(U_z^{(e)})} N = \check{V}_{(U_z^{(e)})_{(r)}} \longrightarrow \check{V}_{U_z^{(e)}}$$

between the two spaces of coinvariants which implies $(\underline{N}_r)_z \neq 0$. It follows that $\mathscr{L}_{r,\rho}(N)_z = \kappa(z) \otimes_L (\underline{N}_r)_z \neq 0$ (8.2.2) which means $\mathscr{L}_{r,\rho}(N)|_{\mathscr{B}} \neq 0$. □

Recall that any $U(\mathfrak{g}_K)$-module $M \in \mathcal{O}$ is of finite length.

Proposition 11.1.10. *Let $M \in \mathcal{O}_{\mathrm{alg},\theta_0}$. Let $n \geq 0$ be the Jordan-Hölder multiplicity of the trivial representation in the module M and let $V = ind_B^G(1)$. There is a (noncanonical) isomorphism of $\mathcal{O}_{\mathscr{B}}$-modules*

$$\mathscr{L}_{\rho,r}(\mathcal{F}_B^G(M)') \xrightarrow{\cong} \mathscr{L}_{\rho,r}(V'^{\oplus n})$$

with both sides equal to zero in case $n = 0$.

Proof. Let $\underset{\approx,r}{\check{V}}$ be the constructible sheaf of K-vector spaces on $\mathscr{B}$ which is constructed in the same way as $\underset{\approx}{\check{V}}$ but using the groups $(U_F^{(e)})_{(r)}$ instead of $U_F^{(e)}$ for all facets F. The very same arguments as in the case $r = r_0$ (Thm. 9.2.4) show that the $\mathcal{O}_{\mathscr{B}}$-module $\mathscr{L}_{\rho,r}(V')$ is isomorphic to the module $\mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx,r}{\check{V}}$. In particular, it is a free $\mathcal{O}_{\mathscr{B}}$-module.

We now prove the claim of the proposition by induction on n. Let $n = 0$. By exactness of the functors $\mathcal{F}_B^G(.)'$ and $\mathscr{L}_{\rho,r}$ a Jordan-Hölder filtration of M induce a filtration of $\mathscr{L}_{\rho,r}(\mathcal{F}_B^G(M)')$ whose graded pieces vanish by the preceding corollary. Thus $\mathscr{L}_{\rho,r}(\mathcal{F}_B^G(M)') = 0$. Let $n = 1$. Using a Jordan-Hölder filtration of M and the case $n = 0$ we may assume that the trivial representation sits in the top graded piece of M. Applying the case $n = 0$ a second time gives the claim. Assume now $n \geq 2$. Using again a Jordan-Hölder filtration of M we have an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\mathcal{O}_{\mathrm{alg},\theta_0}$ where M_i has multiplicity $n_i \geq 1$. Applying the induction hypothesis to M_1 and M_2 yields an exact sequence of $\mathcal{O}_{\mathscr{B}}$-modules

$$0 \to \mathscr{L}_{\rho,r}(V'^{\oplus n_1}) \to \mathscr{L}_{\rho,r}(\mathcal{F}^G_B(M)') \to \mathscr{L}_{\rho,r}(V'^{\oplus n_2}) \to 0\ .$$

By our first remark this sequence is (noncanonically) split. Since $\mathscr{L}_{\rho,r}$ commutes with direct sums, this completes the induction. □

When $r = r_0$ the statement of the preceding Prop. 11.1.10 can be made more concrete, because in this case the sheaf $\mathscr{L}_{\rho,r_0}(V'^{\oplus n})$ equals the sum over n copies of $\mathcal{O}_{\mathscr{B}} \otimes_L \underset{\approx}{\check{V}}$ with $\check{V} = ind^G_B(\delta_B^{-1})$, cf. 9.2.4.

12. Appendix: Analyticity of group actions near points on the building

In this appendix we give a proof of Lemma 6.2.6 about the analyticity of group actions near points on the building. Before doing so we would like to remark that we have not used anything special about these points, except that they correspond to supremum norms on affinoid subdomains. It is certainly possible to prove more general statements in similar settings.

