LOCAL MONODROMY OF GENERALIZED ALEXANDER MODULES

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Abstract. Given a semi-abelian variety G over a field $k \subset \mathbb{C}$, a morphism $f : X\backslash Y \to G$ of algebraic varieties, and a closed sub-variety $Y \subset X$, one can consider the cohomology groups

$$\mathrm{H}^i(X^{an}, Y^{an}; f^*\mathbb{Z}[\pi_1(G)])$$

where $\mathbb{Z}[\pi_1(G)]$ is the 'universal local system of $\mathbb{Z}[\pi_1(G(\mathbb{C}), e)]$-modules' on $G(\mathbb{C})$. If G is a Torus, then classical Alexander modules are examples of such groups. In this article, we study local monodromy as we vary the data (X, Y, G, f) in a family over a curve. In particular, we obtain an analog of the classical local monodromy theorem in this context. Our tools include a Mordell-Lang type result, a parametrized version of the basic lemma due to Beilinson, and a Galois theoretic weight argument.

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1. Introduction

In the following, we fix a field k of $char(k) = 0$ and an embedding $k \hookrightarrow \mathbb{C}$. Let G denote a semi-abelian variety over k and $e \in G(k)$ denote the identity. In this setting, we may consider the local system $\mathbb{Z}[\pi_1(G)]$ on $G(\mathbb{C})$ (cf. 2.1.1). Its stalk at a point $y \in G(\mathbb{C})$ is given by $\mathbb{Z}[\pi_1(G; e, y)]$, i.e., the free abelian group over homotopy classes of paths from e to y (see 2.1.1). It is naturally a local system of (free rank 1)

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$\mathbb{Z}[\pi_1(G,e)]$-modules.[1]

Let X be a variety over k, $Y \subset X$ be a closed subvariety, and $f : X\backslash Y \to G$ a morphism of varieties. One can consider the relative singular cohomology groups

$$\mathrm{H}^i(X^{an}, Y^{an}; f^*\mathbb{Z}[\pi_1(G)]) := \mathrm{H}^i(X^{an}, j_! f^*\mathbb{Z}[\pi_1(G)]), \tag{1.0.0.1}$$

where X^{an} and Y^{an} are the associated complex analytic spaces and $j : X\backslash Y \hookrightarrow X$ is the natural inclusion.[2] In this article, we study the local monodromy of these objects as we vary them in a family. We note that a more general local monodromy theorem in this context was proven in [7] using the theory of nearby cycles. The goal of this article is to provide a different proof via Galois theory and bypassing the theory of nearby cycles.

In a subsequent article, we shall construct a universal abelian category, denoted $\mathrm{GM}(k,G)$, of 'Gamma motives over G'. This category comes equipped with a canonical conservative and exact Betti realization functor

$$R_B : \mathrm{GM}(k;A) \to \mathrm{M}(\mathbb{Z}[\pi_1(G,e)]).$$

Here, $\mathrm{M}(\mathbb{Z}[\pi_1(G,e)])$ denotes the category of finitely generated $\mathbb{Z}[\pi_1(G,e)]$-modules. Any quadruple (X,Y,f,i), with (X,Y,f) as before and i a nonnegative integer gives rise to an object $\mathrm{H}^i(X,Y,f) \in \mathrm{GM}(k,G)$ whose Betti realization is $\mathrm{H}^i(X^{an}, Y^{an}; f^*\mathbb{Z}[\pi_1(G)])$. From this perspective, the present article studies the local monodromy of a family of 'Gamma motives'. We do not discuss Gamma motives in this article, since the monodromy results and tools presented here are likely to be of wider interest.

1.1. **The Local Monodromy theorem.** We begin by recalling the classical local monodromy theorem due to Grothendeick ([1]). In this section, we work in the setting of complex algebraic varieties. Let C be a smooth (connected) curve over $\mathbb{C}$, $c \in C$ be a fixed closed point, and $F : X \to C$ a morphism. Let Δ denote a small disk centered at c. The restriction of $R^iF_*\mathbb{Q}$ to a sufficiently small punctured disk $\Delta^\times$ is a local system with stalk $R^iF_*\mathbb{Q}_t = \mathrm{H}^i(X_t,\mathbb{Q})$ ($t \in \Delta^*$). In particular, for a general point $t \in \Delta^\times$ and a fixed base point $s \in \Delta^\times$, one has the corresponding local monodromy representation:

$$\rho : \pi_1(\Delta^\times, s) \to \mathrm{GL}(\mathrm{H}^i(X_t,\mathbb{Q})).$$

In this setting, the classical local monodromy theorem states that, if σ denotes the canonical generator of $\pi_1(\Delta^\times, s)$ (i.e. the counter-clockwise loop), then the eigenvalues of σ are roots of unity.

One of the main results of this article is an analog of the aforementioned local monodromy theorem in the context of generalized Alexander modules discussed above. For simplicity, we begin by recalling the main statement of our monodromy theorem in the following slightly specialized setting. With C as above, we assume that we are given the following data:

- (D1) A morphism $F : X \to C$ and a closed subscheme $Y \subset X$,
- (D2) A morphism $f : X\backslash Y \to \mathcal{G}$ (over C) where $\mathcal{G} := G \times C$. In particular, $\pi : \mathcal{G} \to C$ is a split semi-abelian scheme.

With these assumptions, we have the constant local system $\mathscr{R} := \mathbb{Z}[\pi_1(G,e)]$ of $R := \mathbb{Z}[\pi_1(G,e)]$-modules on the curve C and a local system $\mathcal{L}$ of $\pi^*(\mathscr{R})$-modules on $\mathcal{G}$, such that $\mathcal{L}|_{\mathcal{G}_t}$ is the local system $\mathbb{Z}[\pi_1(G)]$ on G for each $t \in C$. For each $t \in C$, consider the resulting morphism $f_t : X_t\backslash Y_t \to \mathcal{G}_t \cong G$. The cohomology groups $\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])$ form a local system on a sufficiently small punctured disk $\Delta^\times$ centered on $c \in C$. For simplicity, we assume that the cohomology group $\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])$ is a free R-module for all $t \in \Delta^\times$. Moreover, we assume that the aforementioned cohomology groups vanish in all other degrees (i.e., for all $j \neq i$). In this setting, the cohomology groups $\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])$ form a

[1] We use the algebraic geometry convention for composition of paths i.e. $\gamma_1 \cdot \gamma_2 := \gamma_1 \circ \gamma_2$.

[2] In the following, we often drop the superscript 'an' in the notation for the associated complex analytic space.

local system of free R-modules on $\Delta^\times$. Note that the monodromy action on $\mathscr{R}_t = \mathbb{Z}[\pi_1(G,e)]$ is trivial and therefore the monodromy action on $\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])$ is R-linear. In particular, for a general t (and fixed base point $s \in \Delta^\times$), we have a natural monodromy representation:

$$\rho : \pi_1(\Delta^\times, s) \to \mathrm{GL}_R(\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])).$$

Let K be a fixed algebraic closure of the the fraction field of R. By definition, we have an extension

$$1 \to T \to G \to A \to 1,$$

where T is a torus and A is an abelian variety. In particular, we have a natural inclusion

$$\pi_1(T, e) \hookrightarrow R^\times \hookrightarrow K^\times.$$

Let $P(x) \in R[x]$ denote the characteristic polynomial of the canonical generator σ of $\pi_1(\Delta^\times, s)$.

thm:monv1 **Theorem 1.1.1.** *With notation as above, let $P(x) = (x - \xi_1)\cdots(x - \xi_d)$ be a factorization of P in K. Then there exists a natural number m such that $\xi_1^m, \xi_2^m, \ldots, \xi_d^m$ are in the image of $\pi_1(T, e)$.*

We may identify R with the Laurent polynomial ring $\mathbb{Z}[t_1^\pm, \ldots, t_{r+r'}^\pm]$ where $r = \dim(T)$, $r' = 2\dim(A)$, and the natural inclusion $\pi_1(T, e) \hookrightarrow R$ has image given by the monomials in the t_i for $1 \leqslant i \leqslant r$. With this notation, the theorem posits the existence of a natural number m such that the m-th power of the eigenvalues of monodromy are monomials of the form $t_1^{a_1}\cdots t_r^{a_r}$ where $a_i \in \mathbb{Z}$.

Remark 1.1.2. Note that if G is taken to be a point, then the theorem specializes to the classical monodromy theorem.

Remark 1.1.3. The assumption that $\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])$ is free in degree i and vanishes in all other degrees can be replaced by simply requiring freeness in all degrees.

As an immediate consequence, one has the following corollary.

Corollary 1.1.4. *With notation and assumptions as in Theorem 1.1.1, suppose furthermore that G is an abelian variety. Then the eigenvalues of monodromy are roots of unity.*

Proof. In this case, the torus is trivial and $\pi_1(T, e) = 1$. The result now follows from the previous theorem. □

In the following, we prove the theorem in a more general setting. More precisely, we do not assume that the semi-abelian scheme $\mathcal{G}$ is split and allow for more general semi-abelian schemes $\mathcal{G} \to C$. We also remove the assumptions of free-ness and 'only one non-vanishing degree' on the cohomology groups considered above. More precisely, let Y, X and C be as above. Suppose we are given a semi-abelian scheme $\mathcal{G} \to C$ (with identity section $e : C \to \mathcal{G}$), a morphism $f : X\backslash Y \to \mathcal{G}$ over C such that there is a global extension (over C):

$$1 \to \mathcal{T} \to \mathcal{G} \to \mathcal{A} \to 1$$

where $\mathcal{T}$ is a torus (over C), and $\mathcal{A}$ is an abelian scheme (over C). In this setting, one has the following data (see 2.1.4):

(1) A local system $\mathscr{R}$ on C with fiber $\mathscr{R}_t = \mathbb{Z}[\pi_1(\mathcal{G}_t, e_t)]$.
(2) A local system $\mathcal{L}$ of $\pi^{-1}\mathscr{R}$-modules such that $\mathcal{L}|_{\mathcal{G}_t} = \mathbb{Z}[\pi_1(\mathcal{G}_t)]$.

