ON A LOCALIZATION FORMULA OF EPSILON FACTORS VIA MICROLOCAL GEOMETRY

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ABSTRACT. Given a lisse l-adic sheaf $\mathcal{G}$ on a smooth proper variety X and a lisse sheaf $\mathcal{F}$ on an open dense U in X, Kato and Saito conjectured a localization formula for the global l-adic epsilon factor $\varepsilon_l(X, \mathcal{F} \otimes \mathcal{G})$ in terms of the global epsilon factor of $\mathcal{F}$ and a certain intersection number associated to $\det(\mathcal{G})$ and the Swan class of $\mathcal{F}$. In this article, we prove an analog of this conjecture for global de Rham epsilon factors in the classical setting of $\mathcal{D}_X$-modules on smooth projective varieties over a field of characteristic zero.

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1. INTRODUCTION

Let X denote a smooth proper variety of dimension d over a finite field F of characteristic p, and let $\mathcal{G}$ be a smooth étale $\mathbb{Q}_l$ (resp. $\mathbb{F}_l$) sheaf. Then, one has the usual global l-adic epsilon factor

$$\varepsilon_l(X, \mathcal{F}) := \prod_{q=0}^{2d} \det(-\sigma : \mathrm{H}^q_c(U_{\bar{F}}, \mathcal{F}))^{(-1)^q},$$

where $\sigma \in \mathrm{Gal}(\bar{F}/F)$ is the geometric Frobenius. In this setting, Kato and Saito conjectured the following 'localization' formula for the epsilon factor of the tensor product:

Conjecture. *([3], 4.3.11) Let $\mathcal{F}$ be a constructible sheaf on X, and $\mathcal{G}$ be a smooth sheaf on X. Then one has*

$$\varepsilon_l(X, \mathcal{F} \otimes \mathcal{G}) = \varepsilon_l(X, \mathcal{F})^{\mathrm{rk}(\mathcal{G})} \cdot \langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F}) \rangle.$$

Here $r_{\mathcal{G}}$ denotes the rank of $\mathcal{G}$, and $\langle -, - \rangle$ denotes a pairing defined using the class field theory which we do not recall here.

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When X is a proper smooth variety over a field k of characteristic 0, the second author constructed the de Rham epsilon factor formalism in ([8]). More precisely, let $\mathrm{K}(\mathcal{D}_X)$ denote the K-theory spectrum of coherent $\mathcal{D}_X$-modules, and $\mathrm{K}(T^*X)$ denote the K-theory spectrum of coherent sheaves. Then he constructed a map of spectra

$$\varepsilon\colon \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X).$$

At the level of Grothendieck groups, given a holonomic module $\mathcal{F}$, $[\varepsilon(\mathcal{F})] \in \mathrm{K}_0(T^*X)$ is the class $[\mathrm{gr}^F(\mathcal{F})]$ where F is a good filtration of $\mathcal{F}$. It is well-known that the class is independent of the choice of good filtration. The composition of ε with pullback by certain twist of the zero-section followed by the push-forward $R\Gamma\colon \mathrm{K}(X) \to \mathrm{K}(k)$ is homotopic to the de Rham cohomology map $R\Gamma_{\mathrm{dR}}$ (cf. Lemma 2.7.4). In particular, passing to Grothendieck groups, we may proceed via ε in order to compute the Euler-Poincaré characteristic. Moreover, an automorphism f of $\mathcal{F}$ determines an element in $\pi_1\mathrm{K}(\mathcal{D}_X)$ whose image under the morphism $R\Gamma_{\mathrm{dR}}$ gives an element of $\pi_1\mathrm{K}(k) \cong k^\times$. The latter is precisely the determinant of the induced automorphism on the de Rham cohomology of $\mathcal{F}$. In ([8]), a 'microlocalized' version of ε was also constructed, which allows one to pass to the K-theory of holonomic $\mathcal{D}_X$-modules and construct a morphism of spectra

$$\mathrm{CC} : \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{K}^{(d)}(X, -).$$

Here $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ is the K-theory spectrum of holonomic $\mathcal{D}_X$-modules, and $\mathrm{K}^{(d)}(X,-)$ is part of Levine's homotopy coniveau tower. We do not recall the definition here, but only note that $\pi_0(\mathrm{K}^{(d)}(X,-)) = \mathrm{CH}_0(X)$ and, at the level of π_0, CC associates to the class of a holonomic $\mathcal{D}_X$-modules to the zero cycle given by pulling back it's characteristic cycle by the zero section. Our main result is the following analog of the Kato-Saito localization formula in the de Rham and K-theoretic setting:

Theorem (cf. 4.1.1)**.** *Let d be the dimension of X. The following diagram commutes up to homotopy:*

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "for^{\nabla}\wedge\mathrm{CC}"'] & \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[d, "R\Gamma_{\mathrm{dR}}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\langle -,-\rangle^{\mathrm{K}}_{(d,-)}"'] & \mathrm{K}(k).
\end{tikzcd}$$

The pairing $\langle -,-\rangle^{\mathrm{K}}_{(d,-)}$ is an analog in our setting of the pairing appearing in the conjecture above. The usual dictionary between connective spectra and Picard groupoids allows one to get formulas for determinants of endomorphisms, and, in particular, by taking π_1 of the commutative diagram above we get an equality of actual numbers analogous that in the conjecture above. We refer the reader to Theorem 4.3.1 for a precise statement. We note that this particular consequence can be shown with a much simpler argument as described in *loc. cit.*. On the other hand, by the same method, we also obtain similar formulas in the setting of correspondences (and not just endomorphisms). In particular, suppose we are given an automorphism φ of X. Then a correspondence of a $\mathcal{D}_X$-module $\mathcal{F}$ is an isomorphism $\mathcal{F} \to \varphi_*\mathcal{F}$. Given a correspondence, it induces an automorphism on the cohomology $R\Gamma_{\mathrm{dR}}(X, \mathcal{F})$, and we may again consider the determinant of this automorphism. We also obtain a localization formula in this setting.

We note that, after most of this paper is written, the original conjecture of Kato and Saito has been proven, with some modification of the definition of characteristic cycles and following recent developments in ramification theory for l-adic sheaves, in ([11]). However, following the philosophy of Beilinson (cf. [1]), we believe that the K-theoretic method gives a different perspective on localization formulas for epsilon factors. In principal, proving the formula at the level of K-theory spectra should also give formulas in higher K-theory. At the level of K_0 (resp. K_1) one gets formulas for the Euler characteristic (resp. determinants). It would be interesting to see the consequences at the level of K_2 (or higher K-groups).

Let us explain the structure of the paper. The paper begins with collecting some materials from K-theory used in this paper. In particular, we recall some basic properties of Levine's homotopy coniveau tower. In §3, we define the pairing $\langle -,-\rangle^{\mathrm{K}}_{(d,-)}$, and prove a key vanishing lemma 3.7.1. This allows us to compute the pairing in the setting of correspondences. We formulate and prove the localization formula in the last section. The localization formula as an equality of values is especially easy to prove when we are given actual automorphism of modules. We conclude the paper by providing an elementary proof of this simple case.

2. Background

In this article, we shall make use of K-theory spectra and their associated Picard groupoids. However, our applications will mostly use these constructions in a formal manner. We briefly recall the required concepts and constructions for ease of exposition.

2.1. (Spectra) In the following, we fix a symmetric monoidal category of spectra and denote it by $\mathcal{S}$. For example, one could take for $\mathcal{S}$ Lurie's $(\infty,1)$-category of spectra or the category of symmetric spectra. We shall only make use of this category in a formal manner. Moreover, our results on traces only depend on the associated homotopy category (which are all known to be equivalent for the various models for spectra). Recall that $\mathcal{S}$ is a proper simiplicial model category. In particular, one has functorial fibrant-cofibrant replacements. In the following, we shall assume all our spectra are fibrant-cofibrant. We shall denote by $\wedge$ the monoidal structure in $\mathcal{S}$.

The homotopy category of $\mathcal{S}$ is denoted by $\mathrm{Ho}(\mathcal{S})$. By definition, this is the localization of $\mathcal{S}$ with respect to the weak equivalences. A weak equivalence of spectra $P\to Q$ can be inverted as a morphism in the homotopy category. However, in general such a morphism cannot be inverted as a morphism of spectra. To remedy this situation, one must use the more general notion of a homotopy morphism of spectra. A *homotopy morphism* $P\to Q$ consists of a contractible simplicial set K and a genuine morphism of spectra $f: K\wedge P\to Q$. We refer to K as the base of the homotopy morphism, and by abuse of notation we shall denote the homotopy morphism simply by $f: P\to Q$. Given two homotopy morphisms f,g with bases K_f, K_g, an *identification of f and g* is a homotopy morphism h with base K_h together with morphisms $K_f\to K_h\leftarrow K_g$ such that f,g are the respective pullbacks of h. One can define the composition of two homotopy morphisms $f: P\to Q$ and $g: Q\to R$ as the composition $K_g\wedge K_f\wedge P\to K_g\wedge Q\to R$. A homotopy morphism from a sphere spectrum to a given spectrum P will be referred to as a homotopy point of P. If f and g are identified, then they induce the same maps on homotopy groups. A weak equivalence between fibrant-cofibrant spectra can be canonically inverted as a homotopy morphism. We refer to ([8], 2.1) or ([1], 1.4(ii)) for the details. We note that in the following the language of homotopy morphisms is not necessary, since, for our purposes, we could work directly in the homotopy category. However, it is a convenient notion for constructions at the level of actual spectra (rather than the homotopy category).

2.2. (K-theory Spectra) Let $\mathcal{E}$ be a small exact category. Then Quillen's K-theory construction gives a functor from the category of small exact categories to the category of spectra. If $F_1 : \mathcal{E}_1 \to \mathcal{E}_2$ and $F_2 : \mathcal{E}_2 \to \mathcal{E}_3$ are exact functors, then one has $\mathrm{K}(F_2) \circ \mathrm{K}(F_1) = \mathrm{K}(F_2 \circ F_1)$. More generally, a natural isomorphism of functors induces a homotopy equivalence of the corresponding morphisms of K-theory spectra. By taking a large enough Grothendieck universe, we may assume all our categories are small.

More generally, Waldhausen associates to any category with cofibrations and weak equivalences a corresponding K-theory spectrum. Furthermore, an exact functor between Waldhausen categories induces a morphism between the corresponding spectra. In this article, we shall mostly be interested in complicial bi-Waldhausen categories and complicial exact functors; we refer the reader to ([10]) for details. If $\mathcal{E}$ is an exact category, then $\mathrm{C}^{\mathrm{b}}(\mathcal{E})$ is a complicial bi-Waldhausen category with weak equivalences. A fundamental result of Thomason–Trobaugh–Waldhausen–Gillet ([10]) shows that the inclusion of $\mathcal{E}$ into $\mathrm{C}^{\mathrm{b}}(\mathcal{E})$ as degree zero morphisms induces a canonical weak equivalence of spectra $\mathrm{K}(\mathcal{E}) \to \mathrm{K}(\mathrm{C}^{\mathrm{b}}(\mathcal{E}))$. Here the right side is the Waldhausen K-theory spectrum associated to $\mathrm{C}^{\mathrm{b}}(\mathcal{E})$. This allows us to canonically identify various Quillen and Waldhausen K-theory spectra. In the following, we shall always assume all our spectra to be fibrant-cofibrant. In particular, the machinery from the previous section will allow us to invert various weak equivalences canonically as homotopy morphisms.

Given a Waldhausen category $\mathcal{A}$, we denote by $\mathcal{A}^{\mathrm{tri}}$ the associated homotopy category given by inverting the weak equivalences; note that this is a triangulated category. If $F : \mathcal{A} \to \mathcal{B}$ is a complicial exact functor between two complicial bi-Waldhausen categories such that the induced map on homotopy categories is an equivalence of categories, then the induced map on K-theory spectra is a weak equivalence. We will often consider derived functors which are *a priori* only defined on $\mathcal{A}^{tri}$. Usually, these can be lifted to functors on certain full complicial bi-Waldhausen subcategories $\mathcal{C} \subset \mathcal{A}$ such that the inclusion induces an equivalence on the associated triangulated categories. Using the formalism of homotopy morphisms, we can lift the derived functor to a morphism of K-theory spectra. A typical application is the following: Let X be a proper scheme over k, and let $\mathrm{K}(X)$ be the K-theory spectrum of perfect complexes on X. Since X is proper, we can define $R\Gamma : \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(k)$. The above approach allows us to lift this to a homotopy morphism $R\Gamma : \mathrm{K}(X) \to \mathrm{K}(k)$, where $\mathrm{K}(X)$ is the K-theory spectrum of the category of perfect complexes on X and similarly for $\mathrm{K}(k)$. First, we may consider the (full) complicial bi-Waldhausen sub-category of flasque perfect complexes. On this subcategory, $R\Gamma$ is represented by Γ. Furthermore, the properness assumption implies that Γ preserves perfectness. We refer to the article by Thomason–Trobaugh ([10]) for more details.