Proof of 6.2.6. *Step 1.* Recall that G acts transitively on the set of apartments in $\mathscr{B}$, and that we denote by $A = (X_*(\mathbf{T})/X_*(\mathbf{C})) \otimes_{\mathbb{Z}} \mathbb{R}$ the apartment which corresponds to the torus $\mathbf{T}$, cf. 4.1.2. The affine Weyl group determined by $\mathbf{T}$ acts on A, and this action has a relatively compact fundamental domain, which we denote by D. Because of the identity $gU^{(e)}_z g^{-1} = U^{(e)}_{gz}$, cf. 4.1.6, it suffices to prove the assertion of 6.2.6 for those z which lie in D. For any fixed e, the set of groups $\{U^{(e)}_z \mid z \in D\}$ is finite, as D is relatively compact. Recall that for fixed z the groups $U^{(e)}_z$ form a fundamental system of neighborhoods of 1 in G, cf. 4.1.7. Hence, given any affinoid subgroup $\mathbb{U} \subset \mathbf{G}^{an}$, there is $e_{st} \geq e_{uni}$ such that $\mathbb{U}^{(e)}_z \subset \mathbb{U}(L)$ for all $z \in D$ and all $e \geq e_{st}$. In step 3 below we exhibit a certain condition for an affinoid subgroup $\mathbb{U} \subset \mathbf{G}^{an}$. This condition is fulfilled by any sufficiently small affinoid subgroup $\mathbb{U}$. We will then show that there is a fundamental system $\{V_n\}_{n\geq 0}$ as in 6.2.4 such that $\mathbb{U}$ acts analytically on every V_n, in the sense of 6.2.5.

Step 2. According to [**?**, 2.17], the map $\vartheta_{\mathbf{B}} : \mathscr{B} \to X^{an}$ maps the apartment A into the analytification of the open subscheme $U_1 = \mathbf{N}^-\mathbf{B}/\mathbf{B} \subset X$, which is isomorphic to $\mathbf{N}^-$ (notation as in 5.1.3, 5.1.5). Put $\Psi = -\Phi^+(\mathbf{G},\mathbf{T})$. The choice of a Chevalley basis for $\mathfrak{g}$ gives coordinates $(X_\alpha)_{\alpha\in\Psi}$ on $\mathbf{N}^-$, hence on U_1. The points of the apartment A, considered as a subset of U_1^{an}, can then be described as norms on the algebra $L[(X_\alpha)_{\alpha\in\Psi}] = \mathcal{O}_X(U_1)$ as follows. To $z \in A$ corresponds the norm

$$L[(X_\alpha)_{\alpha\in\Psi}] \ni \sum_{\nu\in\mathbb{N}^\Psi} a_\nu X^\nu \quad\mapsto\quad \sup_\nu |a_\nu| \prod_{\alpha\in\Psi} e^{\nu(\alpha)\langle z,\alpha\rangle}\ ,$$

cf. [**?**, 2.17]. Here $\langle z,\alpha\rangle$ is the canonical pairing between co-characters and characters. The norm just described is the supremum norm on the polydisc $\mathbb{D}(r)$ with polyradius $r=(e^{\langle z,\alpha\rangle})_{\alpha\in\Psi}$. $\mathbb{D}(r)$ is an affinoid domain in U_1^{an}, and it is strictly affinoid if (and only if) all $e^{\langle z,\alpha\rangle}$ are in the extended value group $\sqrt{|L^*|}$. As D is a bounded subset of A, there are numbers $R_0>1>r_0$ in $|L^*|$ such that

$$r_0<\inf_{z\in D,\alpha\in\Psi} e^{\langle z,\alpha\rangle} \quad\text{and}\quad \sup_{z\in D,\alpha\in\Psi} e^{\langle z,\alpha\rangle}<R_0 \ .$$

In particular, D lies in the interior of the (strictly) affinoid polydisc $\mathbb{D}(R_0)$ with polyradius $(R_0,\dots,R_0)$. By [**?**, 3.4.6], U_1^{an} is an open subset of X^{an}. The polydisc $\mathbb{D}(R_0)$ is thus a neighborhood of the fundamental domain D. Because of this we will henceforth work on $\mathbb{D}(R_0)$.

