K-THEORY OF ALGEBRAIC MICRODIFFERENTIAL OPERATORS

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Abstract. Given a smooth variety X over a field k of characteristic zero, we construct a morphism of K-theory spectra $\mathrm{K}(\mathcal{E}_X|_V) \to \mathrm{K}(V)$, where $\mathcal{E}_X$ is the sheaf of microdifferential operators and $V \subset T^*X$ is a conic open subset. This generalizes a result of Quillen for the K-theory of $\mathcal{D}_X$-modules. As an application, we give another construction of de Rham epsilon factors.

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1. Introduction

The main goal of this article is to construct various microlocalization morphisms at the level of K-theory spectra. Furthermore, we shall apply these results to construct a theory of de Rham epsilon factors. The constructions here can also be applied to construct higher K-theoretic microlocal characteristic classes.

The prototype for the sort of microlocalization theorems considered here is a classical theorem of Quillen. We begin by recalling Quillen's theorem. Let (A, F) denote a positively filtered ring. Quillen ([13]) showed that, under certain conditions on (A, F), the extension of scalars morphism of K-theory spectra

$$\mathrm{K}(F^0A) \to \mathrm{K}(A)$$

is a weak equivalence of spectra. We refer to Theorem 3.13 for the precise statement. On the other hand, again under certain conditions, one has an extension of scalars morphism:

$$\mathrm{K}(F^0A) \to \mathrm{K}(gr(A))$$

As an application of Quillen's theorem, one can construct a natural morphism of K-theory spectra

$$\mathrm{K}(A) \to \mathrm{K}(gr(A)),$$

where $gr(A)$ denotes the associated graded ring.[1] Strictly speaking, the construction of the previous morphism requires one to be able to invert the weak equivalence $\mathrm{K}(F^0A) \to \mathrm{K}(A)$. In general, one cannot canonically invert a weak equivalence of spectra. However, one can canonically invert a weak equivalence of fibrant-cofibrant spectra as a *homotopy morphism* (cf. section 2.1). Below, we shall assume all our spectra are fibrant-cofibrant and morphisms mean homotopy morphisms.

We can restate Quillen's result in a more geometric language, and hence view it as an instance of microlocalization. Let X be a smooth affine variety over a field k of characteristic zero and $\mathcal{D}_X$ the sheaf of differential operators on X. The sheaf $\mathcal{D}_X$ comes equipped with a natural increasing filtration $\mathcal{D}_{p,X}$ by $\mathcal{O}_X$-submodules such that $\mathcal{D}_{p,X} = 0$ for all $p < 0$, $\mathcal{D}_{0,X} = \mathcal{O}_X$, and $\cup \mathcal{D}_{p,X} = \mathcal{D}_X$. Furthermore, one has a natural isomorphism $gr(\mathcal{D}_X) \overset{\cong}{\to} \pi_*(\mathcal{O}_{T^*X})$ where $\pi : T^*X \to X$ is the natural projection. Since X is affine, taking global sections induces weak equivalences:

$$\mathrm{K}(gr(\mathcal{D}_X)) \to \mathrm{K}(\Gamma(X, gr(\mathcal{D}_X)))$$

and

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(\Gamma(X, \mathcal{D}_X)).$$

In particular, Quillen's theorem gives a diagram of K-theory spectra:

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(\Gamma(X, \mathcal{D}_X)) \to \mathrm{K}(\Gamma(X, gr(\mathcal{D}_X))) \to \mathrm{K}(gr(\mathcal{D}_X)).$$

Furthermore, since π is a vector bundle, the natural pushforward morphism

$$\mathrm{K}(T^*X) \to \mathrm{K}(gr(\mathcal{D}_X))$$

is a weak equivalence. In particular, we have a natural morphism

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X)$$

of K-theory spectra. Note that this morphism is a weak equivalence.

[1] In the introduction, K-theory spectra will generally refer to the K-theory spectrum of finitely generated (left) modules over the given ring, or the category of coherent sheaves of (left) modules over a given sheaf of coherent rings. Since we will work with regular rings, we can safely replace these by categories of finitely generated projective modules (resp. locally free sheaves of modules).