Remark 2.2.1. Let X be a smooth projective variety over a field k, and $\mathcal{D}_X$ the sheaf of differential operators on X. Let $\mathrm{K}(\mathcal{D}_X)$ denote the K-theory spectrum of complexes of coherent-$\mathcal{D}_X$-modules. Then, via the above procedure, the $\mathcal{D}_X$-module pushforward induces a homotopy morphism $R\Gamma_{\mathrm{dR}} : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$. For example, one can take the usual locally free resolution by the de-Rham complex and restrict to flasque complexes.

2.3. (Picard groupoids, determinants, and traces) We recall some basic facts about Picard groupoids and determinants which will be useful in the following. We refer to the beautiful article ([2]) for the basic theory of Picard groupoids and determinants.

A *Picard groupoid* $\mathcal{P}$ is a symmetric monoidal category in which every object is invertible, which satisfies natural commutativity and associativity constraints. We refer the reader to ([8], 5.2) for a discussion of the definition. In the following, we shall always assume that our Picard groupoids come with a fixed unit $\mathbb{I}$. In order to avoid confusion, we shall denote by $+$ the monoidal structure in a Picard groupoid. The following will be one of our main examples of a Picard groupoid.

Example 2.3.1. Let $\mathrm{Pic}^{\mathbb{Z}}(X)$ denote the category whose objects are pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on X, and $\alpha : X \to \mathbb{Z}$ is a continuous function. We define $\mathrm{Hom}((\mathcal{L}, \alpha), (\mathcal{L}', \alpha'))$ to be the set of isomorphisms $\mathcal{L} \to \mathcal{L}'$ if $\alpha = \alpha'$ and the empty set if $\alpha \neq \alpha'$. The monoidal structure is given by setting

$$(\mathcal{L}, \alpha) + (\mathcal{L}', \alpha') := (\mathcal{L} \otimes \mathcal{L}', \alpha + \alpha').$$

The commutativity constraint

$$c_{\mathcal{L},\mathcal{L}'} \colon (\mathcal{L}, \alpha) + (\mathcal{L}', \alpha') \cong (\mathcal{L}', \alpha') + (\mathcal{L}, \alpha)$$

is given (locally) by sending $l_x \otimes l'_x$ to $(-1)^{\alpha(x)+\alpha'(x)}(l'_x \otimes l_x)$.

Given a vector bundle V on X, one can associate to it an object $\det(V) \in \mathrm{Pic}^{\mathbb{Z}}(X)$, where $\alpha(x)$ is taken to be the rank of V at x. This construction gives rise to a *determinant* functor

$$\det : \mathrm{Vect}(X)^{\mathrm{iso}} \to \mathrm{Pic}^{\mathbb{Z}}(X).$$

Here $\mathrm{Vect}(X)^{\mathrm{iso}}$ denotes the category whose objects are vector bundles on X, and morphisms are isomorphisms of vector bundles. We do not recall the definition of a determinant functor and refer to ([2]) for details. We only note here that there are natural isomorphisms

$$\det(x \oplus y) \cong \det(x) + \det(y)$$

which are compatible with commutativity constraints. In fact, one can define the notion of a $\mathcal{P}$-valued determinant functor for any exact category $\mathcal{E}$ or even derived categories of exact categories (see [4]). Moreover, one can extend the determinant functor det above to the category of coherent sheaves or even derived category of perfect complexes on X ([5], [4]).

One can associate natural homotopy groups to a Picard groupoid. By definition, $\pi_0(\mathcal{P})$ is the group of isomorphism classes of objects in $\mathcal{P}$ and $\pi_1(\mathcal{P}) := \mathrm{End}_{\mathcal{P}}(\mathbb{I})$. We note that if $\mathrm{L} \in \mathcal{P}$, then there is a canonical isomorphism

$$\mathrm{End}_{\mathcal{P}}(\mathrm{L}) \to \pi_1(\mathcal{P})$$

defined as follows. If $f : \mathrm{L} \to \mathrm{L}$ is an endomorphism, then it induces an endomorphism

$$f \otimes \mathrm{Id} : \mathrm{L} \otimes \mathrm{L}^{-1} \to \mathrm{L} \otimes \mathrm{L}^{-1},$$

and composing this with the natural isomorphisms $\mathbb{I} \to \mathrm{L} \otimes \mathrm{L}^{-1}$ and $\mathrm{L} \otimes \mathrm{L}^{-1} \to \mathbb{I}$ gives an element of $\mathrm{End}_{\mathcal{P}}(\mathbb{I})$. We shall call this the trace of f, denoted $\mathrm{Tr}(f|\mathrm{L}) \in \pi_1(\mathcal{P})$. The following example explains this terminology.

Example 2.3.2. One has $\pi_1(\mathrm{Pic}^{\mathbb{Z}}(k)) = k^{\times}$ for a field k. An automorphism $f : V \to V$ of a finite dimensional vector space over k, gives a map

$$\det(f) : (\det(V), \dim(V)) \to (\det(V), \dim(V))$$

in $\mathrm{Pic}^{\mathbb{Z}}(k)$. One can check that the $\mathrm{Tr}(\det(f)|\det(V)) \in k^{\times}$ is the usual determinant of f.

The following Lemma is immediate, and only recorded here for future use:

Lemma 2.3.3. *Let $\mathcal{P}$ be a Picard groupoid, and* $\mathrm{L} \in \mathcal{P}$.
(1) If $\mathrm{Id} : \mathrm{L} \to \mathrm{L}$ *is the identity, then* $\mathrm{Tr}(\mathrm{Id}|\mathrm{L}) = \mathrm{Id} \in \mathrm{End}_{\mathcal{P}}(\mathbb{I})$.
(2) If $f, g : \mathrm{L} \to \mathrm{L}$ *denote two automorphisms, then*

$$\mathrm{Tr}(f \circ g|\mathrm{L}) = \mathrm{Tr}(f|\mathrm{L}) \circ \mathrm{Tr}(g|\mathrm{L}).$$

2.4. (Picard groupoids and Spectra) Let Pic denote the category of Picard groupoids. We let $\mathrm{Ho}(Pic)$ denote the homotopy category of Picard groupoids. This is by definition the category of Picard groupoids localized at equivalences of Picard groupoids. It is well-known that the category of Picard groupoids identifies homotopically with the category of connective (i.e. all negative homotopy groups vanish) spectra ([8], §5). In particular, there are natural adjoint functors $\Pi : \mathcal{S}^{\geq 0} \to Pic$ and $\mathrm{B} : Pic \to \mathcal{S}^{\geq 0}$ which induce an equivalence on the associated homotopy categories. Here B takes a Picard groupoid to its usual classifying space and Π is the fundamental groupoid associated to a connective spectrum.

This construction allows one to view the Picard groupoid associated to K-theory as a universal determinant functor. Let $\mathcal{E}$ be an exact category and $C^{\mathrm{b}}(\mathcal{E})$ denote the corresponding Waldhausen category of bounded chain complexes in $\mathcal{E}$. The homotopy point construction gives rise to a natural universal determinant functor

$$\det : (\mathrm{D}^{\mathrm{b}}(\mathcal{E}), qis) \to \Pi(\mathrm{K}(C^{\mathrm{b}}(\mathcal{E}))).$$

In the following, we shall mostly be interested in applying this construction to the K-theory spectrum of a scheme. In particular, let $\mathrm{K}(X)$ denote the K-theory spectrum vector bundles (or coherent sheaves or perfect complexes) on a smooth scheme X. In that case, there is a natural map, denoted by Det,

$$\Pi(\mathrm{K}(X)) \xrightarrow{\mathrm{Det}} \mathrm{Pic}^{\mathbb{Z}}(X).$$

Moreover, the usual determinant functor $\det : (\mathrm{D}^b(X), qis) \to \mathrm{Pic}^{\mathbb{Z}}(X)$ is compatible with the previous two. In particular, the following diagram is commutative:

$$\begin{tikzcd}
(\mathrm{D}^{\mathrm{b}}(X), qis) \arrow[r] \arrow[dr] & \Pi(\mathrm{K}(X)) \arrow[d] \\
 & \mathrm{Pic}^{\mathbb{Z}}(X).
\end{tikzcd}$$

Remark 2.4.1. Here, and in what follows, by a commutative diagram of categories we mean commutative up to natural transformation.

We note that an explicit construction of a model for the Picard groupoid $\Pi(\mathrm{K}(X))$ can be given by Deligne's virtual categories ([2]).

2.5. (Distributive Functors) In the following, we shall be interested in certain pairings of Picard groupoids. Given two Picard groupoids $\mathcal{P}$ and $\mathcal{P}'$, let $\mathcal{P} \times \mathcal{P}'$ denote the product groupoid. Note that we consider this as a mere groupoid (and not a Picard groupoid). A *distributive functor* is a functor

$$\langle -, - \rangle : \mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$$

which satisfies some natural "bilinearity" or "distributive" conditions. We refer to ([2], 4.11) for the precise definitions. The definition, in particular, implies that for each fixed $\mathrm{L} \in \mathcal{P}$ (resp.

$\mathrm{L}' \in \mathcal{P}$), the induced functor $\langle \mathrm{L}, - \rangle$ (resp. $\langle -, \mathrm{L}' \rangle$) is a morphism of Picard groupoids. This morphism is natural in L (resp. L$'$). Moreover, one also has natural isomorphisms

$$\langle \mathrm{L}_1 + \mathrm{L}_2, \mathrm{L}' \rangle \cong \langle \mathrm{L}_1, \mathrm{L}' \rangle + \langle \mathrm{L}_2, \mathrm{L}' \rangle \text{ and } \langle \mathrm{L}, \mathrm{L}'_1 + \mathrm{L}'_2 \rangle \cong \langle \mathrm{L}, \mathrm{L}'_1 \rangle + \langle \mathrm{L}, \mathrm{L}'_1 \rangle.$$

We shall refer to such a distributive functor simply as a pairing of Picard groupoids. The following will be one of our main examples of a pairing.

Example 2.5.1. Let X be an integral scheme over k. The tensor product $\otimes$ of line bundles induces a distributive functor:

$$(- \otimes -) : \mathrm{Pic}^{\mathbb{Z}}(X) \times \mathrm{Pic}^{\mathbb{Z}}(X) \to \mathrm{Pic}^{\mathbb{Z}}(X).$$

Explicitly, it sends $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') := (\mathcal{L}^{\otimes \alpha'} \otimes \mathcal{L}'^{\otimes \alpha}, \alpha\alpha')$. Note that for vector bundles G and G', one has $\det(G \otimes G') \cong \det(G) \otimes \det(G')$ in $\mathrm{Pic}^{\mathbb{Z}}(X)$.

Lemma 2.5.2. *Let $\langle -, - \rangle : \mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$ be a distributive functor. Given $f : \mathrm{L} \to \mathrm{L} \in \mathcal{P}$ and $g : \mathrm{L}' \to \mathrm{L}' \in \mathcal{P}'$, let $\langle \mathrm{L}, \mathrm{L}' \rangle(f, g) := \mathrm{Tr}(\langle f, g \rangle | \langle \mathrm{L}, \mathrm{L}' \rangle) \in \pi_1(\mathcal{P}'')$.*

(i) For any $\mathrm{L} \in \mathcal{P}$, if f is the identity, then $\langle \mathrm{L}, \mathrm{L}' \rangle(\mathrm{Id}, g)$ is the image of $\mathrm{Tr}(g|\mathrm{L}')$ under the induced map $\langle \mathbb{I}, - \rangle : \pi_1(\mathcal{P}') \to \pi_1(\mathcal{P}'')$.