Step 3. Let $\mathbb{C}_p$ be the completion of an algebraic closure of L. Let $\|\cdot\|$ be the maximum norm on $U_1^{an}(\mathbb{C}_p)={\mathbb{C}_p}^{\Psi}$, i.e., $\|(x_\alpha)_{\alpha\in\Psi}\|=\max_\alpha|x_\alpha|$. Fix $r_1\in(0,r_0)$, and choose a connected strictly affinoid subgroup $\mathbb{U}\subset\mathbf{G}^{an}$ which leaves U_1^{an} stable, and such that for all $g\in\mathbb{U}(\mathbb{C}_p)$ and all $x\in\mathbb{D}(R_0)(\mathbb{C}_p)$ one has $\|g(x)-x\|\le r_1$. Such an affinoid subgroup $\mathbb{U}$ exists because the action of $\mathbf{G}$ on X is algebraic. Put

$$\delta=\frac{r_1}{\inf_{z\in D,\alpha\in\Psi} e^{\langle z,\alpha\rangle}} \ ,$$

which is less than 1.

Step 4. From now on we fix a point $z\in D$ which we think of as a supremum norm on the polydisc $\mathbb{D}(r)$ with polyradius $r=(e^{\langle z,\alpha\rangle})_{\alpha\in\Psi}$. We remark that $|X_\alpha(z)|=e^{\langle z,\alpha\rangle}$. It follows from the very definition of the topology on the affinoid space $\mathbb{D}(R_0)$ that a fundamental system of neighborhoods of z is given by finite intersections of sets of the form

$$V_{f,c,C}=\{x\in\mathbb{D}(R_0)\mid c\le|f(x)|\le C\} \ ,$$

where $f\in\mathcal{O}(\mathbb{D}(R_0))=L\langle R_0^{-1}X\rangle$ and $c<|f(z)|<C$, cf. [**?**, 2.2.3 (iii)]. A particular example of such a neighborhood is the annulus

$$\mathbb{A}_{s,t}=\{x\in\mathbb{D}(R_0)\mid\forall\alpha\in\Psi: s_\alpha\le|X_\alpha(x)|\le t_\alpha\}=\bigcap_\alpha V_{X_\alpha,s_\alpha,t_\alpha} \ ,$$

where $s=(s_\alpha)_{\alpha\in\Psi}$ and $t=(t_\alpha)_{\alpha\in\Psi}$ are such that $s_\alpha<e^{\langle z,\alpha\rangle}<t_\alpha$ for all $\alpha\in\Psi$.

Let $\underline{r}_0=(r_0,\dots,r_0)$ be the tuple indexed by Ψ which has all components equal to r_0. Given a neighborhood $V_{f,c,C}$, we are now going to find real numbers $c'<C'$ in $\sqrt{|L^*|}$, and a tuple $r'=(r'_\alpha)_{\alpha\in\Psi}\in\sqrt{|L^*|}^{\Psi}$ such that

$$V'=V_{f,c',C'}\cap\mathbb{A}_{\underline{r}_0,r'}$$

has the following properties:

(i) V' is a neighborhood of z,

(ii) V' is contained in $V_{f,c,C}$,

(iii) V' is stable under the action of $\mathbb{U}$.

Step 5. It is straightforward to see that one can find real numbers $c' < C'$ in $\sqrt{|L^*|}$ with the following properties

$$c < c' < |f(z)| < C' < C \quad \text{and} \quad C'\delta < c' \ ,$$

where δ is as in step three. Furthermore, as f has supremum norm less than C' on the disk $\mathbb{D}(r)$, we can find $r' = (r'_\alpha)_{\alpha\in\Psi} \in \sqrt{|L^*|}^{\Psi}$ such that

- for all $\alpha \in \Psi$: $r'_\alpha > e^{\langle z,\alpha\rangle}$,
- f has supremum norm less or equal to C' on the disc $\mathbb{D}(r')$.

We remark that the affinoid group $\mathbb{U}$ acts on the strictly affinoid annulus $\mathbb{A}_{\underline{r}_0,r'}$. Moreover, the strictly affinoid domain $V' = V_{f,c',C'} \cap \mathbb{A}_{\underline{r}_0,r'}$ is a neighborhood of z. Our aim is to show that $\mathbb{U}$ also acts on V'. To see this, it is enough to work with $\mathbb{C}_p$-valued points. Write f as a power series

$$f(X) = \sum_{\nu\in\mathbb{N}^\Psi} a_\nu X^\nu \ .$$

Then we have $|a_\nu|(r')^\nu \le C'$. Consider $x \in V'(\mathbb{C}_p)$ and $g \in \mathbb{U}(\mathbb{C}_p)$. Expand f around x