Since $\mathcal{G}$ is globally an extension of $\mathcal{A}$ by $\mathcal{T}$, the morphism $\pi : \mathcal{G} \to C$ is a topological fibration. The local system $\mathscr{R}$ is constant on a small disk Δ centered at c, and we identify it with its fiber $R := \mathscr{R}_c = \mathbb{Z}[\pi_1(\mathcal{G}_c, e)]$. The cohomology groups $\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(\mathcal{G}_t)])$ form a local system of R-modules on a small disk punctured disk $\Delta^\times$ as before. In particular, one has a natural monodromy representation:

$$\rho : \pi_1(\Delta^\times, s) \to Aut_R(\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(\mathscr{G}_t)])).$$

Let $T := \mathcal{T}_c$, and K denote the algebraic closure of the fraction field of R.

thm:monv2 **Theorem 1.1.5.** *Let σ be the canonical generator of $\pi_1(\Delta^\times, s)$. With notation as above, there are natural numbers r, k and $m_1, \ldots, m_k \in \pi_1(T, e)$ such that*

$$(\sigma^r - m_1)\cdots(\sigma^r - m_k)\mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(\mathcal{G}_t)]) = 0.$$

Note that, as before, if $\mathcal{G}$ is an abelian scheme, then one has all $m_i = e$ and the local monodromy is quasi-unipotent.

arbmorphisms **Remark 1.1.6.** The content of the theorem is local around c. In particular, the existence of a global extension $1 \to \mathcal{T} \to \mathcal{G} \to \mathcal{A} \to 1$ is only required in a neighborhood of c.

1.2. **Strategy of Proof.** Our proof of the monodromy theorem is based on three ingredients:

(1) A Mordell-Lang type statement: More precisely, let K denote an algebraically closed field of characteristic 0 and $A = K[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_n^{\pm 1}]$ denote the corresponding Laurent polynomial ring. A *monomial* in A is an element of the form $t_1^{a_1} t_2^{a_2}\cdots t_n^{a_n}$ for some $(a_1, ..., a_n) \in \mathbb{Z}^n$. Finally, let $E := K(t_1, \ldots, t_n)$ denote the fraction field of A, and $\overline{E}$ denote a fixed algebraic closure of E. In this setting, we have the following result.

thm:algebraic **Theorem 1.2.1.** *Let $P(x; t_1, t_2, ..., t_n) \in A[x]$ be a monic polynomial and let $P = (x - \xi_1)\cdots(x - \xi_d)$ be its factorization in $\overline{E}$. The following assumptions on P are equivalent:*

(I) If $b, c_1, c_2, \ldots, c_n \in K$ are such that $P(b; c_1, c_2, \ldots, c_n) = 0$ and $c_1, \ldots, c_n$ are all roots of unity, then b is also a root of unity.

(II) There is a natural number r such that $\xi_1^r, \xi_2^r, \ldots, \xi_d^r$ are all monomials of A.

The above theorem follows from the Mordell-Lang conjecture in the Tori setting due to Laurent ([5]). However, we give a completely self-contained elementary proof of the theorem in section 3.1 below. In would be interesting to see if the proof given here can be extended to cover the more general statement of loc. cit. We hope to come back to this problem in the future.

Given the above theorem, an application of base change and the local monodromy theorem for unipotent local systems in the classical setting immediately allows one to deduce a slightly weaker version of Theorem 1.1.1 from the Theorem above. Namely, one obtains the analog of Theorem 1.1.1 where the eigenvalues are monomials (but not necessarily that they come from the Torus).

(2) A Galois theoretic weight argument: We define an etale analog of our cohomology groups in the setting where all our objects are defined over a field $k \subset \mathbb{C}$ (see 2.1.7). Over the algebraic closure these come equipped with a Galois action, and one has Artin's comparison theorem identifying these etale objects with the relative cohomology groups above (after passing to profinite completions at the level of fundamental groups). A weight argument using the Galois action allows one to deduce that the monomials appearing as eigenvalues must come from the Torus. We refer to sections 3.2 and 3.3 for the weight argument. This completes the proof of Theorem 1.1.1.

(3) A parametrized 'Basic Lemma': Finally, we prove a parametrized version of the 'Basic Lemma' due to Beilinson ([4], Lemma 3.3). In his earlier paper "Notes on Absolute Hodge Cohomology" ([3]), Beilinson produced a functor from Varieties over $\mathbb{C}$ to the derived category of mixed Hodge structures. Whereas Beilinson's Lemma 3.3 produces a functor from Varieties over a field k to the derived category of representations of $\mathrm{Gal}(k_{sep}/k)$ on finite-dimensional vector spaces over $\mathbb{Q}_\ell$ where l is any prime $\neq \mathrm{char}(k)$. The methodology for the proof of the parametrized version is taken from ([6]). This allows us to deduce 1.1.5 from 1.1.1 (see 4.6). We refer to section 4 for the precise statement of the parametrized basic lemma and the application to Theorem 1.1.5.

1.3. **Contents.** In Section 2 we recall some basic background and define our etale objects. In Section 3, we prove Theorems 1.1.1 and 1.2.1. Finally, in Section 4 we prove the parametrized basic lemma and apply it to prove Theorem 1.1.5.

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2. Preliminaries

In this section, we recall some background and basic facts about our cohomological objects. We consider etale analogs of our cohomological objects, and compare these with the Betti analogs.

2.1. **The Betti realization.** As before, let k be a fixed field of characteristic zero equipped with a fixed embedding $\sigma : k \hookrightarrow \mathbb{C}$, G denote a fixed semi-abelian variety over k, and $e \in G(k)$ the identity. We also fix an algebraic closure $k \subset \bar{k} \subset \mathbb{C}$.

allocalsystem 2.1.1. (The local system $\mathbb{Z}[\pi_1(G)]$) The category of local systems on G^{an} is equivalent to the category of $\mathbb{Z}[\pi_1(G,e)]$ modules. In particular, the identity map $\pi_1(G,e) \to \pi_1(G,e)$ gives rise to a natural local system on G^{an}, denoted by $\mathbb{Z}[\pi_1(G)]$. The fiber of $\mathbb{Z}[\pi_1(G)]$ at a point $y \in G(\mathbb{C})$ is given by $\mathbb{Z}[\pi_1(G;e,y)]$. Here, $\pi_1(G;e,y)$ denotes the homotopy classes of paths from e to y. Note that $\mathbb{Z}[\pi_1(G)]$ is a local system of (free rank one, left) $\mathbb{Z}[\pi_1(G),e]$-modules. If $\tilde{G} \xrightarrow{\pi} G^{an}$ is a fixed universal cover, then it is easy to see that $\pi_!\mathbb{Z} \cong \mathbb{Z}[\pi_1(G)]$. For example, consider the usual path space $PG^{an} \to G^{an} \times G^{an}$ and the pullback PG^{an}_e of this path space along the inclusion $G \times e \mapsto G \times G$. This construction gives an explicit model $\pi : PG^{an}_e \to G^{an}$ for the universal cover, and one can check that $\pi_!\mathbb{Z} \cong \mathbb{Z}[\pi_1(G)]$.

ra:prosystem 2.1.2. (Multiplication by m) Let $[m] : G \to G$ denote the isogeny induced by multiplication by m. The corresponding subgroup $\pi_1(G,e)^m \subset \pi_1(G,e)$ gives rise to the local systems

$$V_{m,n} := \mathbb{Z}/n\mathbb{Z}[\pi_1(G,e)/\pi_1(G,e)^m].$$

Note that this is also given by $[m]_*(\mathbb{Z}/n\mathbb{Z})$. These local systems form a natural prosystem, and the inverse limit (over m,n) will be denoted by $\widehat{\mathbb{Z}}[[\pi_1(G)]]$. Taking the limit over (ℓ^k,ℓ^k) (for a fixed prime ℓ) gives rise to a local system denoted by $\mathbb{Z}_\ell[[\pi_1(G)^{(\ell)}]]$. It is a local system of $\mathbb{Z}_\ell[[\pi_1(G,e)^{(\ell)}]] := \varprojlim \mathbb{Z}/\ell^k\mathbb{Z}[\pi_1(G,e)/\pi_1(G,e)^{\ell^k}]$-modules. Note that this ring is naturally a power series in $\dim(G)$-variables. If we choose generators x_i for $\pi_1(G,e)$, then it is a power series ring in the variables $x_i - 1$.

alelocsystem 2.1.3. (Etale local systems) One has an etale analog of the constructions of the previous paragraphs. Consider the (geometric) etale fundamental group $\pi_1^{et}(G_{\bar{k}},e)$. The multiplication by m map is defined over k, and one obtains etale local systems $V^{et}_{m,n}$ exactly as above over $\bar{k}$. Note that the base change of these etale local systems to $\mathbb{C}$ identify with the local systems $V_{m,n}$ on $G(\mathbb{C})$ (as a consequence of Grothendeick's comparison theorem for the etale fundamental groups). For future reference, we note that $\pi_1^{et}(G_{\bar{k}},e)$ has a canonical $\Gamma_k := Gal(\bar{k}/k)$-action.

ocinfamilies 2.1.4. (The local system in families) The constructions of the previous paragraphs can be performed in families as follows. In particular, let $\pi : \mathcal{G} \to S$ be a semi-abelian scheme (over the complex numbers), and $e : S \to \mathcal{G}$ denote the identity section. In this case, the exponential map gives rise to an exact sequence:

$$1 \to \mathcal{K} \to e^*\mathcal{T}(\mathcal{G}/S) \xrightarrow{h} \mathcal{G} \to 1$$

where $\mathcal{T}(\mathcal{G}/S)$ is the relative tangent bundle. This is an exact sequence of group schemes over S. By abuse of notation, we denote by $\mathcal{K}$ the corresponding sheaf of sections. Then the stalk $\mathcal{K}_s$ at $s \in S$ is given by $\pi_1(\mathcal{G}_s, e(s))$. Let $\mathscr{R} := \mathbb{Z}[\mathcal{K}]$. Then $\mathscr{R}$ is a sheaf of rings on S such that $\mathscr{R}_s = \mathbb{Z}[\pi_1(\mathcal{G}_s, e(s))]$. Setting $\mathcal{L} := h_!\mathbb{Z}$ gives a local system on $\mathcal{G}$. Moreover, it is a sheaf of $\pi^{-1}(\mathscr{R})$-modules. By construction, $\mathcal{L}|_{\mathcal{G}_s} = \mathbb{Z}[\pi_1(\mathcal{G}_s)]$ as a local system on $\mathcal{G}_s$. We may also define a families version, denoted by $\mathcal{L}_{m,n}$, of the local systems $V_{m,n}$ of the previous paragraph. Similarly, a family version $\mathcal{L}^{et}_{m,n}$ of $V^{et}_{m,n}$. We leave the details to the reader.