(ii) If $f_i : \mathrm{L}_i \to \mathrm{L}_i \in \mathcal{P}$ $(i = 1, 2)$, then

$$\langle \mathrm{L}_1 + \mathrm{L}_2, \mathrm{L}' \rangle(f_1 + f_2, g) = \langle \mathrm{L}_1, \mathrm{L}' \rangle(f_1, g) \langle \mathrm{L}_2, \mathrm{L}' \rangle(f_2, g)$$

Any abelian group can be considered as a Picard groupoid. We will sometimes be interested in a pairing of a Picard groupoids $\mathcal{P}$ and an abelian group G with values in a Picard groupoid $\mathcal{P}''$. By definition, this means that for each $g \in G$, we have a morphism of Picard groupoids $F_g : \mathcal{P} \to \mathcal{P}'$ such that $F_e = \mathrm{Id}$ (where $e \in G$ is the identity), and there are natural isomorphisms $F_{g+h} \cong F_g + F_h$. Note that $\mathrm{Hom}(\mathcal{P}, \mathcal{P}')$ is also a Picard groupoid, and such a pairing can be interpreted as a morphism of groupoids

$$G \to \mathrm{Hom}(\mathcal{P}, \mathcal{P}').$$

The following will be our central example of such a pairing.

Example 2.5.3. Let $f : X \to \mathrm{Spec}(k)$ denote a smooth proper scheme over a field k, and $\mathrm{Z}_0(X)$ denote the abelian group of 0-cycles on X. Then we have a pairing

$$\langle -, - \rangle : \mathrm{Pic}^{\mathbb{Z}}(X) \times \mathrm{Z}_0(X) \to \mathrm{Pic}^{\mathbb{Z}}(k)$$

defined as follows. If $i_Z : Z \subset X$ is a closed subscheme of dimension 0, let $[Z]$ denote its class in $Z_0(X)$. Then we set

$$\langle (\mathcal{L}, \alpha), [Z] \rangle := (\det(\pi_{Z,*} i_Z^* \mathcal{L}), \deg([Z])\alpha).$$

Here $\pi_Z : Z \to \mathrm{Spec}(k)$ is the structure map, and $\deg([Z])$ is the usual degree of the given zero-cycle. This defines the pairing for all effective cycles, and then we extend by linearity.

Finally, we note that pairings of spectra induce pairings of Picard groupoids. We refer to ([9, I, 5.1]) for details on the notion of bilinear pairings of spectra. Here we only note that a bilinear pairing of spectra K_1 and K_2 with values in K_3 is equivalent to giving a morphism of spectra

$$\mathrm{K}_1 \wedge \mathrm{K}_2 \to \mathrm{K}_3.$$

Furthermore, a bi-exact functor of exact categories (or Waldhausen categories) induces a bilinear pairing of the corresponding K-theory spectra ([10], 3.15). Moreover, under the equivalence of categories between Picard groupoids and spectra, a bilinear map gives rise to a pairing of the associated Picard groupoids. In particular, the usual tensor product of vector bundles induces a pairing of spectra $\mathrm{K}(X) \wedge \mathrm{K}(X) \to \mathrm{K}(X)$ and, therefore, a pairing $\Pi(\mathrm{K}(X)) \times \Pi(\mathrm{K}(X)) \to \Pi(\mathrm{K}(X))$. Moreover, this pairing is compatible with the one from Example 2.5.1 under $\mathrm{Det} : \Pi(\mathrm{K}(X)) \to \mathrm{Pic}^{\mathbb{Z}}(X)$. In the following, we sometimes use the notation

$$\mathcal{P} \wedge \mathcal{P}' \to \mathcal{P}''$$

to mean a pairing $\mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$. We note that this should only be thought of as formal notation, and the Picard groupoid $\mathcal{P} \wedge \mathcal{P}'$ has not been defined.

Remark 2.5.4. One can take the fundamental Picard groupoid associated to $\mathrm{B}\mathcal{P} \wedge \mathrm{B}\mathcal{P}'$ as the definition of $\mathcal{P} \wedge \mathcal{P}'$. Moreover, there is an equivalence between pairings $\mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$ and morphisms of Picard groupoids $\mathcal{P} \wedge \mathcal{P}' \to \mathcal{P}''$. However, we shall not need this in what follows. Note that for strictly commutative Picard groupoids this construction is described in SGA 4 Expose XVIII.

2.6. (Levine's Homotopy Coniveau Tower) In this subsection, X will be a smooth scheme of finite type over a field k. Moreover, $\mathrm{K}(X)$ will denote the K-theory spectrum of coherent sheaves on X. We recall the construction and some basic properties of Levine's homotopy coniveau tower associated to the K-theory of schemes which shall be used in the following. We refer to ([6], [7]) for details.

Let $\Delta^n := Spec(k[t_0, \dots, t_r]/(\Sigma_j t_j - 1))$ denote the usual n-simplex. These form a cosimplicial scheme. A face of Δ^n is a closed subscheme defined by equations of the form $t_{i_1} = \cdots = t_{i_s} = 0$. Then one defines

$$\mathrm{K}^{(q)}(X, p) := \underrightarrow{\mathrm{holim}}_W \mathrm{K}_W(X \times \Delta^p)$$

where the homotopy limit is taken over closed subschemes $W \subset X \times \Delta^p$ such that

$$\mathrm{codim}_{X \times F}(W \cap (X \times F)) \geq q$$

for all faces $F \subset \Delta^p$. These form a simplicial spectrum, and we let $\mathrm{K}^{(q)}(X, -)$ denote the corresponding total spectrum ([6], 1.5). Moreover, one has a tower of spectra

$$\cdots \to \mathrm{K}^{(q)}(X, -) \to \mathrm{K}^{(q-1)}(X, -) \to \cdots \to \mathrm{K}^{(0)}(X, -) \leftarrow \mathrm{K}(X)$$

where the arrow on the right is the natural augmentation map. This tower of spectra is referred to as the homotopy coniveau tower. It satisfies the following properties proved by Levine:

(1) Given a morphism of smooth schemes $F : X \to Y$ there is a natural pull back morphism on the corresponding coniveau towers.
(2) The augmentation map $\epsilon \colon \mathrm{K}(X) \to \mathrm{K}^{(0)}(X, -)$ is a weak equivalence.
(3) The fibers $\mathrm{K}^{(p/p-1)}(X, -)$ of the homotopy coniveau tower are naturally weak equivalent to Bloch's higher Chow groups cycles complex. In particular, there is a natural isomorphism $\mathrm{CH}^d(X) \to \pi_0(\mathrm{K}^{(d)}(X, -))$ if $d = \dim(X)$. Moreover, the naturality implies that this map is compatible with pull backs on Chow groups.
(4) Finally, we note that the tensor product induces natural (functorial) pairings:

$$\mathrm{K}^{(d)}(X, -) \wedge \mathrm{K}^{(d')}(X, -) \to \mathrm{K}^{(d+d')}(X, -).$$

Remark 2.6.1. The existence of pairing as in (4) is a deep theorem and relies on Levine's moving lemma for the homotopy coniveau tower. However, we shall only use the result in the case where $d' = 0$. In that case, $\mathrm{K}^{(0)}(X, -) \cong \mathrm{K}(X)$, and the pairing is simply given by tensor product. In particular, no 'moving' is required.

2.7. (Microlocalization map of K-theory of $\mathcal{D}_X$-modules) In this paragraph, X will denote a smooth projective variety over a field k of characteristic zero. Below we recall the construction of a microlocalization map for K-theory spectra of $\mathcal{D}_X$-modules. We refer to ([8]) for details.

Let $\mathrm{K}(\mathcal{D}_X)$ denote the K-theory spectrum of the abelian category of coherent $\mathcal{D}_X$-modules, and similarly let $\mathrm{K}_S(\mathcal{D}_X)$ denote the K-theory spectrum of the abelian category of coherent $\mathcal{D}_X$-modules with singular support contained in $S \subset T^*X$. Recall, any $\mathcal{D}_X$ module $\mathcal{M}$ has a good filtration $\mathcal{F}^{\cdot}$ such that the associated graded gives rise to a coherent $\mathcal{O}_{T^*X}$-module. This construction gives rise to a well-defined (i.e. independent of the choice of filtration) map $\mathrm{K}_0(\mathcal{D}_X) \to \mathrm{K}_0(T^*X)$. One has an analogous statement in the setting of supports. The following theorem extends this construction to the setting of higher K-theory.

Theorem 2.7.1. *([8]) Let X be as above. There is a natural microlocalization morphism of K-theory spectra:*

$$gr_S : \mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X).$$

Moreover, these are compatible with respect to the inclusions $S \subset S'$.

Let $\mathrm{KF}(\mathcal{D}_X)$ denote the K-theory spectrum of the exact category whose objects are pairs $(\mathcal{M}, \mathcal{F})$ where $(\mathcal{M}, \mathcal{F})$ is a coherent $\mathcal{D}_X$-module and $\mathcal{F}$ is a good filtration. We can similarly define $\mathrm{KF}_S(\mathcal{D}_X)$. There is a natural map $gr_S^{\mathrm{F}} : \mathrm{KF}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X)$ induced by sending a pair $(\mathcal{M}, \mathcal{F})$ to $gr_{\mathcal{F}}(\mathcal{M})$. One also has a natural map $\mathrm{ff} : \mathrm{KF}_S(\mathcal{D}_X) \to \mathrm{K}_S(\mathcal{D}_X)$ given by simply forgetting the filtration. By construction, $gr_S \circ \mathrm{ff}$ is homotopic to gr_S^{F}.

Let $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ denote the K-theory spectrum of the abelian category of holonomic $\mathcal{D}_X$-modules. The preceding theorem immediately gives the following corollary by passing to homotopy colimits.

Corollary 2.7.2. *With notation as above, one has a morphism of spectra:*

$$\varepsilon : \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{K}^{(d)}(T^*X).$$

Proof. By definition, we may view the category of holonomic $\mathcal{D}_X$-modules as a direct limit of the categories of the full sub-categories of $\mathcal{D}_X$-modules with singular support in a fixed codimension d subset $S \subset T^*X$. Since K-theory commutes with direct limits, we may write $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ as the colimit of the corresponding spectra $\mathrm{K}_S(\mathcal{D}_X)$. The result now follows from the previous theorem by taking limits. □

Remark 2.7.3. Note that there is a natural map $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{K}(\mathcal{D}_X)$. Moreover, by the compatibility of gr_S, one has a natural commutative diagram:

$$\begin{tikzcd}
\mathrm{K}(\mathcal{D}_X) \arrow[r, "gr"] & \mathrm{K}(T^*X) \\
\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[u] \arrow[r, "\varepsilon"] & \mathrm{K}^{(d)}(T^*X) \arrow[u]
\end{tikzcd}$$

Let $f : X \to \mathrm{Spec}(k)$ denote the structure map, $\pi : T^*X \to X$ the projection map, and $\sigma : X \to T^*X$ the zero section. Then f and σ induce morphisms of K-theory spectra (2.2)

$$\mathrm{K}(X) \xrightarrow{f_*} \mathrm{K}(k) \text{ and } \mathrm{K}(T^*X) \xrightarrow{\sigma^*} \mathrm{K}(X).$$

The canonical bundle ω_X induces a natural morphism:

$$\mathrm{K}(X) \xrightarrow{(-\otimes\omega_X)} \mathrm{K}(X).$$

We define the twisted pull-back σ^+ as the composition:

$$\mathrm{K}(T^*X) \xrightarrow{\sigma^*} \mathrm{K}(X) \xrightarrow{(-\otimes\omega_X)} \mathrm{K}(X).$$

These give rise to a morphism $f_* \circ \sigma^+ \circ gr : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$. On the other hand, the $\mathcal{D}_X$-module pushforward induces a morphism of K-theory spectra (2.2.1):

$$R\Gamma_{\mathrm{dR}} : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k).$$

We have the following Lemma.