The main result of this note generalizes the previous results to K-theory spectra of sheaves of microdifferential operators on X, and more generally to arbitrary smooth schemes over a field k. In the following, $\mathcal{E}_X$ denotes the sheaf of microdifferential operators on T^*X (cf. section 2.4). Locally, in the conical topology on T^*X, $\mathcal{E}_X$ is given by a certain Ore localization of the filtered sheaf $\mathcal{D}_X$ of differential operators. In particular, the global sections over an open affine conic V are a $\mathbb{Z}$-filtered ring. As a first step toward a version of Quillen's theorem for microdifferential operators, we have the following extension of Quillen's theorem to $\mathbb{Z}$-filtered rings.

Theorem 1.1. *Let (A, F) be a $\mathbb{Z}$-filtered ring. Suppose $B = gr_F(A)$ is noetherian and graded-regular. Then there is a natural morphism*

$$gr_A : \mathrm{K}(A) \to \mathrm{K}(B).$$

Note that, in the $\mathbb{Z}$-filtered case, the analog of Quillen's weak equivalence $\mathrm{K}(F_0A) \to \mathrm{K}(A)$ fails to hold. In particular, the strategy for constructing gr_A in the positively filtered case, via Quillen's theorem, is not applicable. We give a direct construction of this morphism using instead a spectra version of a result due to Van den Bergh (cf. Theorem 3.2). Note that Theorem 1.1 applies also to positively filtered rings. In particular, we get another construction, a priori different from the one via Quillen's theorem, of a microlocalization $\mathrm{K}(A) \to \mathrm{K}(gr(A))$. However, we show that the two are naturally identified as homotopy morphisms (cf. Proposition 3.15).

Next, we show that the microlocalization morphisms are compatible, in an appropriate sense, under zariski localization. This, combined with zariski descent for various presheaves of spectra, allows us to prove the following theorem:

Theorem 1.2. *Let X denote a smooth variety over a field k of characteristic zero. Let $V \subset T^*X$ denote a conic open subset of the cotangent bundle. Then there is a natural morphism of spectra*

$$\mathrm{K}(\mathcal{E}_X|_V) \to \mathrm{K}(V).$$

Furthermore, one has a commutative diagram:

$$\begin{array}{ccc} \mathrm{K}(\mathcal{E}_X|_V) & \longrightarrow & \mathrm{K}(V) \\ \uparrow & & \uparrow \\ \mathrm{K}(\mathcal{D}_X) & \longrightarrow & \mathrm{K}(T^*X). \end{array}$$

The vertical morphisms in this diagram are given by the natural localization maps.

Note that the existence of a morphism

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X)$$

for arbitrary (not necessarily affine) smooth X is part of the theorem above. This result can be obtained via a theorem of Hodges ([7]) for smooth quasi-projective varieties over a field k of characteristic zero. More precisely, Hodges showed that there is a natural weak equivalence of spectra:

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(X).$$

The proof proceeds via an induction and a Bertini argument using Quillen's result in the affine case, and Kashiwara's theorem. This then gives rise to a morphism $\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X)$. Again, this construction is naturally homotopic to the one we construct here.

Let $\mathrm{K}_S(\mathcal{D}_X)$ denote the K-theory spectrum of the category of coherent $\mathcal{D}_X$-modules with support contained in a closed conic $S \subset T^*X$. As an application Theorem 1.2 we construct a natural homotopy morphism:

$$\mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X)$$

Here $\mathrm{K}_S(T^*X)$ denotes the K-theory spectrum of coherent sheaves on T^*X with support in S. Furthermore, this morphism is functorial in S. More precisely, if $S \subset S'$, then one has a commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_S(\mathcal{D}_X) & \longrightarrow & \mathrm{K}_S(T^*X) \\ \downarrow & & \downarrow \\ \mathrm{K}_{S'}(\mathcal{D}_X) & \longrightarrow & \mathrm{K}_{S'}(T^*X) \end{array}$$

The vertical morphisms are induced by the natural inclusion of categories of modules with support contained in S into that of modules with support contained in S'. In addition, we have the following microdifferential operators version of the previous result.