2.1.5. (The Betti realization) Given a morphism $f : X \to G$, we set

$$\mathrm{H}^k(X, f^*\mathbb{Z}[\pi_1(G)]) := \mathrm{H}^k(X^{an}, f^*\mathbb{Z}[\pi_1(G)]) = \mathrm{H}^i(R\Gamma(X^{an}, f^*\mathbb{Z}[\pi_1(G)])),$$

where $f^*\mathbb{Z}[\pi_1(G)]$ is the pull back local system. More generally, for $Y \subset X$ is a closed sub-variety, $j : U := X\backslash Y \hookrightarrow X$ the corresponding open immersion, and a morphism $f : X\backslash Y \to G$, we set

$$\mathrm{H}^k(X, Y; f^*\mathbb{Z}[\pi_1(G)]) := \mathrm{H}^k(X^{an}, Y^{an}; j_! f^*\mathbb{Z}[\pi_1(G)]).$$

We record some standard properties of these cohomology groups in the following remark for future reference.

i_properties **Remark 2.1.6.** With (X, Y, f) and G as above:

(1) The $\mathrm{H}^i(X, Y; f^*\mathbb{Z}[\pi_1(G)])$ are finitely generated $\mathbb{Z}[\pi_1(G, e)]$-modules.

(2) Given a morphism of tuples $F : (X', Y') \to (X, Y)$ over G (i.e. $F : X' \to X$ is a morphism over G where $F(Y') \subset Y$), one has a natural pull back morphism of $\mathbb{Z}[\pi_1(G, e)]$-modules:

$$F^* : \mathrm{H}^i(X, Y; f^*\mathbb{Z}[\pi_1(G)]) \to \mathrm{H}^i(X', Y'; f^*\mathbb{Z}[\pi_1(G)]).$$

(3) Given a closed subvariety $Z \subset Y$, one has a natural long exact sequence of $\mathbb{Z}[\pi_1(G, e)]$-modules

$$\cdots \to \mathrm{H}^i(X, Z; f^*\mathbb{Z}[\pi_1(G)]) \to \mathrm{H}^i(X, Y; f^*\mathbb{Z}[\pi_1(G)])$$
$$\to \mathrm{H}^i(Y, Z; f^*\mathbb{Z}[\pi_1(G)]) \to \mathrm{H}^{i+1}(X, Y; f^*\mathbb{Z}[\pi_1(G)]) \to \cdots$$

lerealizatin 2.1.7. (The etale realization) Consider now a triple (X, Y, f) where X is a scheme of finite type over k, $Y \subset X$ a closed subscheme, and $f : X\backslash Y \to G$ a morphism of schemes over k. Let $j : X\backslash Y \hookrightarrow X$ denote the natural inclusion, and $j_{\bar{k}}$ its base change. In this setting, we can consider the etale cohomology groups:

$$\mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}} V^{et}_{m,n}) := \mathrm{H}^i_{et}(X_{\bar{k}}, j_{\bar{k},!} f^*_{\bar{k}}(V^{et}_{m,n})).$$

We have the following standard properties of these etale cohomology groups:

(1) The cohomology groups $\mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}} V^{et}_{m,n})$ are finite $\mathbb{Z}/n\mathbb{Z}[\pi^{et}_1(G_{\bar{k}}, e)/(\pi^{et}_1(G_{\bar{k}}, e))^m]$-modules. Moreover, since the local systems $V^{et}_{m,n}$ are defined over k, these cohomology groups have a natural Γ_k-action which is semi-linear over the Γ_k-action on $\mathbb{Z}/n\mathbb{Z}[\pi^{et}_1(G_{\bar{k}}, e)/(\pi^{et}_1(G_{\bar{k}}, e))^m]$.

(2) Let $\mathbb{Z}_\ell[[\pi^{et}_1(G_{\bar{k}}, e)^{(\ell)}]] := \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}[\pi^{et}_1(G_{\bar{k}}, e)/(\pi^{et}_1(G_{\bar{k}}, e)^{\ell^n}]$. Taking the inverse limit over n of $\mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}} V^{et}_{\ell^n,\ell^n})$, we obtain a finitely generated $\mathbb{Z}_\ell[[\pi^{et}_1(G_{\bar{k}}, e)^{(\ell)}]]$-module denoted by $\mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}}(\mathbb{Z}_\ell[[\pi^{et}_1(G)^{(\ell)}]]))$. As above, one has an induced (continuous) Γ_k-action which is semi-linear over the Γ_k-action on $\mathbb{Z}_\ell[[\pi^{et}_1(G_{\bar{k}}, e)^{(\ell)}]]$. Note that this construction is functorial in morphisms of triples.

nverselimits **Remark 2.1.8.** Recall that the inverse limit functor is exact on prosystems of modules where each underlying module is a finite set. In our setting, the usual long exact sequences in etale cohomology gives rise to long exact sequences of pro-systems. These remain exact after passing to inverse limits. In particular, various standard long exact sequences in etale cohomology give rise to analogous long exact sequences of $\mathbb{Z}_\ell[[\pi^{et}_1(G_{\bar{k}}, e)^{(\ell)}]]$-modules in the setting above.

Base changing to $\mathbb{C}$, gives a natural map of pro-systems:

$$\mathrm{H}^i(X, Y; f^*(V_{m,n})) \to \mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}} V^{et}_{m,n}).$$

Moreover, by Artin's comparison theorem, this is an isomorphism of prosystems. In the following, we will identify $\mathbb{Z}_\ell[[\pi_1(G, e)^{(\ell)}]]$ with $\mathbb{Z}_\ell[[\pi^{et}_1(G_{\bar{k}}, e)^{(\ell)}]]$ using Grothendieck's comparison theorem. For future reference, we summarize the discussion above in the following proposition.

Proposition 2.1.9. *Let (X, Y, f, i) be as above.*

(1) There is a natural isomorphism of $\mathbb{Z}_\ell[[\pi_1(G,e)^{(\ell)}]]$-modules:

$$comp_{B,et} : \mathrm{H}^i(X, Y f^*(\mathbb{Z}[\pi_1(G)])) \otimes_{\mathbb{Z}[\pi_1(G,e)]} \mathbb{Z}_\ell[[\pi_1(G,e)^{(\ell)}]] \to \mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}}(\mathbb{Z}_\ell[[\pi_1^{et}(G)^{(\ell)}]])).$$

(2) There is a natural continuous $\mathbb{Z}_\ell[[\pi_1^{et}(G_{\bar{k}}, e)^{(\ell)}]]$-semi-linear Γ_k action on the etale cohomology groups $\mathrm{H}^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; f^*_{\bar{k}}(\mathbb{Z}_\ell[[\pi_1^{et}(G)^{(\ell)}]]))$.

2.2. **Comparison of etale and betti monodromy.** Let C be a smooth curve over a field k. We fix embeddings $k \hookrightarrow \bar{k} \hookrightarrow \mathbb{C}$. Let $c \in C(k)$, and $\mathcal{F}$ denote a local system of $\mathbb{Z}/n\mathbb{Z}$-modules on C.

Let B denote the completion of the local ring $\mathcal{O}_{C,c}$ at its maximal ideal. This is a complete dvr with residue field k and fraction field denoted by K. The local system $\mathcal{F}$ gives rise to a local monodromy representation:

$$\rho^{et} : Gal(\bar{K}/K) \to Aut(\mathcal{F}_{\bar{K}}).$$

On the other hand, we may consider the corresponding local system $\mathcal{F}^{an}$ on C^{an}. We fix a small disk Δ centered at c, and consider the corresponding representation

$$\rho : \pi_1(\Delta^*, s) \to Aut(\mathcal{F}^{an}_t)$$

where $s \in \Delta^*$ is a fixed base point, and $t \in \Delta^*$ some generic point. Choosing the standard positive generator allows us to identify $\mathbb{Z} = \pi_1(\Delta^*, s)$.

Following ([2], Expose XIV), we may identify the restriction of ρ^{et} to the geometric etale fundamental group with the corresponding representation ρ. In fact, in loc. cit. a more general result comparing vanishing cycles (defined in the etale and complex analytic settings) is proved. We recall here only the statement needed in the following.

More precisely, suppose $k = \mathbb{C}$. In this setting, the inertia group $I = Gal(\bar{K}/K) = \widehat{\mathbb{Z}}(1)$. We may (canonically) identify the latter with $\widehat{\mathbb{Z}}$, and we have the resulting inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z} = \widehat{\mathbb{Z}}(1)$. With this notation, one has an isomorphism $\mathcal{F}_{\bar{K}} \cong \mathcal{F}^{an}_t$ (functorial in $\mathcal{F}$) such that the resulting representations ρ^{et} and ρ are isomorphic when restricted to $\mathbb{Z}$. In the following, we shall apply this to the local systems $\mathcal{L}^{et}_{m,n}$ and $\mathcal{L}_{m,n}$ defined above (or rather their higher direct images to the base curve C).

3. Proof of Theorem 1.1.1

In this section, we give a proof of Theorem 1.1.1. In the first sub-section, we prove the key Theorem 1.2.1. In the second sub-section, we give a galois theoretic weight argument. Finally, the last sub-section completes the proof of Theorem 1.1.1.

3.1. **Proof of Theorem 1.2.1.** We begin by recalling the statement of the theorem. Let K denote an algebraically closed field of characteristic 0 and $A = K[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_n^{\pm 1}]$ denote the corresponding Laurent polynomial ring. A *monomial* in A is an element of the form $t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ for some $(a_1, ..., a_n) \in \mathbb{Z}^n$. Finally, let $E := K(t_1, \ldots, t_n)$ denote the fraction field of A, and $\overline{E}$ denote a fixed algebraic closure of E.

Theorem 3.1.1. *Let $P(x; t_1, t_2, ..., t_n) \in A[x]$ be a monic polynomial and let $P = (x - \xi_1) \cdots (x - \xi_d)$ be its factorization in $\overline{E}$. The following assumptions on P are equivalent:*

(I) If $b, c_1, c_2, \ldots, c_n \in K$ are such that $P(b; c_1, c_2, \ldots, c_n) = 0$ and $c_1, \ldots, c_n$ are all roots of unity, then b is also a root of unity.
(II) There is a natural number r such that $\xi_1^r, \xi_2^r, \ldots, \xi_d^r$ are all monomials of A.