Lemma 2.7.4. *The morphisms $R\Gamma_{\mathrm{dR}}$ and $f_* \circ \sigma^+ \circ gr$ are homotopic.*

Proof. For the first part, first note that the composition

$$\mathrm{K}(X) \to \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X) \to \mathrm{K}(X)$$

is homotopic to the identity. Here the first map is the natural map induced by $\mathcal{D}_X \otimes (-)$, which is a weak equivalence by a theorem of Quillen ([8]). Then one is reduced to showing that the composition

$$\mathrm{K}(X) \xrightarrow{(-\otimes\omega_X)} \mathrm{K}(X) \xrightarrow{f_*} \mathrm{K}(k)$$

is homotopic to the composition

$$\mathrm{K}(X) \xrightarrow{(\mathcal{D}_X\otimes-)} \mathrm{K}(\mathcal{D}_X) \xrightarrow{R\Gamma_{dR}} \mathrm{K}(k).$$

The latter follows from the fact that the corresponding functors are naturally isomorphic on the corresponding derived categories. $\square$

Remark 2.7.5. We think of $R\Gamma_{\mathrm{dR}}$ as the global de Rham epsilon factor. Recall, at the level of $\mathcal{D}_X$-modules, up to a shift, the $\mathcal{D}_X$-module push-forward computes de Rham cohomology of the corresponding $\mathcal{D}_X$-modules.

2.8. (Global epsilon factors and tensor products) We record an elementary lemma computing the global epsilon factor of a tensor product.

Lemma 2.8.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(\mathcal{D}_X) \arrow[r, "gr\wedge gr"] \arrow[d, "(-\otimes-)"] & \mathrm{K}_X(T^*X) \wedge \mathrm{K}(T^*X) \arrow[d, "(-\otimes-)"] \\
\mathrm{K}(\mathcal{D}_X) \arrow[r, "gr"] & \mathrm{K}(T^*X).
\end{tikzcd}$$

Proof. The commutativity follows from the fact that gr commutes with tensor product if one of the sheaves is a bundle with connection. To be precise, we note that it is enough to show that the following diagram is homotopy commutative:

$$\begin{tikzcd} \mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(X) \arrow[r] \arrow[d, "(-\otimes-)"] & \mathrm{K}_X(T^*X) \wedge \mathrm{K}(T^*X) \arrow[d, "(-\otimes-)"] \\ \mathrm{K}(\mathcal{D}_X) \arrow[r, "gr"] & \mathrm{K}(T^*X) \end{tikzcd}$$

Here the top horizontal is given by $(gr \wedge gr) \circ (\mathrm{Id} \wedge (\mathcal{D}_X \otimes -)$. Since $\mathrm{K}(X) \to \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X)$ is homotopic to π^*, the composition of the top horizontal and right vertical is homotopic to $\pi^* \circ (- \otimes_{\mathcal{O}_X} -)$. On the other hand, there is a natural map $\mathrm{K}_X(\mathcal{D}_X) \to \mathrm{KF}(\mathcal{D}_X)$, and similarly $\mathrm{K}(X) \to \mathrm{KF}(\mathcal{D}_X)$, and the left vertical factors through $\mathrm{KF}(\mathcal{D}_X) \wedge \mathrm{KF}(\mathcal{D}_X)$. The first map here just gives a bundle with connection the trivial filtration. It's now clear that the composition of this with lower horizontal with the left vertical is also homotopic to $\pi^* \circ (- \otimes_{\mathcal{O}_X} -)$. □

3. Comparison of traces for various pairings of Picard groupoids

In this subsection, we recall the construction of some pairings on K-theory spectra at the level of Levine's homotopy coniveau tower, and make some computations of traces of tensor products in this setting.

3.1. (Pairings on K-theory with supports) Given a closed subset $Z \subset X$, there is a natural pairing of K-theory spectra

$$\mathrm{K}(X) \wedge \mathrm{K}_Z(X) \xrightarrow{\otimes} \mathrm{K}_Z(X)$$

induced by the tensor product ([10], 3.15). Since X is smooth, Quillen's localization theorem implies that the natural map $\mathrm{K}(Z) \to \mathrm{K}_Z(X)$, induced by the push-forward, is a weak-equivalence. If $i_Z : Z \hookrightarrow X$ is a reduced closed subscheme of dimension 0, and hence regular, then we may identify $\mathrm{K}(Z)$ with the K-theory spectrum of locally free sheaves. Moreover, one also has a natural pairing

$$\mathrm{K}(X) \wedge \mathrm{K}(Z) \to \mathrm{K}(Z) \wedge \mathrm{K}(Z) \to \mathrm{K}(Z)$$

where the first map is induced by $i_Z^* \wedge \mathrm{Id}$. We record the following standard lemma for future use.

Lemma 3.1.1. *The following diagram is commutative:*

$$\begin{tikzcd} \mathrm{K}(X) \wedge \mathrm{K}_Z(X) \arrow[r] & \mathrm{K}_Z(X) \\ \mathrm{K}(X) \wedge \mathrm{K}(Z) \arrow[r] \arrow[u, "\mathrm{Id}\wedge i_{Z,*}"] & \mathrm{K}(Z) \arrow[u] \end{tikzcd}$$

Proof. This is a special case of the projection formula ([10], 3.17). □

We shall now suppose Z is a reduced closed subscheme of dimension zero. One has a natural norm map (given by the push-forward):

$$N : \mathrm{K}(Z) \to \mathrm{K}(k).$$

Similarly, one has the usual push forward $\pi_* : \mathrm{K}_Z(X) \to \mathrm{K}(k)$. Composing the pairings above with these push-forward maps give rise to pairings:

$$\langle -, - \rangle^{\mathrm{K}} : \mathrm{K}(X) \wedge \mathrm{K}_Z(X) \to \mathrm{K}(k)$$

and

$$\langle -, - \rangle^{\mathrm{K}} : \mathrm{K}(X) \wedge \mathrm{K}(Z) \to \mathrm{K}(k).$$

By the previous lemma these two pairings are identified via the natural weak equivalence $i_{Z,*} : \mathrm{K}(Z) \to \mathrm{K}_Z(X)$. Therefore, in the following we shall use the two interchangeably and use the same notation to denote the two pairings.

Since the pairings above are compatible with respect to inclusions $Z' \subset Z$, we may pass to homotopy limits and deduce a pairing:

$$\langle -, - \rangle^{\mathrm{K}}_{(d)} : \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \to \mathrm{K}(k).$$

Note this pairing is simply the composition

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \xrightarrow{(-\otimes-)} \mathrm{K}^{(d)}(X) \xrightarrow{\pi_*} \mathrm{K}(k).$$

Here we define π_* as the one induced by taking homotopy limits of the maps $\pi_* \colon \mathrm{K}_Z(X) \to \mathrm{K}(k)$.

Remark 3.1.2. We may also define π_* by taking homotopy limits of the compositions $\mathrm{K}_Z(X) \to \mathrm{K}(Z) \xrightarrow{N} \mathrm{K}(k)$. The two constructions are homotopic.

3.2. (Pairings on the homotopy coniveau tower) We now explain how the constructions of the previous paragraph lift to Levine's homotopy coniveau tower. Recall that we have a natural augmentation morphism

$$\epsilon : \mathrm{K}^{(d)}(X) \to \mathrm{K}^{(d)}(X, -).$$

Moreover, there are natural pairings

$$\mathrm{K}^{(r)}(X, -) \wedge \mathrm{K}^{(s)}(X, -) \to \mathrm{K}^{(r+s)}(X, -).$$

In particular, we may consider the composition

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \to \mathrm{K}^{(0)}(X, -) \wedge \mathrm{K}^{(d)}(X, -) \to \mathrm{K}^{(d)}(X, -).$$

Moreover, composing with $\pi_*^{(d)} : \mathrm{K}^{(d)}(X, -) \to \mathrm{K}^{(0)}(X, -) \to \mathrm{K}(X) \xrightarrow{f_*} \mathrm{K}(k)$ gives rise to a pairing:

$$\langle -, - \rangle^{\mathrm{K}}_{(d,-)} : \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X, -) \to \mathrm{K}(k).$$

Here we have inverted the augmentation map $\mathrm{K}(X) \to \mathrm{K}^{(0)}(X, -)$. This pairing is compatible with the pairing constructed in the previous paragraph. Namely, we have the following lemma.

Lemma 3.2.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \arrow[r, "\langle -,- \rangle^{\mathrm{K}}_{(d)}"] \arrow[d, "\mathrm{Id}\wedge\epsilon"] & \mathrm{K}(k) \arrow[d, "\mathrm{Id}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X, -) \arrow[r, "\langle -,- \rangle^{\mathrm{K}}_{(d,-)}"] & \mathrm{K}(k)
\end{tikzcd}$$

Proof. First, recall that the tensor product is compatible with the augmentation. Therefore, one is reduced to showing that $\mathrm{K}^{(d)}(X) \xrightarrow{\epsilon} \mathrm{K}^{(d)}(X,-) \xrightarrow{\pi_*^{(d)}} \mathrm{K}(k)$ is homotopic to $\mathrm{K}^{(d)}(X) \xrightarrow{\pi_*} \mathrm{K}(k)$. The latter is evident from the construction of $\pi_*^{(d)}$. □

Remark 3.2.2. One also has a product $\mathrm{K}(T^*X) \wedge \mathrm{K}^{(d)}(T^*X,-) \to \mathrm{K}^{(d)}(T^*X,-)$. By functoriality, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}(T^*X) \wedge \mathrm{K}^{(d)}(T^*X,-) \arrow[r] \arrow[d, "\sigma^* \wedge \sigma^*"] & \mathrm{K}^{(d)}(T^*X,-) \arrow[d, "\sigma^*"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r] & \mathrm{K}^{(d)}(X,-)
\end{tikzcd}$$

3.3. (Pairings on Picard groupoids) Recall, pairings on spectra give rise to pairings on the corresponding fundamental Picard groupoids (2.5). In particular, the pairing $\langle -,-\rangle^{\mathrm{K}}_{(d)}$ induces a pairing

$$\langle -,-\rangle^{\Pi}_{(d)} : \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}(k)).$$

By definition, it is defined as the composition

$$\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}(k)),$$

where the first map is induced by the tensor product and the second by π_*. On the other hand, one has the following pairing which is a generalization of Example 2.5.1:

$$\langle -,-\rangle : \mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z)) \to \mathrm{Pic}^{\mathbb{Z}}(Z).$$

Explicitly, this pairing sends $(L,\alpha) \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $(M,\beta) \in \mathrm{Pic}^{\mathbb{Z}}(Z)$ to the element $(L|_Z^{\beta} \otimes M^{\alpha}, \alpha|_Z + \beta)$. Recall that we have the universal determinant map $\mathrm{Det} : \Pi(\mathrm{K}(X)) \to \mathrm{Pic}^{\mathbb{Z}}(X)$ and similarly for Z. As before (2.5), this gives rise to a commutative diagram:

$$\begin{tikzcd}
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}(Z)) \arrow[r] \arrow[d] & \Pi(\mathrm{K}(Z)) \arrow[d] \\
\mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) \arrow[r] & \mathrm{Pic}^{\mathbb{Z}}(Z).
\end{tikzcd}$$

The push-forward induces a norm map

$$N : \mathrm{Pic}^{\mathbb{Z}}(Z) \to \mathrm{Pic}^{\mathbb{Z}}(k).$$

In particular, one has a natural pairing

$$N \circ \langle -,-\rangle : \mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) \to \mathrm{Pic}^{\mathbb{Z}}(k)$$

By abuse of notation, we shall also denote this pairing by $\langle -,-\rangle$. Explicitly, this pairing sends $(L,\alpha) \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $(M,\beta) \in \mathrm{Pic}^{\mathbb{Z}}(Z)$ to the element $(\det(\pi_{Z,*}(L|_Z^{\beta} \otimes M^{\alpha}), d(\alpha+\beta))$ where $\pi_Z : Z \to \mathrm{Spec}(k)$ is the natural structure map, and d is the degree of this map. The previous remarks show that the following diagram commutes:

$$\begin{tikzcd}
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}(Z)) \arrow[r, "\langle -,- \rangle^{\Pi}"] \arrow[d] & \Pi(\mathrm{K}(k)) \arrow[d] \\
\mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) \arrow[r, "\langle -,- \rangle"] & \mathrm{Pic}^{\mathbb{Z}}(k).
\end{tikzcd}$$

Remark 3.3.1. Recall that the map $\mathrm{Det}\colon \Pi(\mathrm{K}(k)) \to \mathrm{Pic}^{\mathbb{Z}}(k)$ is an isomorphism of Picard groupoids. In the following, we shall make this identification in our resulting pairings.