Theorem 1.3. *Let X be a smooth variety over a field k of characteristic zero. Then one has a natural morphism*

$$\mathrm{K}_S(\mathcal{E}_X) \to \mathrm{K}_S(T^*X)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}_S(\mathcal{E}_X) & \longrightarrow & \mathrm{K}_S(T^*X) \\ \uparrow & \nearrow & \\ \mathrm{K}_S(\mathcal{D}_X) & & \end{array}$$

The above constructions can be extended to the K-theory spectrum of the derived category of $\mathcal{D}_X$ (resp. $\mathcal{E}_X$) modules with coherent cohomology. In this setting, if X is proper, then one can identify the composition

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X) \to \mathrm{K}(X) \stackrel{R\Gamma(X,-)}{\to} \mathrm{K}(k)$$

with the morphism $\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$ induced by $R\Gamma_{dR}(X, \cdot)$. As a result, we obtain another proof of the following theorem ([11], [12]), giving rise to a theory of de Rham epsilon factors. Let S denote a closed conic in T^*X. Let $U \subset X$ denote an open subset and ν a 1-form on U such that $\nu(U) \cap S = \emptyset$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module (or more generally a perfect complex of $\mathcal{D}_X$-modules) with singular support in S. Then $\mathcal{M}$ gives rise to a homotopy point (cf. section 2.3) of the $\mathrm{K}_S(\mathcal{D}_X)$. On the other hand, it also gives rise to a homotopy point of the spectrum $\mathrm{K}(\mathcal{D}_X)$. The image of this homotopy point under $R\Gamma_{dR}(X,-)$ is the homotopy point $[R\Gamma_{dR}(X,\mathcal{M})]$. Let Y denote the complement of U. Then, since Y is also proper, one has a natural morphism $\mathrm{K}(Y)\to\mathrm{K}(k)$ given by $R\Gamma(Y,-)$.

Theorem 1.4. *For every ν and $\mathcal{M}$ as above, there is a natural homotopy point $\mathcal{E}_{\nu,Y}(\mathcal{M})$ of $\mathrm{K}(Y)$ such that the homotopy point $[R\Gamma(\mathcal{E}_{\nu,Y}(\mathcal{M}))]$ of $\mathrm{K}(k)$, induced via composition with $R\Gamma(Y,-)$, is naturally identified with the homotopy point $[R\Gamma_{dR}(X,\mathcal{M})]$. Furthermore, $\mathcal{E}_{\nu,Y}(\mathcal{M})$ only depends on the values of $\mathcal{M}$ and ν on a formal neighborhood of Y.*

The determinant philosophy (see section 2.3), allows one to associate to any homotopy point A of $\mathrm{K}(k)$ an object of the Picard groupoid, $Pic^{\mathbb{Z}}(k)$, of $\mathbb{Z}$-graded lines. Applying the determinant construction to the homotopy point $[R\Gamma_{dR}(X,\mathcal{M})]$ gives the determinant of de Rham cohomology $\det(R\Gamma_{dR}(X,\mathcal{M}))$. Let $\varepsilon_{\nu,Y}(\mathcal{M})$ denote the object of the Picard groupoid associated to

$$[R\Gamma(\mathcal{E}_{\nu,Y}(\mathcal{M}))].$$

Then the above theorem gives a natural isomorphism:

$$\det(R\Gamma_{dR}(X,\mathcal{M}))\cong\varepsilon_{\nu,Y}(\mathcal{M}).$$

If $Y=\coprod Y_i$ then one has a natural decomposition of epsilon factors:

$$\varepsilon_{\nu,Y}(\mathcal{M})\cong\otimes\varepsilon_{\nu,Y_i}(\mathcal{M}).$$

In particular, if X is a curve, then the above theory gives rise to a theory of de Rham epsilon factors in the sense ([3]).