We begin with an intermediate lemma.

lem:IandI'

Lemma 3.1.2. *With notation as in Theorem 3.1.1, suppose that (I) holds. Then there is a natural number r such that the following statement holds:*

(I'): If $b, c_1, c_2, \ldots, c_n \in K$ are such that $P(b; c_1, c_2, \ldots, c_n) = 0$ and $c_1, \ldots, c_n$ are all m-th roots of unity, then b^r is also an m-th root of unity.

Proof. The required r will be expressed as the product of the terms $r_1(d)$ and $e(F)$ defined below. We define $r_1(d)$ as follows. Let φ denote Euler's totient function. In particular, given a prime p, $\varphi(p^k) = p^{k-1}(p-1)$. For every prime p, let $k(p) := \max\{k \geqslant 0 : \varphi(p^k) \leqslant d\}$. For $p > (1+d)$, we see that $k(p) = 0$. We define $r_1(d)$ to be the product (over all primes p) of all the $p^{k(p)}$.

Claim 1: Let $M, M' \geqslant 1$ be positive integers. If M divides M' and $\varphi(M')/\varphi(M) \leqslant d$, then M' divides $r_1(d)M$.
Proof: The claim follows by noting that $\varphi(M'/M) \leqslant \varphi(M')/\varphi(M)$, and then expressing M'/M as a product of prime powers. If $M'/M = \prod p_i^{k_i}$ with p_i distinct primes, then $\prod \varphi(p_i^{k_i}) = \varphi(M'/M) \leqslant d$. It follows that $\varphi(p_i^{k_i}) \leqslant d$, and therefore $k(p_i) \geqslant k_i$. In particular, $p_i^{k_i} | r_1(d)$.

Next, let $F \subset K$ be the subfield generated by all the coefficients of P. Let $\bar{F}$ be the algebraic closure of F in K. Let $\chi : \mathrm{Gal}(\bar{F}/F) \to \widehat{\mathbb{Z}}^\times$ denote the cyclotomic character. Since F is a finitely generated field extension of $\mathbb{Q}$, the image of χ is an open subgroup $U \subset \widehat{\mathbb{Z}}^\times$. We define $e = e(F)$ to be the smallest natural number for which U contains $\ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/e\mathbb{Z})^\times)$. Let F_k be the cyclotomic extension of F obtained by adjoining all the k-th roots of unity. The very definition of $e(F)$ implies

{thm:eq1}

$$[F_{M'} : F_M] = \varphi(M')/\varphi(M) \text{ whenever } e(F)|M|M'. \quad (3.1.2.1)$$

As a consequence we have:

Claim 2: If $m \mid m'$ and $[F_{m'} : F_m] \leqslant d$, then $\frac{m'}{m} \mid r_1(d)e(F)$.
Proof: Let $M = \mathrm{l.c.m}(m, e(F))$ and $M' = \mathrm{l.c.m.}(m', e(F))$. Since $F_{M'} = F_{m'}F_M$, we have

$$[F_{M'} : F_M] \leqslant [F_{m'} : F_m] \leqslant d.$$

By 3.1.2.1, we get the inequality $\varphi(M')/\varphi(M) \leqslant d$. By Claim 1, we see that M'/M divides $r_1(d)$. Now, it is clear that m'/m divides $(M'/M)e(F)$. It follows that m'/m divides $r_1(d)e(F)$, as desired.

We will now show that (I) implies (I'). Let $P(b; c_1, \ldots, c_n) = 0$ where $c_i \in K$ are primitive m_i-th roots of unity for $i = 1, 2, \ldots, n$. Note that it suffices to prove (I') when $m = \mathrm{l.c.m.}(m_1, \ldots, m_n)$. To prove (I'), it has to be shown that $b^{mr} = 1$ where $r = r_1(d)e(F)$. Now b is a root of the monic degree d polynomial $P(x; c_1, ..., c_n) \in F(c_1, c_2, \ldots, c_n)[x]$. It follows that

$$[F(b, c_1, c_2, \ldots, c_n) : F(c_1, c_2, \ldots, c_n)] \leqslant d.$$

Clearly F_m equals $F(c_1, \ldots, c_n)$. If m' denotes the number of roots of unity in $F(b, c_1, \ldots, c_n)$, then this field is $F_{m'}$. Thus $[F_{m'} : F_m] \leqslant d$. Claim 2 shows m'/m divides r. The m'-th power of b is 1. It follows that $b^{mr} = 1$. This completes the proof of (I) implies (I'). □

Proof. (Theorem 3.1.1)
We first show that (I') implies (II). Consider $P_r := (x - \xi_1^r)(x - \xi_2^r) \cdots (x - \xi_d^r) \in \overline{E}[x]$. Considerations of the universal degree d polynomial show:

(i) $P_r \in A[x]$

(ii) if $(x - b_1) \cdots (x - b_d)$ is the factorization of $P(x; c_1, c_2, \ldots, c_n) \in K[x]$ for $c_1, \ldots, c_n \in K^\times$, then $(x - b_1^r) \cdots (x - b_d^r)$ is the factorization of $P_r(x; c_1, \ldots, c_n)$. In particular, if $P_r(b; c_1, \ldots, c_n) = 0$, then there exists $b' \in K$ such that $P(b'; c_1, \ldots, c_n) = 0$ and $b = (b')^r$.

(iii) Under the assumption of (I'), we deduce:
If $c_1, ..., c_n \in K^\times$ are m-th roots of unity, and if $P_r(b; c_1, \ldots, c_n) = 0$, then b is also an m-th root of unity.

Before proceeding further, we introduce some terminology to facilitate the reduction from n variables to one variable. For a natural number h, let

$$S(h) := \{t_1^{a_1} \cdots t_n^{a_n} : |a_i| \leqslant h, \forall 1 \leqslant i \leqslant (n-1)\}.$$

Let $V(h)$ denote the K-linear span of $S(h)$ in A.

In the following, q will be a 'large' odd natural number (to be specified later). Consider the K-algebra homomorphism $j : A \to A_1 := K[t^{\pm 1}]$ given by setting $j(t_1) = t$, and $j(t_i) = j(t_{i-1})^q$ for all $2 \leqslant i \leqslant n$. Every integer can be expressed uniquely as $a_1 + a_2 q + ... + a_{n-1} q^{n-1} + a_n q^n$ with $|a_i| \leqslant \frac{q-1}{2}$ for $i = 1, 2, ..., n-1$. This justifies statement (iv) below.

(iv) The ring homomorphism $j : A \to A_1$ restricts to a bijection from $S(\frac{q-1}{2})$ to $\{t^a | a \in \mathbb{Z}\}$. Consequently, j also restricts to an isomorphism of vector spaces from $V(\frac{q-1}{2})$ to A_1.

We have $s_1, s_2, \ldots, s_d \in A$ such that $P_r = x^d - s_1 x^{d-1} + \cdots + (-1)^d s_d$. Choose the least k_1 and k_2 such that $s_1 \in V(k_1)$ and $s_i \in V(k_2)$ for all $2 \leqslant i \leqslant d$. Let $q = 1 + 2\max(k_2, k_1 d)$.

(v) If there are integers $a_1, \ldots, a_d$ such that $j(P_r) = (x - t^{a_1}) \cdots (x - t^{a_d})$ then there are monomials $g_1, \ldots, g_d \in A$ such that the product $Q = (x - g_1) \cdots (x - g_d)$ is equal to P_r.
Proof of (v). By (iv), we obtain $g_i \in S(\frac{q-1}{2})$ such that $j(g_i) = t^{a_i}$ for $1 \leqslant i \leqslant d$. We now have $j(P_r) = j(Q)$ where

$$Q = (x - g_1)(x - g_2)...(x - g_d) = x^d - s_1' x^{d-1} + s_2' x^{d-2} + ... + (-1)^d s_d'$$

In particular, $j(s_1) = j(s_1')$. This implies $s_1 = s_1'$ by the second assertion of (iv), once it has been noted that
(a) $s_1 \in V(\frac{q-1}{2})$ because $k_1 \leqslant \frac{q-1}{2}$, and
(b) $s_1' \in V(\frac{q-1}{2})$, because s_1' is the sum of the g_i which lie in the same vector space.
Because $V(k_1)$ is spanned by monomials, and because $s_1 \in V(k_1)$ is itsef the sum of monomials g_i, it follows that all the g_i belong to $V(k_1)$. We see that $s_i' \in V(ik_1) \subset V(dk_1)$. Now $s_1, \ldots, s_d, s_1', \ldots, s_d'$ belong to $V(dk_1) + V(k_2)$, which by our choice of q, is contained in $V(\frac{q-1}{2})$. By assumption $j(s_i) = j(s_i')$, and by (iv) we conclude that $s_i = s_i'$ for all $1 \leqslant i \leqslant d$. This proves that $P_r = Q$.

We retain the F and $e(d)$ introduced in the proof of Lemma 3.1.2 above. Since $P \in F[t_1^{\pm 1}, \ldots t_n^{\pm 1}][x]$, it follows that P_r also belongs to the same ring. In particular, $H := j(P_r)$ belongs to $F[t^{\pm 1}][x]$. It remains to verify the hypothesis of (v) above for H.

By definition, $H(x; c) = P_r(x; c, c^q, c^{q^2}, \ldots)$ and by (iii), if c is a primitive m-th root of unity and $H(b; c) = 0$, then b is also an m-th root of unity. It follows that $H(x; c) = (x - c^{a_1}) \cdots (x - c^{a_d})$ with $|a_i| \leqslant \frac{m}{2}$ for all $i = 1, 2, \ldots, d$. Let $H'(x; t) := (x - t^{a_1}) \cdots (x - t^{a_d})$. We can write

$$H(x; t) = \Sigma_{i=0}^d (-1)^i u_i(t) x^{d-i} \text{ and } H'(x; t) = \Sigma_{i=0}^d (-1)^i u_i'(t) x^{d-i} \quad (3.1.2.2)$$

We will now take $m = p$ to be a prime number $> e(F)$. This ensures that the minimal polynomial of c over F is $\Phi_p(t) = 1 + t + ... + t^{p-1}$. Because $H(x, t) - H'(x, t)$ vanishes when $t = c$ and $t = 1$, we see that

$u_i(t) - u'_i(t)$ is divisible by $\mathrm{lcm}(\Phi_p(t), t-1) = t^p - 1$.