3.4. (Picard groupoid pairings coming from coniveau tower) We now descend the pairings $\langle -,- \rangle^{K}_{(d,-)}$ to the level of Picard groupoids. In particular, taking fundamental groupoids gives a pairing:

$$\langle -,- \rangle^{\Pi}_{(d,-)} : \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \to \mathrm{Pic}^{\mathbb{Z}}(k).$$

Combining everything gives rise to the following commutative diagrams which we record as a lemma for future use:

Lemma 3.4.1. *The following diagrams commute (up to natural transformations):*

$$\begin{tikzcd}
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X)) \arrow[r] \arrow[d] & \mathrm{Pic}^{\mathbb{Z}}(k) \arrow[d] \\
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[r] & \mathrm{Pic}^{\mathbb{Z}}(k),
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}(Z)) \arrow[r] \arrow[d] & \mathrm{Pic}^{\mathbb{Z}}(k) \arrow[d] \\
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X)) \arrow[r] & \mathrm{Pic}^{\mathbb{Z}}(k).
\end{tikzcd}$$

Proof. The commutativity of the first diagram follows from Lemma 3.2.1 and that of the second follows from the remarks in 3.1. □

3.5. (Compatibility of various traces of endomorphisms) We explain how the constructions of the previous paragraph pass to traces in the presence of endomorphisms. Given $\mathcal{G} \in \mathrm{Pic}^{\mathbb{Z}}(X)$ (resp. $\mathcal{F} \in \mathrm{Pic}^{\mathbb{Z}}(X)$) and an endomorphism g (resp. f) of $\mathcal{G}$ (resp. $\mathcal{F}$) we have an induced endomorphism $g \otimes f$ of $\langle \mathcal{G}, \mathcal{F} \rangle \in \mathrm{Pic}^{\mathbb{Z}}(k)$. Therefore, we have an element $Tr(g \otimes f) \in \mathrm{Pic}^{\mathbb{Z}}(k) = k^{\times}$. We shall denote the latter trace by $\langle \mathcal{G}, \mathcal{F} \rangle(g, f)$. Similarly, given $\mathcal{G} \in \Pi(\mathrm{K}(X))$ (resp. $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$, $\mathcal{F}' \in \Pi(\mathrm{K}^{(d)}(X,-))$) and g (resp. f, f') an endomorphism of $\mathcal{G}$ (resp. $\mathcal{F}$, $\mathcal{F}'$), we can define the trace $\langle \mathcal{G}, \mathcal{F} \rangle^{\Pi}_{(d)}(g, f) \in k^{\times}$ (resp. $\langle \mathcal{G}, \mathcal{F}' \rangle^{\Pi}_{(d,-)}(g, f') \in k^{\times}$). Note, that a pair $(\mathcal{F}, f)$ of an object and endomorphism in $\Pi(\mathrm{K}^{(d)}(X))$ can also be considered as an object and endomorphism of $\Pi(\mathrm{K}^{(d)}(X,-))$ simply by considering its image under the natural map $\Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}^{(d)}(X,-))$. We record the following corollary of the previous result for future reference.

Corollary 3.5.1. *Let $\mathcal{G} \in \Pi(\mathrm{K}(X))$ and $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$, and g (resp. f) denote an endomorphism of $\mathcal{G}$ (resp. $\mathcal{F}$). Then one has*

$$\langle \mathcal{G}, \mathcal{F} \rangle^{\Pi}_{(d,-)}(g, f) = \langle \mathcal{G}, \mathcal{F} \rangle^{\Pi}_{(d)}(g, f).$$

Proof. This is a direct consequence of Lemma 3.4.1. □

By definition, $\pi_i(\Pi(\mathrm{K}^{(d)}(X))) = \lim \pi_i(\mathrm{K}(Z))$ where the direct limit is over closed reduced subschemes Z of dimension zero. Therefore, for any object $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$ and endomorphism

$f : \mathcal{F} \to \mathcal{F}$, we can choose a Z such that the pair $(\mathcal{F}, f)$ lifts to $\Pi(\mathrm{K}(Z))$. In particular, there is a pair $(\mathcal{F}_Z, f_Z)$ consisting of an object and an endomorphism in $\Pi(\mathrm{K}(Z))$, and an isomorphism h of the image of this pair in $\Pi(\mathrm{K}^{(d)}(X))$ (under the natural map $\Pi(\mathrm{K}(Z)) \to \Pi(\mathrm{K}^{(d)}(X)))$ with the pair $(\mathcal{F}, f)$. In this setting, we have the following lemma;

Lemma 3.5.2. *With notation as above, $\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, f) = \langle \mathcal{G}, \mathcal{F}_Z\rangle^{\Pi}(g, f_Z)$. Moreover, we have an equality $\langle \mathcal{G}, \mathcal{F}_Z\rangle^{\Pi}(g, f_Z) = \langle \mathrm{Det}(\mathcal{G}), \mathrm{Det}(\mathcal{F}_Z)\rangle(g, f_Z)$.*

Proof. The first statement follows from the second commutative diagram in 3.4.1 after passing to Picard groupoids and traces. The second similarly follows from the remarks in 3.3. □

3.6. (Formula for traces of tensor products of endomorphisms) In this paragraph, we prove an elementary formula for traces of tensor products of endomorphisms. In 3.9, shall prove a similar formula in the more general the setting of correspondences. The formula presented in this paragraph will be an easy corollary of that more general formula. However, we present the simpler version here since the proof has some features of the more general situation and might be useful in understanding the more complicated version presented later.

Suppose we are given $\mathcal{G} \in \Pi(\mathrm{K}(X))$ (resp. $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X)))$ with an endomorphism $f : \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{G}$ as before. For any element of $\mathcal{G} \in \Pi(\mathrm{K}(X))$, we let $r_{\mathcal{G}}$ (the rank of $\mathcal{G}$) denote its image in $\pi_0(\mathrm{Pic}^{\mathbb{Z}}(X)) = \mathbb{Z}$. In this setting, one has the following standard formula at the level of traces.

Proposition 3.6.1. *With notation as above:*

$$\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, f) = \mathrm{Tr}(\pi_*(f)|\pi_*(\mathcal{F}))^{r_{\mathcal{G}}} \langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, \mathrm{Id}).$$

Here, $\pi_*(f)$ and $\pi_*(\mathcal{F})$ are the images under the natural map $\Pi(\mathrm{K}^{(d)}(X)) \xrightarrow{\pi_*} \mathrm{Pic}^{\mathbb{Z}}(k)$.

Proof. By corollary 3.5.2, we are reduced to showing that

$$\langle \mathcal{G}, \mathcal{F}\rangle(g, f) = \mathrm{Tr}(\det(\pi_{Z,*}(f))| \det(\pi_{Z,*}(\mathcal{F})))^{r_{\mathcal{G}}} \langle \mathcal{G}, \mathcal{F}\rangle(g, \mathrm{Id}),$$

where $\mathcal{G} \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $\mathcal{F} \in \mathrm{Pic}^{\mathbb{Z}}(Z)$. Here $i_Z : Z \hookrightarrow X$ is a closed integral subscheme of dimension zero and $\pi_Z : Z \to \mathrm{Spec}(k)$ is the structure map. Since (by 2.3.3)

$$\langle \mathcal{G}, \mathcal{F}\rangle(g, f) = \langle \mathcal{G}, \mathcal{F}\rangle(g, \mathrm{Id})\langle \mathcal{G}, \mathcal{F}\rangle(\mathrm{Id}, f),$$

we are reduced to showing that

$$\langle \mathcal{G}, \mathcal{F}\rangle(\mathrm{Id}, f) = \mathrm{Tr}(\det(\pi_{Z,*}(f)| \det(\pi_{Z,*}(\mathcal{F})))^{r_{\mathcal{G}}}.$$

We may further assume Z is irreducible. Then by definition (ignoring the grading)

$$\langle \mathcal{G}, \mathcal{F}\rangle = \det(i_{Z,*}(\mathcal{G}|_Z \otimes \mathcal{F})) = \det(i_{Z,*}(\mathcal{G}|_Z))^{r_{\mathcal{F}}} \otimes \det(i_{Z,*}(\mathcal{F}))^{r_{\mathcal{G}}}.$$

The first equality here is consequence of the fact that $i_{Z,*}$ commutes with tensor products and sends locally free sheaves to locally free sheaves (since i_Z is finite and flat). The second is the usual formula for determinant of a tensor product (i.e. the fact that det is a distributive functor). Moreover, we are computing the trace of the morphism induced by $\mathrm{Id} \otimes f$ on this object. Since the trace of the determinant of Id is one, we have

$$\langle \mathcal{G}, \mathcal{F}\rangle(\mathrm{Id}, f) = \mathrm{Tr}(\det(i_{Z,*}(f)| \det(i_{Z,*}(\mathcal{F})))^{r_{\mathcal{G}}},$$

which concludes the proof □

3.7. (A key vanishing Lemma) We would like a formula similar to that of the last paragraph for $\langle\mathcal{G},\mathcal{F}\rangle^{\Pi}_{(d,-)}(g,f)$ where $\mathcal{F}\in\Pi(\mathrm{K}^{(d)}(X))$. If the pair $(\mathcal{F},f)$ can be lifted to $\Pi(\mathrm{K}^{(d)}(X))$, then we would get such a formula as a consequence of the previous proposition. Unfortunately, while we may lift any such object $\mathcal{F}$ to $\Pi(\mathrm{K}^{(d)}(X))$, it is not always possible to lift the endomorphism f. However, we shall see that the desired formula (in the more general setting of correspondences) will be an easy consequence of the following lemma.

Lemma 3.7.1. *Let X be a smooth projective variety of dimension d and W be a closed subscheme of codimension >0. The map $\pi_1\mathrm{K}^{(d)}(X,-)\to\pi_1\mathrm{K}^{(d)}(X,-)$ induced by $\otimes\mathcal{O}_W$ is trivial.*

Proof of Lemma 3.7.1. First, we remark that ([6], Thm 2.6.2) holds when X is projective. Below, we shall follow the notation of ([6], Thm 2.6.2). Using that theorem for $\mathcal{C}=\{W\}$ and $e=0$, we get a weak equivalence $\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e}\xrightarrow{\sim}\mathrm{K}^{(d)}(X,-)$. Now, the map $\otimes\mathcal{O}_W$ factors through $\mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e}\to\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e}$, and we have the commutative diagram

$$\begin{tikzcd}
\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e} \arrow[r, "\otimes\mathcal{O}_W"] \arrow[d] & \mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e} \arrow[r] \arrow[d] & \mathrm{K}^{(d)}(X,-)_{\mathcal{C},e} \arrow[d] \\
\mathrm{K}^{(d)}(X,-) \arrow[rr, bend right, "\otimes\mathcal{O}_W"'] & \mathrm{K}^{(d+1)}(X,-) \arrow[r] & \mathrm{K}^{(d)}(X,-).
\end{tikzcd}$$

For $n\in\{d,d+1\}$, consider the spectral sequences

$$\begin{tikzcd}
(E_1^{p,q})^{(n)}_{\mathcal{C},e}=\pi_{-q}\mathrm{K}^{(n)}(X,-p)_{\mathcal{C},e} \arrow[r, Rightarrow] \arrow[d] & \pi_{-p-q}\mathrm{K}^{(n)}(X,-)_{\mathcal{C},e} \arrow[d, "\sim"] \\
(E_1^{p,q})^{(n)}=\pi_{-q}\mathrm{K}^{(n)}(X,-p) \arrow[r, Rightarrow] & \pi_{-p-q}\mathrm{K}^{(n)}(X,-).
\end{tikzcd}$$

By dimension reasons, we have $\mathrm{K}^{(d+1)}(X,0)_{\mathcal{C},e}=\mathrm{K}^{(d+1)}(X,0)=\{*\}$, which implies $(E_1^{0,q})^{(d+1)}_{\mathcal{C},e}=(E_1^{0,q})^{(d+1)}=0$ for any q. Thus, $(E_2^{-1,0})^{(d+1)}_{\mathcal{C},e}\cong(E_2^{-1,0})^{(d+1)}\cong\pi_1\mathrm{K}^{(d+1)}(X,-)$.