$$f(x') = f(x) + \sum_{\nu\neq 0} b_\nu (x'-x)^\nu \ .$$

Then we also have $|b_\nu|(r')^\nu \le C'$ for all ν. Put $x' = g(x)$ and get $f(g(x)) = f(x) + \sum_{\nu\neq 0} b_\nu (g(x)-x)^\nu$. Using the inequality $\|g(x)-x\| \le r_1$ we find

$$|b_\nu(g(x)-x)^\nu| = |b_\nu|(r')^\nu \frac{|g(x_0)-x_0|^\nu}{(r')^\nu} \le C' \cdot \frac{r_1^{|\nu|}}{(r')^\nu} < C'\delta < c' \ .$$

We conclude that $|f(g(x)) - f(x)| < c'$ and thus

$$|f(g(x))| = |f(x) + f(g(x)) - f(x)| = |f(x)| \ .$$

This shows that $\mathbb{U}$ acts on the (strictly) affinoid neighborhood V'.

Step 6. In the general case, consider a neighborhood of z which is of the form $V = V_1 \cap \ldots \cap V_m$ with $V_i = V_{f_i,c_i,C_i}$. Then we find for each V_i a neighborhood V_i' stable under $\mathbb{U}$, as in step five. The intersection $V' = V_1' \cap \ldots \cap V_m'$ will then be a neighborhood which is stable by the action of $\mathbb{U}$.

Step 7. Now let $W_1 \supset W_2 \supset \ldots$ be a sequence of neighborhoods of z as in 6.2.4. Use step 6 to find an strictly affinoid neighborhood $W_1' \subset W_1$ of z on which $\mathbb{U}$ acts. W_1' is not necessarily irreducible. But note that irreducible and connected components coincide here, cf. [?, 3.1.8] (use that X^{an} is a normal space, by [?, 3.4.3]), and $\mathbb{U}$, being connected, will stabilize the connected component of W_1' containing z. Call this connected component V_1. It is again a strictly affinoid neighborhood of z. Then choose n such that W_n is contained in the topological interior of V_1, and let $W_n' \subset W_n$ be a neighborhood of z on which $\mathbb{U}$ acts (by step 6). Let V_2 be the connected component of W_n' containing z. Continuing this way we construct from $(W_n)_n$ a descending sequence of irreducible strictly affinoid neighborhoods $(V_n)_n$ with the same properties a s that in 6.2.4, but with the additional property that $\mathbb{U}$ acts on each V_n. This finishes the proof of 6.2.6. □

13. Appendix: Inductive limits of Banach spaces

Let K be a non-archimedian spherically complete field. All topological tensor products between locally convex K-vector spaces will be taken with respect to the projective tensor product topology. All inductive limits of locally convex spaces will be endowed with the locally convex inductive limit topology. In the following, $(V_n)_{n\in\mathbb{N}}$ will denote a countable inductive system of K-Banach spaces such that the continuous linear transition maps

$$i_n : V_n \longrightarrow V_{n+1}$$

are compact and injective. Furthermore, W will denote a fixed K-Banach space. In this set-up, one has a natural continuous linear map

$$\pi : \varinjlim V_n \hat{\otimes}_K W \longrightarrow (\varinjlim V_n)\hat{\otimes}_K W.$$

The main goal of this appendix is to prove the following proposition.

Proposition 13.0.11. *The continuous map π is an isomorphism of topological vector spaces.*

We begin with some preliminary lemmas. In the following all tensor products will be over K, and we let $V_n'' := ((V_n')_b)_b'$ denote the strong double dual.