Let $W(k)$ denote the F-linear span of t^a with $|a| \leqslant k$, and choose $l_i \geqslant 0$ so that $u_i(t) \in W(l_i)$ for $i = 1, 2, ..., d$. Let $\frac{p-1}{2} \geqslant \max\{dl_1, l_2, l_3, ..., l_d\}$ for all $i = 1, 2, ..., d$. To proceed, we observe

(a) $W(\frac{p-1}{2}) \cap (t^p - 1)F[t^{\pm 1}]$ is zero
(b) $u_i(t) \in W(\frac{p-1}{2})$ for all $i = 1, 2, \ldots, d$
(c) $u'_1(t) \in W(\frac{p-1}{2})$

Taking $i = 1$ in (b), we employ (a) and (c) to deduce that $u_1(t) = u'_1(t)$. This implies that $u'_i(t)$ belongs to $W(il_1)$ (for all i), which is contained in $W(\frac{p-1}{2})$ by our choice of p. Now (a) and (b) imply that $u_i(t) = u'_i(t)$ for all i. This completes the proof of (I') implies (II).

Finally, note that the implication II implies I is standard: it follows, for instance, from property (ii) of P_r in the proof of I' implies II.

□

We believe that a stronger statement should be true. With K and A as before, let $P \in A[x]$ be a monic polynomial. For every natural number m, consider the hypothesis $H(m)$ below:

$H(m)$: If $c_1, c_2, ..., c_n \in K$ are m-th roots of unity and if $P(b; c_1, ..., c_n) = 0$, then b is a root of unity.

Let $F(P)$ be the field generated by the coefficients of P, $X(P)$ be the finite set of monomials that appear in P, and let $d(P) = \deg(P)$.

Conjecture 3.1.3. *There is a constant $C(d, F, X)$ defined for all natural numbers d, all finite subsets X of the set of monomials of A, and for all finitely generated subfields F of K with the following property: If there exists a natural number $m > C(d(P), F(P), X(P))$ for which $H(m)$ is valid, then some power of every root of P is a monomial in A.*

Our proof of the theorem proves the following weaker statement:

If there is a prime $p > C(d(P), F(P), X(P))$ for which $H(p)$ is valid, then the same conclusion holds.

In reality, the proof uses the set of monomials $X(P_r)$ rather than $X(P)$ itself. But the former gives an upper bound of the latter, so we obtain such a constant $C(d, P, X)$.

eightargument

3.2. **The Galois theoretic weight argument.** Let B denote a complete discrete valuation ring, k its residue field, and K its fraction field. Let K^s denote a fixed separable closure of K, and $\Gamma := \mathrm{Gal}(K^s/K)$. If C is the integral closure of B in K^s, then the residue field $\bar{k}$ of C is an algebraic closure of k, and we denote by $\Gamma' = \mathrm{Aut}_k(\bar{k})$ the group of k-automorphisms of the residue field. In this setting, we have the usual short exact sequence

$$1 \to I \to \Gamma \xrightarrow{s} \Gamma' \to 1$$

where $I := \ker(s)$ is the inertia subgroup, and s is the natural quotient map.

Let $\ell \neq \mathrm{char}(k)$ be a fixed prime, and $\chi_\ell : \Gamma' \to \mathbb{Z}_\ell^\times$ denote the cyclotomic character. On the other hand, usual Kummer theory gives rise to the standard Γ'-equivariant surjection:

$$c : I \to \mathbb{Z}_\ell(1).$$

In particular, one has the following relation:

-cyclotomic} (3.2.0.1) $$c(ghg^{-1}) = \chi_\ell(s(g))c(h) \ \forall g \in G, h \in I.$$

The following hypothesis will be assumed for the rest of the section:

The image of χ_ℓ is open.

For example, this is true if k is finitely generated over its prime field.

Let $(A, \mathfrak{m}_A)$ be a complete local ring with residue characteristic ℓ, $\mathrm{Aut}(A)$ the group of ring automorphisms, and $\sigma : \Gamma' \to \mathrm{Aut}(A)$ a continuous group homomorphism. Note that $\mathrm{Aut}(A) = \varprojlim \mathrm{Aut}(A/\mathfrak{m}_A^n A)$ and therefore has the natural inverse limit topology. In the following, we shall assume that the residue field of A is finite. Given a finitely generated A-module M, let $\mathrm{Aut}_{inv}(M) := \varprojlim \mathrm{Aut}(M/\mathfrak{m}_A^n M)$, where $\mathrm{Aut}(M/\mathfrak{m}_A^n M)$ denotes the group of automorphisms of the finite abelian group $M/\mathfrak{m}_A^n M$. Note that $\mathrm{Aut}_{inv}(M)$ also comes equipped with the natural inverse limit topology.

patibleaction **Definition 3.2.1.** A σ-compatible pair (M, ρ) is a finitely generated A-module M equipped with a continuous group homomorphism $\rho : \Gamma \to \mathrm{Aut}_{inv}(M)$ such that the following relation holds:

i-linearity} (3.2.1.1)
$$\rho(g)av = (\sigma(s(g))a)\rho(g)v \ \forall g \in \Gamma, a \in A, \text{ and } v \in M.$$

Remark 3.2.2. Since $I := \ker(s)$, it follows that $\rho(h)$ is an A-module automorphism of M for all $h \in I$.

In particular, we have an induced homomorphism $\rho|_I : I \to \mathrm{Aut}_A(M)$, where $\mathrm{Aut}_A(M)$ is the group of A-module automorphisms of M. Note that $\mathrm{Aut}_A(M) = \varprojlim \mathrm{Aut}_A(M/\mathfrak{m}_A^n M)$ has the natural inverse limit topology, and $\rho|_I$ is continuous with this topology (the subspace topology on $\mathrm{Aut}_A(M) \hookrightarrow \mathrm{Aut}_{inv}(M)$ is the aforementioned inverse limit topology). Since the residue field of A is finite, it follows that the kernel of the natural homomorphism

$$\mathrm{Aut}_A(M) \to \mathrm{Aut}_A(M/\mathfrak{m}_A M)$$

is a pro-ℓ-group. Therefore, the image of $\rho|_I$ is a pro-ℓ group. In particular, after passing to a finite separable extension K' of K, and replacing B by the integral closure of B in K', we may assume that the image of $\rho|_I$ in $\mathrm{Aut}_A(M/\mathfrak{m}_A M)$ is trivial. On the other hand, no subquotient of the kernel of $c : I \to \mathbb{Z}_\ell(1)$ has non-trivial ℓ-torsion. It follows that $\rho|_I$ factors factors through c i.e. there exists

$$\bar{\rho} : \mathbb{Z}_\ell(1) \to \mathrm{Aut}_A(M)$$

such that $\bar{\rho} \circ c = \rho|_I$.

In the following, we assume that the σ-compatible pair (M, ρ) satisfies the hypotheses above, and in particular a $\bar{\rho}$ is fixed for such a pair.

'equivariant **Lemma 3.2.3.** *Let (M, ρ) be a σ-compatible pair as above. In particular, we assume that $\rho|_I$ factors via $\bar{\rho}$ as above. Let $v \in M$ such that $\rho(I)(Av) \subset Av$, and $P_v := Ann(v) \subset A$ denote its annihilator.*

(1) For $g \in \Gamma'$, set $P_{gv} := Ann(\rho(g_1)v)$ for some lift $g_1 \in \Gamma$ of g. Then P_{gv} is independent of the chosen lift.
(2) Let $g \in \Gamma'$. Then $\sigma(g)P_v = P_{gv}$. One has an induced homomorphism

$$g \cdot [-] : (A/P_v)^\times \to (A/P_{gv})^\times$$

given by sending the class $[a]$ of $a \in A$ to $g \cdot [a] := [\sigma(g)a]$. This construction is compatible with the group structure of Γ'. In particular, $(g' \cdot [-]) \circ (g \cdot [-]) = g'g \cdot [-]$.
(3) Let $\theta_v : \mathbb{Z}_\ell(1) \to (A/P_v)^\times$ be defined as follows. For $h \in \mathbb{Z}_\ell(1)$, $\bar{\rho}(h)v = av$ for some $a \in A$. Let $\theta_v(h) := [a]$ where $[a]$ is the class of a in A/P_v. Then θ_v is a well-defined homomorphism.

(4) *Let $g \in \Gamma'$. Then $\rho(I)(A(\rho(g)v)) \subset A(\rho(g)v)$. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}_\ell(1) & \xrightarrow{\theta_v} & (A/P_v)^\times \\ \downarrow g & & \downarrow g\cdot[-] \\ \mathbb{Z}_\ell(1) & \xrightarrow{\theta_{gv}} & (A/P_{gv})^\times \end{array}$$

Here the Γ' action on the left is via the cyclotomic character.

Proof. (1) Let g_2 be another lift. We claim that $Ann(\rho(g_1)v) = Ann(\rho(g_2)v)$. Let $a \in A$ such that $a\rho(g_i)v = 0$. By σ-compatibility, this is true iff $\sigma(s(g_i^{-1}))(a)v = 0$. On the other hand, $\sigma(s(g_1^{-1})) = \sigma(s(g_2^{-1}))$.

(2) The first part results from a direct computation using the definition of σ-compatibility. If $g \in \Gamma'$ and $a \in P_v$, then $\sigma(g)(a)(\rho(g)(v)) = \rho(g)(av)$. It follows that $\sigma(g)P_v \subset P_{gv}$. A similar argument gives the reverse inequality. The compatibility with group structure follows easily from the definition.

(3) In order to see that θ_v is well defined, suppose $\bar{\rho}(h)v = a'v$ for some other $a' \in A$. Then $a' - a \in P_v$. It follows that $\theta_v(h) = [a]$ is well-defined. Note that $[a] \in (A/P_v)^\times$. If $\bar{\rho}(h^{-1})v = \tilde{a}v$, then

$$v = \bar{\rho}(h)(\bar{\rho}(h^{-1})v)) = \bar{\rho}(h)(\tilde{a}v) = a\tilde{a}v.$$

Here the last equality follows since $\bar{\rho}$ is A-linear. It follows that $a\tilde{a} - 1 \in P_v$, and therefore $[a] \in (A/P_v)^\times$. We leave it to the reader to show that θ_v is a group homomorphism.