Now, we have the following big commutative diagram:

$$\begin{tikzcd}
\pi_1\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e} \arrow[rr, "\otimes\mathcal{O}_W"] \arrow[d, two heads] & & \pi_1\mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e} \arrow[r] \arrow[d, dash, "\sim"] & \pi_1\mathrm{K}^{(d+1)}(X,-) \arrow[r] \arrow[d, dash, "\sim"] & \pi_1\mathrm{K}^{(d)}(X,-) \\
(E_\infty^{-1,0})^{(d)}_{\mathcal{C},e} \arrow[r, dash, "\sim"] & (E_2^{-1,0})^{(d)}_{\mathcal{C},e} \arrow[r, "\otimes\mathcal{O}_W"'] & (E_2^{-1,0})^{(d+1)}_{\mathcal{C},e} \arrow[r] & (E_2^{-1,0})^{(d+1)} & \\
& K \arrow[u, two heads] \arrow[r] & (E_1^{-1,0})^{(d+1)}_{\mathcal{C},e} \arrow[u, two heads] & &
\end{tikzcd}$$

where $K:=\mathrm{Ker}\big((E_1^{-1,0})^{(d)}_{\mathcal{C},e}\to(E_1^{0,0})^{(d)}_{\mathcal{C},e}\big)$. Take $\alpha\in\pi_1\mathrm{K}^{(d)}(X,-)\cong\pi_1\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e}$. Our goal is to show that the image of α by the composition of the homomorphisms of the first row is trivial. By the diagram above, there exists

$$\widetilde{\alpha} \in K \subset (E_1^{-1,0})^{(d)}_{\mathcal{C},e} = \pi_0 \mathrm{K}^{(d)}(X,1)_{\mathcal{C},e}$$

such that $\alpha \otimes \mathcal{O}_W$ coincides with $\widetilde{\alpha} \otimes \mathcal{O}_W$ in $\pi_1 \mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e}$. It suffices to show that the image of $\widetilde{\alpha} \otimes \mathcal{O}_W$ in $(E_2^{-1,0})^{(d+1)}$ is 0.

There exists a closed subscheme $Z \subset X \times \Delta^1$ belonging to $\mathcal{S}^{(d)}_{X,\mathcal{C},e}(1)$ (in particular, dimension 1) such that $\widetilde{\alpha}$ can be lifted to $\mathrm{K}_Z(X,1)$, which we denote by $\widetilde{\alpha}'$. We have $\widetilde{\alpha}' \otimes \mathcal{O}_W \in \pi_0 \mathrm{K}_{Z \cap \mathrm{pr}^{-1}(W)}(X \times \Delta^1)$ (where $\mathrm{pr}\colon X \times \Delta^1 \to X$ is the projection). Since $Z \in \mathcal{S}^{(d)}_{X,\mathcal{C},e}(1)$, the intersection $Z \cap \mathrm{pr}^{-1}(W)$ is 0-dimensional. By definition of $\mathcal{S}^{(d)}_{X,\mathcal{C},e}(1)$, note that $Z \cap \mathrm{pr}^{-1}(W) \subset X \times (\Delta^1 \setminus \{0,1\})$. The canonical coordinates of Δ^2 is denoted by t_1, t_2. Take a closed point $(w,s) \in X \times (\Delta^1 \setminus \{0,1\})$. Let

$$H_{(w,s)} := \{w\} \times \big\{t_1 + st_2 - s = 0\big\} \subset X \times \Delta^2,$$

namely the closed subscheme in $\{w\} \times \Delta^2$ which is the line connecting $(s,0)$ and $(0,1)$. We have the morphism $\rho_{(w,s)}\colon H_{(w,s)} \to \{(w,s)\} \hookrightarrow X \times \Delta^1$. Now, put

$$\beta := \bigoplus_{(w,s) \in Z \cap \mathrm{pr}^{-1}(W)} \rho^*_{(w,s)}(\widetilde{\alpha}' \otimes \mathcal{O}_W) \in \pi_0 \mathrm{K}_H(X \times \Delta^2) \qquad (H := \bigcup_{(w,s) \in Z \cap \mathrm{pr}^{-1}(W)} H_{(w,s)})$$

By construction, this will give a homotopy between $\widetilde{\alpha}' \otimes \mathcal{O}_W$ and 0. Indeed, let $f_1\colon X \times \Delta^1 \hookrightarrow X \times \Delta^2$ be the map defined by $t_2 = 1$, f_3 by $t_1 = 0$, f_2 by $t_1 + t_2 = 1$ sending 0, 1 to $(0,1)$, $(1,0)$ respectively. The homotopy β defines

$$f_1^*(\beta) + f_2^*(\beta) \sim f_3^*(\beta).$$

On the other hand, $f_1^*(\beta) = \widetilde{\alpha}' \otimes \mathcal{O}_W$ and $f_2^*(\beta) = f_3^*(\beta)$, thus $\widetilde{\alpha}' \otimes \mathcal{O}_W$ is homotopic to 0. □

3.8. (An elementary projection formula) In this paragraph, we recall an elementary projection formula, which will be used in the following paragraph in the discussion correspondences. Let $\varphi\colon X \to X$ be an automorphism. In this setting, we have the following elementary projective formula.

Lemma 3.8.1. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) & \xrightarrow{\mathrm{Id} \wedge \varphi_*} & \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \\ \big\downarrow{\scriptstyle \varphi^* \wedge \mathrm{Id}} & & \big\downarrow{\scriptstyle (-\otimes-)} \\ \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) & \xrightarrow{\varphi_* \circ (-\otimes-)} & \mathrm{K}^{(d)}(X,-) \end{array}$$

Proof. By definition of $(-\otimes-)$, we are reduced to showing the corresponding statement for each level of the simplicial spectrum corresponding to $\mathrm{K}^{(d)}(X,-)$. In that case, it follows directly from Thomason's projection formula for K-theory spectra ([10]). □

It follows that we have a commutative diagram at the level of Picard groupoids:

$$\begin{tikzcd}
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[r, "\mathrm{Id}\wedge\varphi_*"] \arrow[d, "\varphi^*\wedge\mathrm{Id}"] & \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[d, "(-\otimes-)"] \\
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[r, "\varphi_*\circ(-\otimes-)"] & \Pi(\mathrm{K}^{(d)}(X,-))
\end{tikzcd}$$

In particular, for $\mathcal{G} \in \Pi(\mathrm{K}^{(d)}(X,-))$ and $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X,-))$ we have natural isomorphisms

$$\mathrm{proj}_{\mathcal{G},\mathcal{F}} : \varphi_*(\varphi^*(\mathcal{G}) \otimes \mathcal{F}) \to \mathcal{G} \otimes \varphi_*\mathcal{F}.$$

3.9. (Formula for traces of tensor products of correspondences) We shall now prove a formula for the traces of tensor products of correspondences. Let $\varphi\colon X \to X$ be an endomorphism. Then one has an induced map $\varphi_*\colon \mathrm{K}^{(d)}(X,-) \to \mathrm{K}^{(d)}(X,-)$. Moreover, we also have the pushforward map

$$\pi_*^{(d)} : \mathrm{K}^{(d)}(X,-) \to \mathrm{K}(k).$$

Note that $\pi_*^{(d)} \circ \varphi_*$ is homotopic to $(\pi_*^{(d)} \circ \varphi)_* = \pi_*^{(d)}$. Below, we use the same notation to denote the corresponding induced morphisms on the associated Picard groupoids.

Definition 3.9.1. Let $\mathcal{F} \in \Pi\mathrm{K}^{(d)}(X,-)$. A *correspondence* $\Phi_{\mathcal{F}}$ on $\mathcal{F}$ is a morphism $\Phi_{\mathcal{F}}\colon \mathcal{F} \to \varphi_*\mathcal{F}$ in $\Pi\mathrm{K}^{(d)}(X,-)$.

Let $(\mathcal{F}, \Phi_{\mathcal{F}})$ be an object in $\Pi\mathrm{K}^{(d)}(X,-)$ endowed with a correspondence. Then we have morphisms

$$\pi_*^{(d)}(\Phi_{\mathcal{F}})\colon \pi_*^{(d)}(\mathcal{F}) \to \pi_*^{(d)}(\varphi_*\mathcal{F}) \cong \pi_*^{(d)}(\mathcal{F})$$

in $\Pi\mathrm{K}(k)$.

Suppose now we are given $\mathcal{G} \in \Pi\mathrm{K}(X)$ and a morphism $\Psi_{\mathcal{G}}\colon \varphi^*\mathcal{G} \to \mathcal{G}$. Then $\mathcal{F} \otimes \mathcal{G} \in \Pi\mathrm{K}^{(d)}(X,-)$ is endowed with a correspondence as follows:

$$\Psi_{\mathcal{G}} \otimes \Phi_{\mathcal{F}}\colon \mathcal{G} \otimes \mathcal{F} \xrightarrow{\mathrm{Id}\otimes\Phi_{\mathcal{F}}} \mathcal{G} \otimes \varphi_*\mathcal{F} \xleftarrow{\mathrm{proj}_{\mathcal{G},\mathcal{F}}} \varphi_*(\varphi^*(\mathcal{G}) \otimes \mathcal{F}) \xrightarrow{\varphi_*(\Psi_{\mathcal{G}}\otimes\mathrm{Id})} \varphi_*(\mathcal{G} \otimes \mathcal{F}).$$

In the following, we shall sometimes denote the trace $\mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{G}} \otimes \Phi_{\mathcal{F}})) \in k^\times$ by $\langle \mathcal{G}, \mathcal{F} \rangle^{\Pi}_{(d,-)} (\Psi_{\mathcal{G}}, \Phi_{\mathcal{F}})$.

Before stating the next proposition, we first set-up some notation. Recall, that $\mathrm{CH}_0(X) = \pi_0(\mathrm{K}^{(d)}(X,-))$ (2.6), and therefore we have a surjection

$$\mathrm{Z}_0(X) \to \pi_0(\mathrm{K}^{(d)}(X,-)).$$

It follows that given $[\mathcal{F}] \in \pi_0(\mathrm{K}^{(d)}(X,-))$, there exists $Z \in \mathrm{Z}_0(X)$ such that $[Z] = [\mathcal{F}] \in \mathrm{CH}_0(X)$. We fix such a Z, and the corresponding element $\mathcal{O}_Z \in \mathrm{K}_0(X)$. Note Z gives rise to an object, also denoted $\mathcal{O}_Z$, in $\Pi(\mathrm{K}^d(X))$ and $\Pi(\mathrm{K}^d(X,-)$. Furthermore, $[Z] = [\mathcal{F}]$ can be interpreted as saying that there is an isomorphism $\mathcal{O}_Z \cong \mathcal{F}$ in $\Pi(\mathrm{K}^d(X,-)$.

Remark 3.9.2. Note that here $\mathcal{O}_Z$ is a virtual object. If Z is an effective cycle then $\mathcal{O}_Z$ is precisely the structure sheaf of the corresponding artinian scheme.

Recall that for an object $\mathcal{G} \in \Pi(\mathrm{K}(X))$ we can consider its generic rank $r_{\mathcal{G}}$ (i.e. the image of $\mathcal{G}$ in $\pi_0(\mathrm{Pic}^{\mathbb{Z}}(X)) = \mathbb{Z}$), and that there is an object $\mathcal{G}_0 \in \Pi(\mathrm{K}(X))$ of generic rank 0 such that $\mathcal{G} \cong \det(\mathcal{O}_X^{r_{\mathcal{G}}}) + \mathcal{G}_0 \in \Pi(\mathrm{K}(X))$. For the latter statement, note that by a Lemma of Serre, any object $[\mathcal{G}] \in \mathrm{K}_0(X)$ can be written as $[\mathcal{G}_0] + [\mathcal{O}_X^{r_{\mathcal{G}}}] \in \mathrm{K}_0(X)$. It follows that one has an isomorphism between $\mathcal{G}$ and $\det(\mathcal{G}_0) + \det(\mathcal{O}_X^{r_{\mathcal{G}}})$ in $\Pi(\mathrm{K}(X))$. In the following, we denote $\det(\mathcal{O}_X^{r_{\mathcal{G}}})$ simply by $\mathcal{O}_X^{r_{\mathcal{G}}}$ and $\det(\mathcal{G}_0)$ by $\mathcal{G}_0$. Note that $\mathcal{O}_X$ comes equipped with a canonical correspondence $\mathrm{can} : \pi^*\mathcal{O}_X \to \mathcal{O}_X$, and therefore $\mathcal{O}_X^{r_{\mathcal{G}}}$ also comes equipped with a canonical correspondence (also denoted can). As a result, $\mathcal{G}_0 = \mathcal{G} + (\mathcal{O}_X^{r_{\mathcal{G}}})^{-1}$ also comes equipped with a correspondence. We shall denote the it by $\Psi_{\mathcal{G}_0}$.