Lemma 13.0.12. *With notation and assumptions as above, the double duality map*

$$V_n \longrightarrow V_n''$$

induces a continuous linear bijection:

$$\varinjlim V_n \hat{\otimes} W \longrightarrow \varinjlim V_n'' \hat{\otimes} W.$$

Proof. The double duality map induces a continuous linear map

$$V_n \hat{\otimes} W \longrightarrow V_n'' \hat{\otimes} W,$$

given by tensoring with the identity on the second factor. Therefore we have an induced continuous linear map

$$\varinjlim V_n \hat{\otimes} W \longrightarrow \varinjlim V_n'' \hat{\otimes} W.$$

By [?, Lemma 16.4] the double dual

$$V_n'' \longrightarrow V_{n+1}''$$

of the compact map i_n factors through (the image of) V_{n+1}. Therefore, the induced map

$$V_n''\hat{\otimes}W \longrightarrow V_{n+1}''\hat{\otimes}W$$

factors through (the image of) $V_{n+1}\hat{\otimes}W$. As a result,

$$\varinjlim V_n\hat{\otimes}W \longrightarrow \varinjlim V_n''\hat{\otimes}W$$

is a bijection. □

For two locally convex K-vector spaces U, V we denote by $\mathcal{L}_b(U,V)$ the space of continuous linear maps $U \to V$ endowed with the strong topology. According to [**?**, Lemma 18.1] the linear map

$$U_b' \otimes V \longrightarrow \mathcal{L}_b(U,V),\ \sum_{i=1,\ldots,r} \ell_i \otimes v_i \mapsto [u \mapsto \sum_{i=1,\ldots,r} \ell_i(u)\cdot v_i]$$

is a topological isomorphism onto its image. We let $V_n' := (V_n)_b'$ denote the strong dual of V_n. It is again a Banach space.

Lemma 13.0.13. *The continuous linear maps*

$$V_n''\hat{\otimes}W \longrightarrow \mathcal{L}_b(V_n',W)$$

induce a continuous linear bijection

$$\varinjlim V_n''\hat{\otimes}W \longrightarrow \varinjlim \mathcal{L}_b(V_n',W).$$

Proof. The transition map

$$\mathcal{L}_b(V_n',W) \longrightarrow \mathcal{L}_b(V_{n+1}',W)$$

is given by sending an element $\ell \in \mathcal{L}_b(V_n',W)$ to the composition

$$V_{n+1}' \xrightarrow{i_n'} V_n' \xrightarrow{\ell} W,$$

where i_n' is the dual of the compact map i_n. According to [**?**, 16.4 and 16.7] the linear maps i_n' and $\ell \circ i_n'$ are again compact. This means that the image of the transition map

$$\mathcal{L}_b(V_n',W) \longrightarrow \mathcal{L}_b(V_{n+1}',W)$$

is contained in the subspace of compact operators. On the other hand, by [**?**, 18.11], this subspace equals (the image of) $V_{n+1}''\hat{\otimes}W$. It follows that the transition map factors through $V_{n+1}''\hat{\otimes}W$, and therefore

$$\varinjlim V_n''\hat{\otimes}W \longrightarrow \varinjlim \mathcal{L}_b(V_n',W)$$

is a bijection. □

The dual maps $i_n' : V_{n+1}' \to V_n'$ give rise to a countable projective system of K-Banach spaces with compact continuous linear transition maps. We equip the projective limit $\varprojlim V_n'$ with the initial topology with respect to the projection maps

$$p_m : \varprojlim V_n' \longrightarrow V_m'$$

for any $m \in \mathbb{N}$.

Lemma 13.0.14. *The natural map*

$$\varinjlim \mathcal{L}_b(V_n', W) \to \mathcal{L}_b(\varprojlim V_n', W)$$

is a continuous linear bijection.

Proof. By the discussion following [?, 16.5] we may pass to a suitable cofinal projective system and may therefore assume additionally that each map p_n has dense image. By definition of the strong topology, the continuous linear map p_n induces a continuous linear map

$$\mathcal{L}_b(V_n', W) \to \mathcal{L}_b(\varprojlim V_n', W)$$

and, in the limit, a continuous linear map

$$\varinjlim \mathcal{L}_b(V_n', W) \to \mathcal{L}_b(\varprojlim V_n', W).$$

Since each p_n has dense image, the latter map is injective. To prove its surjectivity, we argue along the lines of the proof of [?, 19.9]: let $f : U \to W$ be a continuous linear map where $U := \varprojlim V_n'$. The inverse image $L := f^{-1}(B)$ of the unit ball $B \subseteq W$ is an open lattice in U and, consequently, f factores through the Hausdorff completion $U \to \widehat{U_L}$. By definition of the initial topology on U, there is $m \in \mathbb{N}$ such that the inverse image $L_m := p_m^{-1}(B_m)$ of the unit ball $B_m \subseteq V_m'$ is contained in L. The identity on U extends therefore to a continuous linear map $\widehat{U_{L_m}} \to \widehat{U_L}$. It remains to observe that, because p_m has dense image, it identifies with the completion map $U \to \widehat{U_{L_m}}$. □