(4) Let $k \in I$, and $a \in A$. Then $k' = g^{-1}kg \in I$, and we have $\rho(k')(v) = av$ for some $a \in A$. It follows that $\rho(k)(\rho(g)v) = \sigma(g)(a)(\rho(g)(v))$. Therefore, $\rho(I)(\rho(g)v) \subset A(\rho(g)v)$. This proves the first assertion. Now let $h \in \mathbb{Z}_\ell(1)$ and $g \in \Gamma'$. Then $g \cdot \theta_v(h) = [\sigma(g)a]$ where $\theta_v(h) = [a]$. On the other hand, $\theta_{gv}(g \cdot h) = [b]$ where $\bar{\rho}(g \cdot h)(\rho(g)v) = b(\rho(g)v)$. Here $g \cdot h$ denotes the action by the cyclotomic character. Let $\tilde{h} \in I$ be a lift of h. Then, by 3.2.0.1, we have $\bar{\rho}(g \cdot h)(gv) = \rho(g\tilde{h}g^{-1})(\rho(g)v) = \rho(g)(av) = \sigma(g)(a)(\rho(g)v)$. It follows that $[b] = [\sigma(g)(a)]$. □

eteproofthm1

3.3. **Completion of Proof of Theorem 1.1.1.** Recall, we are in the setting of Theorem 1.1.1. In particular, we are given a tuple (X, Y, f, G) and a curve C, all over $\mathbb{C}$, such that:

(1) Let $\mathcal{G} := G \times C$, and $\pi : \mathcal{G} \to C$ denote the projection map
(2) $F : X \to C$ is a morphism, and $c \in C(\mathbb{C})$ a fixed point
(3) $f : X \backslash Y \to \mathcal{G}$ is a morphism over C

We assume that there is an exact sequence

$$1 \to T \to G \to A \to 1$$

where $T = \mathbb{G}_m^r$ is an r-dimensional torus, and A is an abelian variety.

Remark 3.3.1. We fix a finitely generated subfield $k \subset \bar{k} \subset \mathbb{C}$ (where $\bar{k}$ is the algebraic closure of k in $\mathbb{C}$), so that our data above is defined over k. In particular, (X, Y, f, G), F, C are defined over k and $c \in C(k)$. Below, we shall add a subscript k or $\bar{k}$ to denote the corresponding object. For example, X_k is the variety over k, and X is its base change to $\mathbb{C}$ etc.

We now restrict to a small disk Δ centered at c, and consider the constant local system $\mathscr{R} = \mathbb{Z}[\pi_1(G, e)]$ on Δ, and the corresponding local system $\mathcal{L}$ of $\pi^*(\mathscr{R})$-modules on $\mathcal{G}$. Recall, $\mathcal{L}_t$ is identified with the local system $\mathbb{Z}[\pi_1(G)]$ for $t \in \Delta$. Let $j : X\backslash Y \hookrightarrow X$ denote the inclusion and consider $R^iF_*(j_!f^*\mathcal{L})$. First, note that on a small enough punctured disk $\Delta^\times$ this is a local system of $R := \mathbb{Z}[\pi_1(G, e)]$-modules. Moreover, its stalk at a point t is $\mathrm{H}^i(X_t, (j_t)_!f_t^*\mathcal{L}_t) = \mathrm{H}^i(X_t, Y_t; f_t^*(\mathbb{Z}[\pi_1(G)])$. Here $j_t : X_t\backslash Y_t \hookrightarrow X_t$ is the natural inclusion. Hence, we have a local system of R-modules, and the corresponding monodromy representation.

Suppose now that the cohomology groups above vanish for all $j \neq i$, and is a free R-module in degree i. We would like to show that the eigenvalues of monodromy are monomials. In particular, we have the monodromy representation

$$\rho : \pi_1(\Delta^\times, s) \to \mathrm{GL}_R(\mathrm{H}^i(X_t, Y_t, f_t^*\mathbb{Z}[\pi_1(G)])).$$

Let $P(x) = (x - \xi_1)\cdots(x - \xi_d)$ be a factorization of the characteristic polynomial of the canonical generator $\sigma \in \pi_1(\Delta^\times, s)$ as in Theorem 1.1.1. We wish to show that there is an r such that $\xi_i^r \in \pi_1(T, e)$ for all i.

(1) We first show that there is a natural number r such that ξ_j^r is a monomial for all $1 \leqslant j \leqslant d$ i.e. $\xi_j^r \in \pi_1(G)$. We only need to verify that $P(x)$ verifies hypothesis (1) of Theorem 3.1.1. We will see that this follows from an application of the classical local monodromy theorem. Consider the etale covering $[m] : G \to G$ given by multiplication by m, and let G_m denote the kernel. We have the resulting covering of $\mathcal{G}$. As before, let $(X\backslash Y)_m \to X\backslash Y$ denote the corresponding etale covering (given by base-change along f) with Galois group G_m, and denote the normalization of the above covering by $X_m \to X$. Note that $(X\backslash Y)_m = X_m\backslash Y_m$ where $Y_m : X_m \times_X Y$. Let R_m denote the group-ring of G_m, and $j_m : X_m\backslash Y_m \hookrightarrow X_m$ denote the inclusion. The sheaf $R_m \otimes_R j_! f^*\mathcal{L}$ is identified with the push-forward to X of $(j_m)_!\mathbb{Z}$. This identifies $R_m \otimes_R \mathrm{H}^i(X_t, (j_t)_! f_t^*\mathcal{L}_t)$ with the i-th cohomology of the inverse image of t of the pair (X_m, Y_m) i.e. $\mathrm{H}^i(X_{m,t}, Y_{m,t}; \mathbb{Z})$. In order to see this, note that $R_m \otimes^{\mathbb{L}} R\Gamma(X_t, (j_t)_! f_t^*\mathcal{L}_t) \cong R\Gamma(X_t, R_m \otimes (j_t)_! f_t^*\mathcal{L}_t)$. On the other hand, by our assumption of freeness and only one non-vanishing cohomology group, $R_m \otimes^{\mathbb{L}} R\Gamma(X_t, (j_t)_! f_t^*\mathcal{L}_t) = R_m \otimes_R \mathrm{H}^i(X_t, (j_t)_! f_t^*\mathcal{L}_t)$.[3] The classical monodromy theorem applies here; in particular, if we denote by $P_m(x) \in \mathbb{C}[x]$ the characteristic polynomial of monodromy action on $\mathrm{H}^i(X_{m,t}, Y_{m,t}; \mathbb{Z})$, then the roots of $P_m(x)$ are roots of unity. Let $\bar{P} \in R_m[x]$ be the image of $P_T \in R[x]$. The tensor product $\mathbb{C} \otimes R_m$ is the product of copies of $\mathbb{C}$ indexed by the $(c_1, \ldots, c_n)$ where $c_i^m = 1$ for all i. Therefore, under this identification, the image of $P_m(x)$ (in $\mathbb{C}[x]$) is the product of $P_T(x; c_1, \ldots, c_n)$ over all $(c_1, \ldots, c_n)$ where $c_i^m = 1$ for all i. On the other hand, by the discussion above, this product of polynomials is precisely $P_m(x)$. As we have seen, the latter has roots given by roots of unity. We conclude that $P(x) \in R[x]$ satisfies hypothesis (1) of Theorem 3.1.1.

(2) In order to deduce that the monomial must lie in $\pi_1(T, e)$, we shall apply the comparison with etale cohomology (as in 2.2) and use a weight argument. Specifically, let $M_b := \mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(G)])$, $R = \mathbb{Z}[\pi_1(G, e)]$, and $J \subset R$ denote the augmentation ideal. Let B denote the completion of $\mathcal{O}_{C_k,c}$ at its maximal ideal. By choosing a local parameter t (at c), we may identify $B = k[[t]]$ with fraction field $K = k((t))$. Let $\bar{K}$ be a fixed algebraic closure of K with residue field $\bar{k}$. Consider now the etale cohomology group $M_{et,\ell} := \varprojlim \mathrm{H}^i_{et}(X_{\bar{K}}, Y_{\bar{K}}; f_{\bar{K}}^* V^{et}_{\ell^k,\ell^k})$ with $V^{et}_{\ell^k,\ell^k}$ as in 2.1.3. Recall, $M_{et,\ell}$ is naturally a $\mathbb{Z}_\ell[[\pi_1^{et}(G_{\bar{K}}, e)^{(\ell)}]]$-module (see 2.1.7 for the definition) with an action of $\Gamma = Gal(\bar{K}/K)$ semi-linear action. Since etale fundamental groups are invariant under algebraically closed field extensions, we may identify $\mathbb{Z}_\ell[[\pi_1^{et}(G_{\bar{K}}, e)^{(\ell)}]] = \mathbb{Z}_\ell[[\pi_1^{et}(G_{\mathbb{C}}, e)^{(\ell)}]] =: \hat{R}$. Note that $\hat{R}$ is the $\ell R + J$-adic completion of R, and set $M = M_b \otimes_R \hat{R}$. We consider $M_{et,\ell}$ as an $\hat{R}$-module (as before via Grothendieck's comparison theorem for the etale fundamental group) and following the discussion in section 2.2, we have a natural isomorphism $M \cong M_{et,\ell}$ of $\hat{R}$-modules compatible with the monodromy action. More precisely, the monodromy induces a natural $\mathbb{Z}$-action on M. On the other hand, one has an action of inertia $I = \widehat{\mathbb{Z}}(1)$ on $M_{et,\ell}$, and the two actions are compatible under the canonical inclusion $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}(1)$.

(3) Recall that $\hat{R}$ has a natural (continuous) Γ-action, which factors through $\Gamma' = Gal(\bar{k}/k)$ (with notation as in section 3.2). We now view M as an $\hat{R}$-module with a Γ-semi-linear action as in Definition 3.2.1. Note that $\mathbb{Z}_\ell \otimes \pi_1(G, e)$ has the natural Γ'-action (identifying it with the

[3] Alternatively, we could instead assume all cohomology groups are free modules in order to obtain the same conclusion.

corresponding Tate module). On the other hand $\hat{R}$, and therefore $\hat{R}^\times$ has the natural Γ'-action; the inclusion $\mathbb{Z}_\ell \otimes \pi_1(G,e) \subset \hat{R}^\times$ is Γ'-equivariant.

(4) We now work with M, $\hat{R}$, B, and Γ as above. We assume that our data satisfy the hypotheses of section 3.2 (where the A in loc. cit. is taken to be $\hat{R}$). In particular, the action of inertia factors through $\mathbb{Z}_\ell(1)$ (recall, we can always achieve this upto replacing K by a finite extension). Our loop is the element $1 \in \mathbb{Z}$, and we consider it as an element $\gamma \in \mathbb{Z}_\ell(1)$ by the discussion above. We are concerned with the eigenvalues of $\rho(\gamma)$ (in the notation of 3.2).