Proposition 3.9.3. *Let X and $\varphi : X \to X$ be as above. Suppose $\mathcal{G} \in \Pi\mathrm{K}(X)$, $\mathcal{F} \in \Pi\mathrm{K}^{(d)}(X, -)$, and both are endowed with correspondences $\Psi_{\mathcal{F}} \colon \varphi^*\mathcal{G} \to \mathcal{G}$ and $\Phi_{\mathcal{F}} \colon \mathcal{F} \to \varphi_*\mathcal{F}$. Moreover, we fix an $\mathcal{O}_Z$ as above and a correspondence Φ_Z of $\mathcal{O}_Z$. Then one has the following formula:*

$$\mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{F}} \otimes \Phi_{\mathcal{G}})) = \mathrm{Tr}(\pi_*^{(d)}(\Phi_{\mathcal{F}}))^{r_{\mathcal{G}}} \times \langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} (\Psi_{\mathcal{G}_0}, \Phi_Z).$$

Moreover, $\langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} (\Psi_{\mathcal{G}_0}, \Phi_Z)$ is independent of the choice of Z and Φ_Z.

Proof. Let $\mathcal{G}_1 := \mathcal{O}_X^{r_{\mathcal{G}}}$. Then $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_0 \in \Pi(\mathrm{K}(X))$. Moreover, by definition, $\Psi_{\mathcal{G}} = \mathrm{Can} + \Psi_{\mathcal{G}_0}$. Since $\langle -, -\rangle^{\Pi}_{(d,-)}$ is distributive we have:

$$\mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{F}} \otimes \Phi_{\mathcal{G}})) = \mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{F}} \otimes \mathrm{Can})) \times \mathrm{Tr}(\pi_*^{(d)}(\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}_0}))$$

Since $\mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{F}} \otimes \mathrm{can})) = \mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{F}}))^{r_{\mathcal{G}}}$, we are reduced to showing that $\mathrm{Tr}(\pi_*^{(d)}(\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}_0})) = \langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} (\Psi_{\mathcal{G}_0}, \Phi_Z)$ and the latter is independent of the choice of Φ_Z. We fix an isomorphism $P : \mathcal{F} \to \mathcal{O}_Z$ in $\Pi(\mathrm{K}^{(d)}(X, -))$. The result will follow if we show that the following diagram is commutative:

$$\begin{tikzcd}
\langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} \arrow[r, "\langle \mathrm{Id}, \Phi_{\mathcal{F}}\rangle^{\Pi}_{(d,-)}"] \arrow[d, "\langle \mathrm{Id}, P\rangle"'] \arrow[dr, phantom, "\mathrm{a}"] & \langle \mathcal{G}_0, \varphi_*\mathcal{F}\rangle^{\Pi}_{(d,-)} \arrow[r, "\mathrm{proj}"] \arrow[d, "\langle \mathrm{Id}, \varphi_* P\rangle"] & \langle \varphi^*\mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} \arrow[r, "\langle \Psi_{\mathcal{G}_0}, \mathrm{Id}\rangle"] \arrow[d, "\langle \mathrm{Id}, P\rangle"] & \langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} \arrow[d, "\langle \mathrm{Id}, P\rangle"] \\
\langle \mathcal{G}_0, [\mathcal{O}_Z]\rangle^{\Pi}_{(d,-)} \arrow[r, "\langle \mathrm{Id}, \Phi_Z\rangle"'] & \langle \mathcal{G}_0, \varphi_*[\mathcal{O}_Z]\rangle^{\Pi}_{(d,-)} \arrow[r, "\mathrm{proj}"'] & \langle \varphi^*(\mathcal{L}, 0), [\mathcal{O}_Z]\rangle^{\Pi}_{(d,-)} \arrow[r, "\langle \Psi_{\mathcal{G}_0}, \mathrm{Id}\rangle"'] & \langle \mathcal{G}_0, [\mathcal{O}_Z]\rangle^{\Pi}_{(d,-)}
\end{tikzcd}$$

To check this, we only need to show the commutativity for the square "a". Since $\mathcal{G}_0$ has rank zero, there exists $C \in \Pi\mathrm{K}^{(1)}(X)$ such that $C \cong \mathcal{G}_0$. Then the diagram above is isomorphic to

$$\begin{tikzcd}
\langle C, \mathcal{F}\rangle^{\Pi}_{(d,-)} \arrow[r, "\langle \mathrm{Id}, \Phi_{\mathcal{F}}\rangle"] \arrow[d, "\langle \mathrm{Id}, P\rangle"'] & \langle C, \varphi_*\mathcal{F}\rangle^{\Pi}_{(d,-)} \arrow[d, "\langle \mathrm{Id}, \varphi_* P\rangle"] \\
\langle C, [\mathcal{O}_Z]\rangle^{\Pi}_{(d,-)} \arrow[r, "\langle \mathrm{Id}, \Phi_Z\rangle"'] & \langle C, \varphi_*[\mathcal{O}_Z]\rangle^{\Pi}_{(d,-)} .
\end{tikzcd}$$

All the morphisms in this diagram are isomorphisms, and so it is enough to show that the path $P^{-1} \circ \Phi_Z^{-1} \circ \varphi_* P \circ \Phi \in \pi_1\mathrm{K}^{(d)}(X, -)$ tensored with C is homotopic to the identity. In particular, we just need to show that it maps to the identity when viewed as an element of $\Pi(\mathrm{K}^{(d)}(X, -))$. But, this is precisely the content of Lemma 3.7.1. □

Corollary 3.9.4. *Suppose* $f:\mathcal{F}\to\mathcal{F}\in\Pi(\mathrm{K}^{(d)}(X,-))$ *and* $g:\mathcal{G}\to\mathcal{G}\in\Pi(\mathrm{K}(X))$*. Then*

$$\langle\mathcal{G},\mathcal{F}\rangle^{\Pi}_{(d,-)}(g,f)=\mathrm{Tr}(\pi^{(d)}_*(f)|\pi^{(d)}_*(\mathcal{F}))^{rg}\langle\mathcal{G},\mathcal{F}\rangle^{\Pi}_{(d,-)}(g,\mathrm{Id}).$$

Here, $\pi^{(d)}_*(f)$ *and* $\pi^{(d)}_*(\mathcal{F})$ *are the images under the natural map* $\Pi(\mathrm{K}^{(d)}(X,-))\xrightarrow{\pi_*}\mathrm{Pic}^{\mathbb{Z}}(k)$*.*

Proof. Note that if $\varphi=\mathrm{Id}:X\to X$, then a correspondence on $\mathcal{F}$ (resp. $\mathcal{G}$) just amounts to giving an endomorphism of $\mathcal{F}$ (resp. $\mathcal{G}$). The result is then a direct consequence of the previous proposition. □

4. Localization formula for holonomic $\mathcal{D}_X$-modules

In this section, we prove our main results on the global epsilon factors of tensor products of holonomic $\mathcal{D}_X$-modules and flat connections. In the following, $\pi:X\to\mathrm{Spec}(k)$ is a smooth projective variety over a field of characteristic zero.

4.1. (The main theorem) Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. We set

$$\varepsilon_{\mathrm{dR}}(X,\mathcal{F}):=\det\big(R\Gamma_{\mathrm{dR}}(X,\mathcal{F})\big)\in\mathrm{Pic}^{\mathbb{Z}}(k).$$

Below, we consider the following variant of the microlocalization map of Corollary 2.7.2:

$$\mathrm{CC}^{\mathrm{K}}\colon\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)\xrightarrow{\varepsilon}\mathrm{K}^{(d)}(T^*X)\to\mathrm{K}^{(d)}(T^*X,-)$$

where the second map is the natural augmentation map. Recall that we have defined a twisted pull-back map:

$$\sigma^+:\mathrm{K}(T^*X)\to\mathrm{K}(X).$$

In a completely analogous manner we can define the twisted pull-backs:

$$\sigma^+\colon\mathrm{K}^{(d)}(T^*X)\xrightarrow{\sigma^*}\mathrm{K}^{(d)}(X)\xrightarrow{\otimes\omega_X}\mathrm{K}^{(d)}(X),$$

and

$$\sigma^+\colon\mathrm{K}^{(d)}(T^*X,-)\xrightarrow{\sigma^*}\mathrm{K}^{(d)}(X,-)\xrightarrow{\otimes\omega_X}\mathrm{K}^{(d)}(X,-).$$

We set $\mathrm{CC}:=\sigma^+\circ\mathrm{CC}^{\mathrm{K}}$, and let $for^{\nabla}:\mathrm{K}_X(\mathcal{D}_X)\to\mathrm{K}(X)$ denote the morphism induced by forgetting the $\mathcal{D}_X$ module structure. Recall, this is well defined since any $\mathcal{D}_X$-module with singular support in the zero section is coherent as an $\mathcal{O}_X$-module. The following is the main result of this section.

Theorem 4.1.1. *The following diagram commutes up to homotopy equivalence:*

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X)\wedge\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "for^{\nabla}\wedge\mathrm{CC}"'] & \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[d, "R\Gamma_{\mathrm{dR}}"] \\
\mathrm{K}(X)\wedge\mathrm{K}^{(d)}(X,-) \arrow[r, "\langle -,-\rangle^{\mathrm{K}}_{(d,-)}"'] & \mathrm{K}(k).
\end{tikzcd}$$

Proof. We only need to show that the following two diagrams commute:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X)\wedge\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "for^{\nabla}\wedge\mathrm{CC}"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\sigma^+\circ gr"] \\
\mathrm{K}(X)\wedge\mathrm{K}^{(d)}(X,-) \arrow[r, "\otimes"'] & \mathrm{K}(X),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{K}(\mathcal{D}_X) \arrow[r, "\sigma^+\circ gr"] \arrow[d, "R\Gamma_{\mathrm{dR}}"'] & \mathrm{K}(X) \arrow[dl, "f_*"] \\
\mathrm{K}(k).
\end{tikzcd}$$

The commutativity of the right hand diagram follows from Proposition 2.7.4. Therefore, it is enough to verify that the diagram on the left is commutative. The bottom horizontal in this diagram is by definition the composition:

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \xrightarrow{\otimes} \mathrm{K}^{(d)}(X,-) \to \mathrm{K}^{(0)}(X,-) \to \mathrm{K}(X).$$

Since σ^+ commutes with $\otimes$ and augmentation, we are reduced to showing that the following diagram commutes:

$$\begin{tikzcd} \mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "for^\nabla \wedge \mathrm{CC}^{\mathrm{K}}"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\sigma^* \circ gr"] \\ \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\otimes"'] & \mathrm{K}(X), \end{tikzcd}$$

Note that for^∇ is homotopic to $\sigma^* \circ gr : \mathrm{K}_X(\mathcal{D}_X) \to \mathrm{K}_X(T^*X)$. Therefore, by 3.2.1 and 3.2.2, we are reduced to showing that the following diagram commutes:

$$\begin{tikzcd} \mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "gr \wedge \varepsilon"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "gr"] \\ \mathrm{K}_X(T^*X) \wedge \mathrm{K}^{(d)}(T^*X) \arrow[r, "\otimes"'] & \mathrm{K}(T^*X) \end{tikzcd}$$

By definition, this commutative diagram factors as:

$$\begin{tikzcd} \mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r] \arrow[d, "gr \wedge \varepsilon"'] & \mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "gr \wedge gr"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "gr"] \\ \mathrm{K}_X(T^*X) \wedge \mathrm{K}^{(d)}(T^*X) \arrow[r] & \mathrm{K}_X(T^*X) \wedge \mathrm{K}^(T^*X) \arrow[r, "\otimes"'] & \mathrm{K}(T^*X) \end{tikzcd}$$

The left square in this diagram commutes by Remark 2.7.3, and the right square commutes by Lemma 2.8.1. □

The theorem has the following direct consequence for the pairing $\langle -,-\rangle^{\Pi}_{(d,-)}$. Namely, let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-modules, and $\mathcal{G}$ a vector bundle with connection. Then, forgetting the connection, $\mathcal{G}$ gives rise to a natural object $\det(\mathcal{G}) \in \Pi(\mathrm{K}(X))$. On the other hand, $\mathcal{F}$ gives rise to an object of the Picard groupoid associated to $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$, and therefore, an object of $\Pi(\mathrm{K}^{(d)}(X,-))$ via the morphism CC. We denote the corresponding object by $\mathrm{CC}(\mathcal{F}) \in \Pi(\mathrm{K}^{(d)}(X,-))$. Applying the pairing

$$\langle -,-\rangle_{(d)} : \Pi(\mathrm{K}(X)) \wedge \mathrm{K}^{(d)}(X,-) \to \mathrm{Pic}^{\mathbb{Z}}(k)$$

to $\det(\mathcal{G})$ and $\mathrm{CC}(\mathcal{F})$ gives rise to an object $\langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F})\rangle \in \mathrm{Pic}^{\mathbb{Z}}(k)$.