A different treatment of de Rham epsilon factors was given in ([12]), where a microlocalization morphism

$$\mathrm{K}_S(\mathcal{D}_X)\to\mathrm{K}_S(T^*X)$$

was constructed directly via the theory of filtered $\mathcal{D}_X$-modules. The approach via filtered $\mathcal{D}_X$-modules seems more suitable for comparison with the betti epsilon factors ([2]), since such a comparison would first require a construction of an analogous theory for analytic $\mathcal{D}_X$-modules. Furthermore, it seems easier to study the behavior of epsilon factors under push-forward and pull-back in the setting of filtered $\mathcal{D}_X$-modules ([12], 3.27, 3.28, 3.30). On the other hand, the approach via $\mathcal{E}_X$ seems to apply to the setting of Berthelot's arithmetic D-modules. The theory of arithmetic microdifferential operators has been developed recently by Abe ([1]). Another upshot of using microdifferential operators is that, as a result of 1.3, the local epsilon factors only depend on the $\mathcal{E}_X$-module $\mathcal{E}_X\otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M}$.

In the setting of algebraic $\mathcal{D}_X$-modules considered here, we expect that the two homotopy morphisms $\mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X)$, the one constructed here via microdifferential operators and the one constructed in ([12]) via filtered $\mathcal{D}_X$-modules, are naturally identified as homotopy morphisms. It is easy to see that the two morphisms give the same morphism at the level of K-groups (i.e. at the level of homotopy groups). It is also possible to construct a theory of epsilon factors for $\mathcal{E}_X$-modules using the constructions of this article by essentially repeating the proof for $\mathcal{D}_X$-modules. The crucial ingredient is the existence of a microlocalization map $\mathrm{K}_S(\mathcal{E}_X) \to \mathrm{K}_S(T^*X)$.

One can also use the morphisms $\mathcal{E}_S$ to construct microlocal characteristic classes with supports. For example, the natural composition

$$\mathrm{K}_0(\mathcal{E}_X) \to \mathrm{K}_0(T^*X) \to \mathrm{CH}^*(T^*X)$$

sends $\mathcal{M}$ to its characteristic cycle. Therefore, one can use $\mathcal{E}_S$ to construct various micro local characteristic classes on Higher K-theory. We hope to investigate this further elsewhere.

We now describe the contents of each section. In section 2, we recall some background material on spectra, K-theory, sheaves of spectra, and microdifferential operators. All the material in this section is standard, and recalled here for the reader's convenience. In section 3.1, we recall a theorem of van den bergh at the level of spectra. In section 3.2, we apply this result to prove Theorem 1.1. In section 3.3, we sheafify the constructions of section 3.3. In particular, we give a proof of Theorem 1.3. In section 3.4, we apply these results to construct a theory of de Rham epsilon factors.

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2. Preliminaries

In this section, we recall some background material needed in the rest of the paper. The first two sub-sections recall some standard results on spectra and sheaves of spectra. These sections are essentially a recap of the corresponding sections in ([2], [12]) and we refer the reader to loc. cit. for details. The third sub-section recalls some parts of the theory of K-theory spectra and their relation to determinants. Again, the reader is referred to ([2], [12]) for details. Finally, in the last sub-section we recall the construction of (algebraic) microdifferential operators following Laumon ([9]).

2.1. **Spectra.** In this section, we review some basic facts and constructions about the category of simplicial spectra (as in [4]). Recall that a *spectrum* is a sequence of pointed simplicial sets $(P_n)_{n\geq 0}$ together with structure maps $\sigma_n : S^1 \wedge P_n \to P_{n+1}$, where S^1 denotes the one sphere. We denote by $\mathcal{S}$ the category of spectra. The category $\mathcal{S}$ comes equipped with a simplicial structure. Given a simplicial set K, we can define a spectrum $K \wedge P$ whose i-th space is given by $K \wedge P_i = K_+ \wedge P_i$ with the obvious structure maps. Here K_+ is the simplicial set given by K disjoint union a base point. This gives rise to the functor $K \wedge \cdot : \mathcal{S} \to \mathcal{S}$ which has a natural right adjoint given by $P \to P^K$. Then for two spectra P, Q one can define a simplicial set $Map(P, Q)$ whose n-simplices are given by:

$$Map(P, Q)(n) = \mathrm{Hom}_{\mathcal{S}}(P, Q^{\Delta_n}) = \mathrm{Hom}_{\mathcal{S}}(\Delta_n \wedge P, Q).$$