(5) Consider an eigenvalue λ of γ, and $v \in M_b$ the corresponding eigenvector. We use the same notation v to denote $1 \otimes v \in M$. After passing to a finite extension, we may assume that $\lambda \in \pi_1(G,e)$ i.e. that it is a monomial. We would like to show that $\lambda \in \pi_1(T,e)$. This data gives rise to a morphism $\theta : \mathbb{Z}_\ell(1) \to \hat{R}^\times$ with notation as in Lemma 3.2.3. Note that in the current setting M is a free $\hat{R}$-module, and $\hat{R}$ is a domain. In particular, $P_v = P_{gv} = 0$. One has $\theta(\gamma) = \lambda$ and therefore $\theta(\gamma^m) = \lambda^m$ for all $m \in \mathbb{Z}$. By continuity, we see that this holds for all $m \in \mathbb{Z}_\ell$. In particular, θ factorizes as:

$$\mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell \otimes \pi_1(G,e)) \subset \hat{R}^\times.$$

On the other hand, Γ' 'equivariance' (i.e. the last part of Lemma 3.2.3) now shows that the action on λ must be via the cyclotomic character, and therefore $\lambda \in \pi_1(T,e)$. More precisely, one has an exact sequence of Tate modules

$$1 \to \mathbb{Z}_\ell \otimes \pi_1(T,e) \to \mathbb{Z}_\ell \otimes \pi_1(G,e) \to \mathbb{Z}_\ell \otimes \pi_1(A,e) \to 1,$$

and any Γ'-equivariant morphism $\mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell \otimes \pi_1(A,e)$ is zero (due to weight considerations).

4. Parametrized Basic Lemma

In this section, we prove Theorems 4.5.3 and 4.5.4. These statements are applications of Proposition 4.4.1 below, which is a parametrized version of Beilinson's lemma ([4]). More precisely, we prove below a parametrized version of the Basic Lemma (second form) in ([6], page 475). The last sub-section applies these results to prove Theorem 1.1.5.

4.1. **Conventions.** A subfield k of $\mathbb{C}$ remains fixed throughout, and we work with the category of separated k-schemes of finite type. Given such a k-scheme X, the set $X(\mathbb{C})$ of its $\mathbb{C}$-rational points, inherits its classical topology and the sheaf of holomorphic functions. This local ringed space will be denoted by X^{an}. We will utilize only the topology of X^{an} but not the sheaf $\mathcal{O}^{an}$ of holomorphic functions. The main objects considered below are sheaves of abelian groups on X^{an}; these will be referred to simply as "sheaves on X". Given a morphism of scheme $f : X \to Y$ over $\mathrm{Spec}(k)$, we will abuse notation further and write $R^q f_* \mathcal{F}$ in place of $R^q(f_{an})_* \mathcal{F}$.

4.2. **Base Change, Cohomological Base Change, and the sheaf $\mathcal{F}^Y$.**

A : We will often consider pairs $(X \xrightarrow{f} S, \mathcal{F})$ where f is a morphism of k-schemes and $\mathcal{F}$ is a sheaf on X. Given an S-scheme T, we obtain the **base-changed pair** $(X_T \to T, \mathcal{F}|_{X_T})$ where $X_T = X \times_S T$ and $\mathcal{F}|_{X_T} = p^{-1}\mathcal{F}$ with $p : X_T \to X$ denoting the first projection. Denoting by $f : X \to S$, $g : T \to S$ the given morphisms and by $f_T : X_T \to T$ the second projection, we have a natural base change morphism $g^{-1} R^q f_* \mathcal{F} \to R^q(f_T)_* \mathcal{F}|_{X_T}$.

B : We say that the pair $(X \to S, \mathcal{F})$ satisfies *cohomological base-change* (CBC) if

$$g^{-1} R^q f_* \mathcal{F} \to R^q(f_T)_* \mathcal{F}|_{X_T}$$

is an isomorphism for all $q \geqslant 0$ and for all S-schemes T.

C : If any two of the members of a short exact sequence of sheaves on X satisfy CBC with respect to a given morphism $X \to S$, it follows from the definition that the third also satisfies CBC with respect to $X \to S$.

D : Given a closed subscheme Y of X, and a sheaf $\mathcal{F}$ on X we define $\mathcal{F}^Y$ by

$$\mathcal{F}^Y := \ker(\mathcal{F} \to i_* \mathcal{F}|_Y),$$

where $i : Y \hookrightarrow X$ is the inclusion morphism.

E : If Y is closed in X and if CBC holds for both $(X \to S, \mathcal{F})$ and $(Y \to S, \mathcal{F}|_Y)$, it follows from 4.2 C above that CBC holds for $(X \to S, \mathcal{F}^Y)$.

F : As a corollary, if CBC holds for $(X \xrightarrow{f} S, \mathcal{F})$, then CBC holds for $(X \to S, \mathcal{F}^N)$ where $N = f^{-1}M$ and M is a closed subscheme of S.

G : If CBC holds for $(X \xrightarrow{f} S, \mathcal{F})$ and also for $(S \xrightarrow{g} T, R^q f_* \mathcal{F})$ for all $q \geqslant 0$, then CBC holds for $(X \xrightarrow{g \circ f} T, \mathcal{F})$. This assertion is immediate from the Leray spectral sequence.

H : We remind the reader that CBC holds for $(X \xrightarrow{f} S, \mathcal{F})$ whenever f is a proper morphism; in particular, when f is a finite morphism.

line

4.3. The affine line.

Lemma 4.3.1. *Let $\pi : \mathbb{A}^1_S \to S$ be the projection and Z be a closed subscheme of $\mathbb{A}^1_S$ such that $Z \to S$ is a finite surjective morphism. Let $\mathcal{F}$ be a sheaf on $\mathbb{A}^1_S$ such that*

(i) the restriction of $\mathcal{F}$ to the complement of Z is a locally constant sheaf, and

(ii) $\mathcal{F}|_Z = 0$.

Finally, let M be a closed subscheme of S and let $N := \pi^{-1}M$. Then the following assertions hold:

I : $(\mathbb{A}^1_S \xrightarrow{\pi} S, \mathcal{F})$ satisfies CBC.

II : $R^q \pi_* \mathcal{F} = 0$ for all $q \neq 1$.

III : For every $s \in S(\mathbb{C})$, there are x_i $(1 \leqslant i \leqslant r)$ in $\pi^{-1}(s)$ and a non-canonical isomorphism

$$\varphi(s) : (R^1 \pi_* \mathcal{F})_s \to \oplus_{i=1}^r \mathcal{F}_{x_i}.$$

If $\mathcal{R}$ is a sheaf of rings on S, and if $\mathcal{F}$ is a sheaf of $\pi^{-1}\mathcal{R}$-modules, then $\varphi(s)$ is an isomorphism of $\mathcal{R}_s$-modules.

IV : With M and N as above, properties I and II also hold for $(\mathbb{A}^1_S \xrightarrow{\pi} S, \mathcal{F}^N)$; see 4.2 D for the definition of $\mathcal{F}^N$.

V : Assume that $Z \smallsetminus Z \cap N \to S \smallsetminus M$ is etale. Then $R^1 \pi_* \mathcal{F}$ restricts to a locally constant sheaf on $S \smallsetminus M$.

Proof. Parts I and II are special cases of proposition1.3A and corollary 1.3B of ([6], page 477). Moreover, Remark 1.4 of ([6], page 479) proves part III. The isomorphism $\varphi(s)$ depends on the choice of a tree in $\pi^{-1}(s)$ with $\pi^{-1}(s) \cap Z$ as its vertices, and the x_i are interior points of its edges. The isomorphism is obtained from a Mayer-Vietoris sequence, and therefore commutes with all endomorphisms of the given sheaf. In particular, if $\mathcal{F}$ is a sheaf of $\pi^{-1}\mathcal{R}$-modules, the isomorphism chosen is in fact an isomorphism of $\mathcal{R}_s$-modules. Part IV follows from 4.2 F. The point is that I and II are valid for the base-changed pair $(N \to M, \mathcal{F}|_N)$ since this pair satisfies (i) and (ii). Finally, Remark 1.5 on page 479 of ([6]), proves V. □

4.4. The general case.

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Proposition 4.4.1. *Let S be an irreducible scheme and η its generic point. Let $\pi : X \to S$ be an affine morphism such that $\dim X_\eta = n$. Let $\mathcal{F}$ be a sheaf on X such that $\mathcal{F}|_U$ is locally constant for some open dense $U \subset X$. Then there is a closed subscheme Y of X with $\dim Y_\eta < n$ and a nonempty open $T \subset S$ for which the following statements are valid for $\mathcal{F}^Y$:*

(A) The base-changed pair $(X_T \to T, \mathcal{F}^Y|_{X_T})$ satisfies CBC, with $\mathcal{F}^Y$ as in 4.2 D.
(B) $\mathrm{R}^q\pi_*(\mathcal{F}^Y)|_{X_T} = 0$ for all $q \neq n$, and $\mathrm{R}^n\pi_*(\mathcal{F}^Y)|_{X_T}$ is a locally constant sheaf.
(C) For every $s \in S(\mathbb{C})$, there are $x_1, x_2, ..., x_r \in \pi^{-1}(s)$ and an isomorphism

$$\varphi(s) : \mathrm{R}^n\pi_*(\mathcal{F}^Y) \xrightarrow{\cong} \oplus_{i=1}^r \mathcal{F}_x.$$

If $\mathcal{F}$ is a sheaf of $\pi^{-1}\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings on S, then $\varphi(s)$ can be chosen to be an isomorphism of $\mathcal{R}_s$-modules.
(D) In particular, if $\mathcal{F}|_U$ is a locally free sheaf of $\pi^{-1}\mathcal{R}$-modules of finite rank, then $\mathrm{R}^n\pi_*(\mathcal{F}^Y)$ is a locally free sheaf of $\mathcal{R}$-modules of finite rank, when restricted to T.