Corollary 4.1.2. *There is a natural (in $\mathcal{F}$ and $\mathcal{G}$) isomorphism in* $\mathrm{Pic}^{\mathbb{Z}}(k)$*:*

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{G} \otimes \mathcal{F}) \cong \langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F})\rangle.$$

We now apply the previous corollary to compute traces of correspondences and endomorphisms. Let $\mathcal{F}$ denote a holonomic $\mathcal{D}_X$-module and $\mathcal{G}$ a flat connection as above, and fix an automorphism $\varphi : X \to X$.

Definition 4.1.3. A *correspondence* $\Phi_{\mathcal{F}}$ on $\mathcal{F}$ is an isomorphism $\Phi_{\mathcal{F}}\colon \mathcal{F} \to \varphi_*\mathcal{F}$ of $\mathcal{D}_X$-modules. Since φ is assumed to be an automorphism, this is equivalent to giving an isomorphism $\Psi_{\mathcal{F}}\colon \varphi^*\mathcal{F} \to \mathcal{F}$.

We fix correspondences $\Phi_{\mathcal{F}}$ and $\Psi_{\mathcal{G}}$ on $\mathcal{F}$ and $\mathcal{G}$. Note that if $\varphi = \mathrm{id}$ is the identity, then a correspondence is simply an automorphism. Moreover, just as in 3.9, one has an induced correspondence

$$\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}} : \mathcal{F} \otimes \mathcal{G} \to \varphi_*(\mathcal{F} \otimes \mathcal{G}).$$

It follows that one has an induced quasi-isomorphism:

$$R\Gamma(\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}}) : R\Gamma_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}) \to R\Gamma_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}).$$

We let $\varepsilon_{\mathrm{dR}}(X, \mathcal{F}\otimes\mathcal{G}; \Phi_{\mathcal{F}}\otimes\Psi_{\mathcal{G}}) := \mathrm{Tr}(\Phi_{\mathcal{F}}\otimes\Psi_{\mathcal{G}} : \det(R\Gamma_{\mathrm{dR}}(X, \mathcal{F}\otimes\mathcal{G})) \in k^\times$. If φ is the identity, we have simply automorphisms $f : \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{G}$ (as $\mathcal{D}_X$-modules). In this case we denote the corresponding epsilon factor by $\varepsilon_{\mathrm{dR}}(X, \mathcal{F}\otimes\mathcal{G}; f\otimes g) := \mathrm{Tr}(f\otimes g : \det(R\Gamma_{\mathrm{dR}}(X, \mathcal{F}\otimes\mathcal{G})) \in k^\times$. In the following, we fix a lift $\mathrm{SS}(\mathcal{F}) \in \mathrm{Z}_0(X)$ of $[\mathrm{CC}(\mathcal{F})] \in \mathrm{CH}_0(X)$. Moreover, we fix an object (as in 3.9), also denote by $\mathrm{SS}(\mathcal{F})$, of $\Pi(\mathrm{K}^{(d)}(X))$ whose image in $\Pi(\mathrm{K}^{(d)}(X,-))$ is isomorphic to $\mathrm{CC}(\mathcal{F})$ and finally $\mathcal{G}_0$ as in 3.9.3 (corresponding to $\det(\mathcal{G}) \in \Pi(\mathrm{K}(X))$). By Proposition 3.9.3, this data allows us to define $\langle \mathcal{G}_0, \mathrm{SS}(\mathcal{F})\rangle(\Psi_{\mathcal{G}})^{\Pi}_{(d,-)} \in k^\times$.

Theorem 4.1.4. *With notation as above:*

(i) One has

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; \Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}}) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}; \Phi_{\mathcal{F}})^{r_{\mathcal{G}}} \times \langle \mathcal{G}_0, \mathrm{SS}(\mathcal{F})\rangle(\Psi_{\mathcal{G}})^{\Pi}_{(d,-)}$$

(ii) In the setting of endomorphisms (i.e. $\varphi = \mathrm{id}$*), we have*

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}; f)^{r_{\mathcal{G}}} \times \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle(g)$$

Proof. The first equality follows directly from the theorem and Proposition 3.9.3. For the second statement, we note that in the setting of endomorphisms one has by Proposition 3.9.4:

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}; f)^{r_{\mathcal{G}}} \times \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle^{\Pi}_{(d,-)}(g, \mathrm{Id})$$

On the other hand, the latter is

$$\langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle^{\Pi}_{(d,-)}(g, \mathrm{Id}) = \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle^{\Pi}_{(d)}(g, \mathrm{Id}) = \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle(g)$$

by Corollary 3.5.1. $\square$

Remark 4.1.5. We note that $\mathrm{CC}(\mathcal{F}) \in \mathrm{CH}^d(X)$ is precisely the pull back of the characteristic cycle of $\mathcal{F}$ under the zero section $\sigma^* : \mathrm{CH}^d(T^*X) \to \mathrm{CH}^d(X)$.

4.2. (A formula for the local pairing) Let $\mathcal{G} \in \Pi\mathrm{K}(X)$, and $\Psi_{\mathcal{G}}\colon \varphi^*\mathcal{G} \to \mathcal{G}$ be a correspondence in $\Pi\mathrm{K}(X)$. As before, we shall let $\mathcal{G}_0$ denote the associated rank 0 object and $\Psi_{\mathcal{G}_0}$ the induced correspondence. Recall, the choice of such $\mathcal{G}_0$ is unique up to isomorphism. Assume given a cycle $z \in \mathrm{CH}^d(X)$ such that $z = \varphi_*(z)$. We take an object $\mathcal{O}_Z \in \Pi\mathrm{K}^{(d)}(X,-)$ which corresponds to z via the isomorphism $\pi_0\mathrm{K}^{(d)}(X,-) \cong \mathrm{CH}_0(X)$, and take a correspondence $P\colon \mathcal{O}_Z \to \varphi_*\mathcal{O}_Z$ as well. Since $z = \varphi_*(z)$, such a correspondence must exist (though it may not be unique).

In this setting, we have seen that in the proof of Proposition 3.9.3 that

$$\langle \mathcal{G}_0, \mathcal{O}_Z\rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}_0}, P) \in k^\times$$

is independent of the choice of $\mathcal{O}_Z$ and P. When z is represented by a $z_0 \in \mathrm{Z}^d(X)$ such that $\varphi_*(z_0) = z_0$, we may take P such that the description of the pair is especially simple. For simplicity, we assume that z_0 is a effective cycle. In the general case, we can proceed by writing it as a difference of two effective cycles. In this case, let W be the underlying reduced scheme of z_0 in X. Note that W is a smooth scheme of dimension 0. Since, by assumption, $\varphi_*(z_0) = z_0$, there exists an endomorphism φ_W of W such that

$$\begin{tikzcd} W \arrow[r, "\varphi_W"] \arrow[d, hook, "i"'] & W \arrow[d, hook, "i"] \\ X \arrow[r, "\varphi"'] & X \end{tikzcd}$$

is commutative. Since z_0 is an effective cycle, we may write $z_0 = \sum_{w\in|W|} n_w \cdot [w]$ where $n_w > 0$. We set $\mathcal{O}_{z_0} := \bigoplus_{w\in|W|} \mathcal{O}_w^{\oplus n_w}$. The endomorphism φ_W yields a correspondence $P\colon \mathcal{O}_{z_0} \to \varphi_*\mathcal{O}_{z_0}$ We can pull back the correspondence $\varphi^*A \to A$ by i, and get a correspondence $i^*\Psi\colon \varphi_W^*(i^*A) \to i^*A$. One can check that

$$\langle A, z\rangle\, (\Psi, P) = \mathrm{Tr}(R\Gamma(i^*\Psi)).$$

4.3. (Elementary proof of localization formula for endomorphisms) In this section, we give an elementary proof of the main theorem when the correspondence is merely automorphisms. While the proof below is elementary, it doesn't seem to generalize to the setting of correspondences (unlike the K-theoretic approach of the previous sections). We only give an outline of the proof below, and leave the details to the reader.

We begin by recalling the statement for the reader's convenience. Let X denotes a smooth projective variety over an algebraically closed field k of characteristic zero. Let $\mathcal{G}$ denote a flat connection on X, and $\mathcal{F}$ a holonomic $\mathcal{D}_X$-module. Let f (resp. g) denote a $\mathcal{D}_X$-module automorphism of $\mathcal{F}$ (resp. $\mathcal{G}$). Given a cycle $\mathrm{S}(\mathcal{F}) \in \mathrm{CH}_0(X)$ representing the pull-back (by the zero section) of the characteristic cycle of $\mathcal{F}$, we have defined the trace $\langle \det(\mathcal{G}), \mathrm{S}(\mathcal{F})\rangle(g) \in k^\times$. Note that $\mathrm{S}(\mathcal{F}) = [\mathrm{CC}(\mathcal{F})]$ using the previous notation.

Theorem 4.3.1. *With notation as above:*

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F}\otimes\mathcal{G}, f\otimes g) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}, f)^{r_\mathcal{G}} \times \langle \det(\mathcal{G})(g), \mathrm{S}(\mathcal{F})\rangle.$$

Proof. Suppose $0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$ is a finite filtration of $\mathcal{F}$ and that f is an endomorphism which preserves this filtration. Since both sides of the formula are compatible with exact sequences (i.e. are multiplicative), we are reduced to showing the validity of the given formula for $\mathcal{F}$ replaced by $gr_i(\mathcal{F})$ with the morphism induced by f. A similar assertion holds for $\mathcal{G}$. In particular, we can assume that $\mathcal{F}$ is a simple holonomic $\mathcal{D}_X$-module. Then, f is given by multiplication by a scalar. A similar assertion holds for $\mathcal{G}$ and g. Suppose $f = \alpha \in k^\times$ and $g = \beta \in k^\times$. Then the left hand side of the formula is given by $(\alpha\beta)^{\chi(\mathcal{F}\otimes\mathcal{G})}$. The right hand side is given by $\alpha^{\chi(\mathcal{F})r_\mathcal{G}}\beta^{r_\mathcal{G}\chi(\mathcal{F})}$. Therefore, we are reduced to showing that $\chi(\mathcal{F}\otimes\mathcal{G}) = \chi(\mathcal{F})r_\mathcal{G}$. This follows from a direct computation or by the Dubson-Kashiwara formula once one notes that the associated graded (with respect to a good filtration) commutes with the tensor product since $\mathcal{G}$ is $\mathcal{O}_X$-coherent. $\square$

References

[1] A. Beilinson. Topological e-factors. *Pure Appl. Math. Q.*, 3(1, part 3):357–391, 2007.
[2] P. Deligne. Le déterminant de la cohomologie. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume 67 of *Contemp. Math.*, pages 93–177. Amer. Math. Soc., Providence, RI, 1987.
[3] K. Kato and T. Saito. Ramification theory for varieties over a perfect field. *Ann. of Math. (2)*, 168(1):33–96, 2008.
[4] F.F. Knudsen. Determinant functors on exact categories and their extensions to categories of bounded complexes. *Michigan Math. J.*, 50(2):407–444, 2002.
[5] F.F Knudsen and D. Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.*, 39(1):19–55, 1976.
[6] M. Levine. Chow's moving lemma and the homotopy coniveau tower. *K-Theory*, 37(1-2):129–209, 2006.
[7] M. Levine. The homotopy coniveau tower. *J. Topol.*, 1(1):217–267, 2008.
[8] D. Patel. de Rham $\mathcal{D}$-factors. *Invent. Math.*, 190(2):299–355, 2012.
[9] S. Schwede. Symmetric spectra. *preprint.*
[10] R.W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.
[11] N. Umezaki, E. Yang, and Y. Zhao. Characteristic class and the ε-factor of an étale sheaf. *preprint.*

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