Given a (pointed) simplicial set K, we denote by $|K|$ its geometric realization. Then the *homotopy groups* of a spectrum P are defined by $\pi_i(P) = \lim\limits_{\to} \pi_{i+n}(|P_n|)$ for all $i \in \mathbb{Z}$; here the limit is taken over the maps induced by the structure maps. A morphism of spectra is a *weak equivalence* if it induces an isomorphism on the corresponding homotopy groups. A morphism $f : P \to Q$ is a cofibration if the induced maps $P_0 \to Q_0$ and $P_n \cup_{S^1 \wedge P_{n-1}} (S^1 \wedge Q_{n-1}) \to Q_n$ are inclusions. The above notions of weak equivalence and cofibration give $\mathcal{S}$ the structure of a stable and proper simplicial model category (see [4], [6]). The category of spectra has an initial and terminal object. A spectrum is *fibrant* if the natural morphism to the terminal object is a fibration and it is *cofibrant* if the natural map from the initial object is a cofibration. Finally, the category of spectra has functorial fibrant-cofibrant replacements.

The homotopy category of $\mathcal{S}$ is denoted by $\mathrm{Ho}(\mathcal{S})$. By definition, this is the localization of $\mathcal{S}$ with respect to the weak equivalences. A weak equivalence of spectra $P \to Q$ can be inverted as a morphism in the homotopy category. However, in general such a morphism cannot be inverted as a morphism of spectra. To remedy this situation, one must use the more general notion of a homotopy morphism of spectra. A *homotopy morphism* $P \to Q$ consists of a contractible simplicial set K and a genuine morphism of spectra $f : K \wedge P \to Q$. We refer to K as the base of the homotopy morphism, and by abuse of notation we shall denote the homotopy morphism by $f : P \to Q$. Given two homotopy morphisms f, g with bases K_f, K_g, an *identification of f and g* is a homotopy morphism h with base K_h together with morphisms $K_f \to K_h \leftarrow K_g$ such that f, g are the respective pullbacks of h. One can define the composition of two homotopy morphisms $f : P \to Q$ and $g : Q \to R$ as the composition $K_g \wedge K_f \wedge P \to K_g \wedge Q \to R$. A homotopy morphism from a sphere spectrum to a given spectrum P will be referred to as a homotopy point of P. If f and g are identified, then they induce the same maps on homotopy groups.

A weak equivalence between fibrant-cofirant spectra can be canonically inverted as a homotopy morphism. This is the main reason for working with homotopy morphisms. More precisely, let P, Q be fibrant-cofibrant spectra and $f : P \to Q$ a weak equivalence of spectra. Then, a right homotopy inverse to f is a pair (g_r, h_r), where g_r is a morphism $Q \to P$ and h_r is a homotopy $\Delta_1 \wedge Q \to Q$ between fg_r and Id_Q. Dually, one can define the notion of left homotopy inverses. One has analogs of these definitions for homotopy morphisms.

Lemma 2.1. *Let $f : P \to Q$ be a weak equivalence of fibrant-cofibrant objects. Then there exists a canonical right homotopy inverse g_r and left homotopy inverse g_l. Furthermore, there is a natural identification of g_r with g_l.*

Proof. See ([12], Lemma 2.1) □

One also has a notion of *homotopy sum* for spectra. Let I be a finite set. Then one has a canonical morphism of spectra $e_I : \vee_I P \to P^I$ induced by the identity on the (i,i)-th component and trivial on other components. For $k \in I$, let $i_k : P \to \vee_I P$ denote the inclusion onto the k-th component.

Lemma 2.2. *(*[12]*, Lemma 2.3) Suppose P is a fibrant-cofibrant spectrum. Then one has a canonical homotopy morphism (the sum) $\Sigma_I : P^I \to P$ such that the composition $\Sigma_I e_I i_k : P \to P$ is given by id_P.*

2.2. **Presheaves of spectra.** In this section, we give a basic overview of some model structures on the category of presheaves of spectra, and recall some elementary facts. We refer the reader to ([2]) and ([5], section 3) for an excellent overview of these matters.