Proof. The proposition is a statement about affine morphisms $\pi : X \to S$ where the target S is irreducible. Given a sheaf $\mathcal{F}$ on X, we express $\pi : X \to S$ as a composite $X \xrightarrow{f} \mathbb{A}^r_S \xrightarrow{p} S$. We will assume the proposition has been proved for p, and that an even stronger form of the proposition is valid for $f : X \to \mathbb{A}^r_S$. More precisely, we will assume that there is a closed subvariety $Y \subset X$ ($\dim(Y_\eta) < n$) and a nonempty open $T \subset S$ such that the following hold:

(A') The base-changed pair $(X_T \to \mathbb{A}^r_T, \mathcal{F}^Y|_{X_T})$ satisfies CBC,
(B') $\mathrm{R}^q f_*(\mathcal{F}^Y)_{X_T} = 0$ for all $q \neq n-r$, and $\mathrm{R}^{n-r} f_*(\mathcal{F}^Y)_{X_T}$ is a locally constant sheaf on $\mathbb{A}^r_T$.
(C') The obvious analogue of part (C) is valid for $(X_T \to \mathbb{A}^r_T, \mathcal{F})$.

The proposition for $f : X \to \mathbb{A}^r_S$ would require an *arbitrary nonempty subset* of $\mathbb{A}^r_S$, whereas the above stronger form allows only open subsets of the type $\mathbb{A}^r_T$.
To continue, in view of (B'), we are in a position to apply the proposition to the sheaf $\mathcal{G} := \mathrm{R}^{n-r} f_*\mathcal{F}^Y$ and the morphism $p : \mathbb{A}^r_S \to S$ and obtain a closed subvariety $M \subsetneq \mathbb{A}^r_S$ and a nonempty $T' \subset S$ satisfying

(A") The restriction of $\mathcal{G}$ to $\mathbb{A}^r_{T'}$ satisfies CBC for $p_{T'} : \mathbb{A}^r_{T'} \to T'$
(B") $\mathrm{R}^q p_*\mathcal{G}|_{T'}$ is zero for $q \neq r$, and is locally constant for $q = n-r$.
(C") the obvious analogue of part(C) of the proposition holds for $\mathrm{R}^{n-r}p_*\mathcal{G}$.

Let $N := f^{-1}M$ and $W := Y \cup N$. From (A') and 4.2 F, we see that CBC holds for the restriction of the sheaf $\mathcal{F}^W = (\mathcal{F}^Y)^N$ to X_T and the morphism $X_T \to \mathbb{A}^r_T$. Clearly $\mathrm{R}^{n-r} f_*\mathcal{F}^N = \mathcal{G}^M$. In view of (A",B',B") it follows that parts (A) and (B) of the proposition hold for $\pi : X \to S$ with the open $T \cap T' \subset S$ and the closed $W \subset X$. That (C') and (C") imply (C) is evident.
We will complete the proof of the proposition for $X = \mathbb{A}^n_S$ in Step 1 below and the general case in Step 2. Both cases are essentially Noether normalization at η.
Step 1. We will prove the proposition for $\pi : \mathbb{A}^n_S \to S$ by induction on $n \geqslant 0$, the case $n = 0$ being trivial. The sheaf $\mathcal{F}$ on $\mathbb{A}^n_S$ is locally constant on the complement of a (reduced) hypersurface V of $\mathbb{A}^n_S$. After a linear change of variables, we may assume that the projection $f : \mathbb{A}^n_S \to \mathbb{A}^{n-1}_S$ restricts to a finite morphism $V_T \to \mathbb{A}^{n-1}_T$ for some nonempty open $T \subset S$. In view of 4.3, we see that $\mathcal{F}^Y$ satisfies the above (A',B',C') if $Y = f^{-1}D$ and $(V \backslash V \cap Y)_T \to (\mathbb{A}^{n-1} \backslash D)_T$ is etale. Thus the stronger form has been proved for $(\mathbb{A}^n_S, \mathcal{F})$ and the result follows by induction on n.
Step 2. Noether normalization for X_η yields a morphism $f : X \to \mathbb{A}^n_S$, such that $f_T : X_T \to \mathbb{A}^n_T$ is a finite morphism. Choose a a closed subvariety $D \subsetneq \mathbb{A}^n_S$ such that
(a) $(X \backslash f^{-1}D)_T \to \mathbb{A}^n \backslash D)_T$ is etale,
(b) the restriction of $\mathcal{F}$ to the complement $(\mathbb{A}^n \backslash D)_T$ is locally constant. We see then that $Y = f^{-1}D$ satisfies the requirements of (A',B',C'). Because the proposition has been proved for $\mathbb{A}^n_S \to S$, the proposition for the sheaf $\mathcal{F}$ on X follows. □

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4.5. **Weakly constructible sheaves.**

Definition 4.5.1. A sheaf $\mathcal{F}$ on X (in the sense of 4.1) is weakly constructible if there is a finite collection $Z_1, Z_2, ..., Z_r$ of locally closed subvarieties of X such that

(i) $\mathcal{F}$, when restricted to Z_i, is a locally constant sheaf, for every $i = 1, 2, ..., r$
(ii) $X = Z_1 \cup Z_2 \cup \cdots \cup Z_r$.

Remark 4.5.2. (1) Weakly constructible sheaves form an abelian category.
(2) If $\mathcal{R}$ is an arbitrary sheaf of rings on X, then weakly constructible sheaves on $\mathcal{R}$-modules on X also form an abelian category.

appthm **Theorem 4.5.3.** *Let $\pi : X \to S$ be an affine morphism, where S is irreducible with generic point η. Let $n = \dim X_\eta$. Let $\mathcal{F}$ be a weakly constructible sheaf on X. We consider increasing sequences of closed subvarieties $X_0 \subset X_1 \subset ... \subset X_n = X$ with $\dim(X_i)_\eta \leqslant i$ for all $i = 0, 1, 2, ...$ We denote by π_i the composite $X_i \hookrightarrow X \to S$. Such a sequence is admissible if the sheaves $\mathcal{F}_i := \mathcal{F}^{X_{i-1}}|_{X_i}$ satisfy the conditions below for some nonempty $T \subset S$:*

(A) The sheaf $\mathcal{F}_i$ satisfies CBC for the given morphism $(X_i)_T \to T$
(B) $R^q(\pi_i)_*\mathcal{F}_i$ is zero for $q \neq i$ and is locally constant for $q = i$, when restricted to T.

We then have:

(I) There exists an admissible increasing sequence.
(II) More generally, given an increasing sequence $X'_i : i = 0, 1, ..., n$ that satisfies the dimension restriction, there is an admissible increasing sequence $X_i : i = 0, 1, ..., n$ such that $X'_i \subset X_i$ for all $i = 0, ..., n$

Proof. Part (I) follows from part (II) by taking $X'_i = \varnothing$ for $i < n$ For part (II), one constructs the sequence $X_i : 0 \leqslant i \leqslant n$ by decreasing induction on i. Assume that X_i has been chosen so that $X'_i \subset X_i$. If $\dim(X_i)_\eta < i$, we take $X_{i-1} = X_i$. We now assume $\dim(X_i)_\eta = i$. Because $\mathcal{F}$ is weakly constructible, it follows that its restriction to X_i is also weakly constructible. Thus, $\mathcal{F}$ restricts to a locally constant sheaf on an open dense subset U of X_i. Let $Z = X'_{i-1} \cup (X_i \backslash U)$. We apply Proposition 4.4.1 to $(\pi_i : X_i \to S, \mathcal{F}^Z|_{X_i}$ and obtain a closed subvariety $Y \subset X_i$ such that the sheaf $\mathcal{F}^{Z \cup Y}|_{X_i}$ satisfies all parts of the proposition. We define $X_{i-1} = Y \cup Z$. □

appthmm **Theorem 4.5.4.** *With notation and assumptions as in the previous theorem, assume that $\mathcal{F}$ is a sheaf of $\pi^{-1}\mathcal{R}$-modules, where $\mathcal{R}$ is a locally constant sheaf of rings on S. Assume also that the stalk of $\mathcal{F}$ at every point of $x \in X(\mathbb{C})$ is a finitely generated free $\mathcal{R}_{\pi(x)}$-module. Define a sequence to be strongly admissible if the extra condition: "$R^i(\pi_i)_*\mathcal{F}_i$ is a locally constant sheaf of finitely generated free $\mathcal{R}$-modules of finite rank, when restricted to a nonempty open subset of S" holds.*
Then both parts of the above theorem are valid with 'admissible' replaced by 'strongly admissible'.

The proof of theorem 4.5.3 is valid for theorem 4.5.4 as well.

ofgeneralthm 4.6. **Proof of Theorem 1.1.5.** In the setting of Theorem 1.1.5, $M := \mathrm{H}^i(X_t, Y_t; f_t^*\mathbb{Z}[\pi_1(\mathcal{G}_t)])$ is no longer assumed to be a free R-module (concentrated in one cohomological degree) and our abelian scheme $\mathcal{G}$ is no longer assumed to be split. We now explain how to deduce Theorem 1.1.5 from Theorem 1.1.1 and Theorem 4.5.4.

(1) First, note that the proof of Theorem 1.1.1 is valid in the setting where the semi-abelian scheme is not necessarily split (still under the assumptions that the relevant cohomology group is free in degree i and zero in other degrees). The key point is that the local system $\mathscr{R}$ is constant in the small disk, and $\mathcal{G} \to C$ is a topological fibration, and therefore the cohomology group M has an R-linear monodromy action.
(2) Note that R-modules with a $\pi_1(\Delta^\times, s)$-action satisfying the conditions of Theorem 1.1.5 (i.e. annihilated by an element of the form given in loc. cit.) form an abelian category.
(3) Suppose now that $F : X \to C$ is an affine morphism. By Theorem 4.5.4, our M is a sub-quotient of an object appearing of the type appearing in 1.1.1. Therefore, by Theorem 1.1.1 (in view of (1) above), it is annihilated by an element of the type given in 1.1.5.

(4) In general, first fix a finite affine cover $(U_i)_{i\in I}$ of X. Then we may consider the Cech spectral sequence for RF_* (recall $F : X \to C$ is the structure map) for this cover. After restricting to a small enough punctured disk around c, we may assume that all derived push-forwards appearing in the spectral sequence are local systems on this punctured disk. One can achieve this since we have a finite cover, and our complexes are cohomologically bounded. Now one uses the fact that the objects being dealt with form an abelian category.

(5) Note that if X is quasi-projective, then we can immediately conclude from (3) via Jouanalou's trick. More precisely, the statement is local around $c \in C$, and therefore we may assume C is affine. We now use Jouanalou's trick to replace X by an affine scheme which is an affine torsor over X.

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