Lemma 13.0.15. *The space $\mathcal{L}_b(\varprojlim V_n', W)$ is Hausdorff and complete.*

Proof. It follows from [?, 16.10] that $\varprojlim V_n'$ is a Fréchet space. Since Fréchet spaces are bornological by [?, 6.14], the assertion follows directly from [?, 7.16]. □

Lemma 13.0.16. *Let V and W denote locally convex topological vector spaces over K, and let $f : V \to W$ be a continuous linear bijection. If W is Hausdorff and complete, then V is Hausdorff and complete.*

Proof. Let W be Hausdorff and let $v \in V$ be a nonzero vector. There is an open lattice $L \subseteq W$ such that $f(v) \notin L$. Then $f^{-1}(L) \subseteq V$ is an open lattice with $v \notin f^{-1}(L)$. It follows from [?, 4.6] that V is Hausdorff. Now assume that W is complete and let $(v_i)_{i\in I} \subset V$ be a Cauchy net. Then $(f(v_i))_{i\in I} \subset W$ is also Cauchy. To prove this, first note that if $L \subset W$ is an open lattice, then $M := f^{-1}(L) \subset V$ is an open lattice. Since $(v_i)_{i\in I}$ is Cauchy, there exists $i \in I$ such that for all $k, j \geq i$, one has $v_k - v_j \in M$. It follows that for all $k, j \geq i$, one has $f(v_k) - f(v_i) \in L$. Therefore, $(f(v_i))_{i\in I}$ is a Cauchy net. Let w denote its limit. Then, since f is a bijection, there is a unique v such that $f(v) = w$. We claim that the net $(v_i)_{i\in I}$ converges to the vector v. To see this, note that for every $i \in I$ there is an open lattice $L_i \subset W$, such that for all $k \geq i$, one has $f(v_k) \in L_i$, and $\cap_{i\in I} L_i = \{f(v)\}$. Then $M_i := f^{-1}(L_i)$ give open lattices in V such that $v_k \in M_i$ for all $k \geq i$ and, since f is a bijection, $\cap_{i \in I} M_i = f^{-1}(\cap_{i \in I} L_i) = f^{-1}(f(v)) = v$. This proves the claim. $\square$

Proof. (Proposition 13.0.11) Let $V := \varinjlim V_n$ which is complete according to [?, 16.10]. We proceed as in the proof of [?, 1.1.32]. There is a commutative diagram:

$$\begin{array}{ccc} \varinjlim V_n \otimes W & \longrightarrow & V \otimes W \\ \downarrow & & \downarrow \\ \varinjlim V_n \hat{\otimes} W & \longrightarrow & V \hat{\otimes} W \end{array}$$

of locally convex spaces with continuous linear maps. By [?, 1.1.30 and 1.1.31] the top horizontal arrow is a topological isomorphism and the right vertical arrow is a topological embedding such that the target is identified with the Hausdorff completion of the source. Therefore, it suffices to prove the analogous statement for the left-hand vertical arrow. By the commutativity of the diagram, the left-hand vertical arrow is also a topological embedding. It clearly has dense image. Therefore, it suffices to show that $\varinjlim V_n \hat{\otimes} W$ is Hausdorff and complete. On the other hand, we have a chain of linear maps

$$\varinjlim V_n \hat{\otimes} W \to \varinjlim V_n'' \hat{\otimes} W \to \varinjlim \mathcal{L}_b(V_n', W) \to \mathcal{L}_b(\varprojlim V_n', W).$$

By the previous discussion, all these maps are continuous linear bijections, and $\mathcal{L}_b(\varprojlim V_n', W)$ is Hausdorff and complete. The assertion follows therefore from the preceding lemma. $\square$

Department of Mathematics, VU University Amsterdam - Faculty of Sciences, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
E-mail address: deeppatel1981@gmail.com

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany
E-mail address: toschmid@math.uni-muenster.de

Indiana University, Department of Mathematics, Rawles Hall, Bloomington, IN 47405, U.S.A.
E-mail address: mstrauch@indiana.edu