Given a category $\mathcal{T}$, let $\mathrm{Psh}(\mathcal{T}, \mathcal{S})$ denote the category of presheaves of spectra on $\mathcal{T}$. If $\mathcal{P} \in \mathrm{Psh}(\mathcal{T}, \mathcal{S})$, let $\pi_i \mathcal{P}$ denote the associated presheaf of homotopy groups. If $\mathcal{T}$ comes equipped with a Grothendeick topology, then we denote by $\pi_i^s \mathcal{P}$ the sheaf associated to the presheaf of homotopy groups. A morphism of presheaves of spectra $f : \mathcal{P} \to \mathcal{Q}$ is a *global weak equivalence* if the induced morphism $f : \pi_i(\mathcal{P}) \to \pi_i(\mathcal{Q})$ is an isomorphism. A morphism $f : \mathcal{P} \to \mathcal{Q}$ is a *local weak equivalence* if the induced morphism $f : \pi_i^s(\mathcal{P}) \to \pi_i^s(\mathcal{Q})$ is an isomorphism. We say that f is a *global cofibration* if $f : \mathcal{P}(U) \to \mathcal{Q}(U)$ is a cofibration for all $U \in \mathcal{T}$. A morphism f is a *global fibration* if $f : \mathcal{P}(U) \to \mathcal{Q}(U)$ is a fibration for all $U \in \mathcal{T}$.

The *global projective model structure* on $\mathrm{Psh}(\mathcal{T}, \mathcal{S})$ is given by the global weak equivalences and global fibrations. The cofibrations are then defined to be those morphisms which have the left lifting property with respect to the acyclic fibrations. This gives $\mathrm{Psh}(\mathcal{T}, \mathcal{S})$ the structure of a stable proper simplicial model category. The cofibrations in this model structure will be referred to as global projective cofibrations. If f is a global projective cofibration, then the induced morphisms $f : \mathcal{P}(U) \to \mathcal{Q}(U)$ are cofibrations. (However, the converse is false.)

The *local injective model structure* on $\mathrm{Psh}(\mathcal{T}, \mathcal{S})$ is given by the local weak equivalences and the global cofibrations. The fibrations are then defined to be those morphisms which satisfy the right lifting property with respect to the acyclic cofibrations. This gives $\mathrm{Psh}(\mathcal{T}, \mathcal{S})$ the structure of a stable proper simplicial model category. We fix a functorial fibrant replacement $\mathcal{P} \to \mathbb{H}(-, \mathcal{P})$ in the local injective model structure. We will say that $\mathcal{P}$ satisfies descent for $\mathcal{T}$ if the canonical morphism $\mathcal{P} \to \mathbb{H}(-, \mathcal{P})$ is a global weak equivalence. We let $R\Gamma(\mathcal{T}, \mathcal{P}) = \lim_{\mathcal{T}^o} \mathbb{H}(-, \mathcal{P})$. The fibrant objects in the injective model structure are characterized by a homotopy descent property.

In the following, we shall mostly play with the zariski topology on a smooth scheme X or the conical topology on the corresponding cotangent bundle. If $\mathcal{F}$ is a presheaf of spectra in either of these two topologies, then to check that it satisfies descent it enough to check that if U, V are open affines in X (resp. of T^*X), then one has a cartesian square of spectra:

$$\begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

In practice, this amounts to having a localization sequence for closed subschemes and an excision property.

If $\mathcal{T}' \subset \mathcal{T}$ is a subcategory (with induced topology) then we have a canonical restriction functor

$$\mathrm{Psh}(\mathcal{T}, \mathcal{S}) \to \mathrm{Psh}(\mathcal{T}', \mathcal{S}).$$

Suppose every object of $\mathcal{T}$ has an open covering by objects in $\mathcal{T}'$, and every $\mathcal{T}$-covering of an object in $\mathcal{T}'$ has a $\mathcal{T}'$-refinement. Then the induced functor on homotopy categories is an equivalence. In particular, a fibrant-cofibrant object of $\mathrm{Psh}(\mathcal{T}', \mathcal{S})$ (in the local injective model structure) object can be canonically lifted to a fibrant-cofibrant object of $\mathrm{Psh}(\mathcal{T}', \mathcal{S})$.

Suppose $\mathcal{T}' \subset \mathcal{T}$ is as in the previous paragraph. Let $\mathcal{F}$ denote a fibrant-cofibrant object of $\mathrm{Psh}(\mathcal{T}, \mathcal{S})$. Then, let $\mathcal{F}^{\mathcal{T}'}$ denote the presheaf on $\mathcal{T}$ whose sections over an open U are given by the homotopy limit of $\mathcal{F}(V)$ over all open V in $\mathcal{T}'$ such that V is in a $\mathcal{T}'$ covering family of U (i.e. V is a $\mathcal{T}'$-open of U). Then the natural restriction map $\mathcal{F} \to \mathcal{F}^{\mathcal{T}'}$ is a global weak equivalence. We shall use this statement to reduce constructions on various presheaves of K-theory spectra on a scheme X to their counterparts over affine opens in X. For example, suppose $\mathcal{F}$ is a presheaf of spectra on the Zariski site of a scheme X such that $\mathcal{F}$ satisfies descent. Let $\mathcal{F}^{aff}$ denote the presheaf whose sections over an open $U \subset X$ are given by the homotopy limit over all V such that V is an affine open in U. Then the natural morphism $\mathcal{F} \to \mathcal{F}^{aff}$, given by restriction, is a global weak equivalence.

Let $p : X \to Y$ denote a morphism of schemes and $\mathcal{F}$ a presheaf of spectra on X. Note that the push-forward $p_*\mathcal{F}$ satisfies Zariski descent iff $\mathcal{F}$ satisfies Zariski descent. If X is smooth, then one can apply this to the natural projection $T^*X \to X$, where T^*X is the cotangent bundle.

Let $\mathcal{F}$ and $\mathcal{G}$ denote presheaves of spectra on a site. A homotopy morphism $\phi : \mathcal{F} \to \mathcal{G}$ is given by a datum of homotopy morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, for each open U, such that for each open V of U, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V). \end{array}$$

Recall, a diagram of homotopy morphisms, as above, is said to commute if the two resulting homotopy morphisms

$$\mathcal{F}(U) \to \mathcal{G}(V)$$

are identified.

2.3. **K-theory spectra.** Let $\mathcal{E}$ be a small exact category. Then Quillen's K-theory construction gives a functor from the category of small exact categories to the category of spectra. Since $\mathcal{S}$ has functorial fibrant-cofibrant replacements, we assume from now on that the associated spectrum $\mathrm{K}(\mathcal{E})$ is fibrant-cofibrant. If $F_1 : \mathcal{E}_1 \to \mathcal{E}_2$ and $F_2 : \mathcal{E}_2 \to \mathcal{E}_3$ are exact functors, then one has $\mathrm{K}(F_2) \circ \mathrm{K}(F_1) = \mathrm{K}(F_2 \circ F_1)$. More generally, a natural isomorphism of functors induces a canonical homotopy identification of the corresponding morphisms of K-theory spectra. By taking a large enough Grothendieck universe, we may assume all our categories are small.

More generally, Waldhausen associates to any category with cofibrations and weak equivalences a corresponding K-theory spectrum. Furthermore, an exact functor between Waldhausen categories induces a morphism between the corresponding spectra. In this article, we shall mostly be interested in complicial bi-Waldhausen categories and complicial exact functors; we refer the reader to ([14]) for details. If $\mathcal{E}$ is an exact category, then $C^b(\mathcal{E})$ is a complicial bi-Waldhausen category with weak equivalences. We refer the reader to ([14]) for details. A fundamental result of Thomason–Trobaugh–Waldhausen–Gillet ([14]) shows that the inclusion of $\mathcal{E}$ into $C^b(\mathcal{E})$ as degree zero morphisms induces a canonical weak equivalence of spectra $\mathrm{K}(\mathcal{E}) \to \mathrm{K}(C^b(\mathcal{E}))$. Here the right side is the Waldhausen K-theory spectrum associated to $C^b(\mathcal{E})$. This allows us to canonically identify various Quillen and Waldhausen K-theory spectra. In the following, we shall always assume all our spectra to be fibrant-cofibrant. In particular, the machinery from the previous section will allow us to invert various weak equivalences canonically as homotopy morphisms.

Given a Waldhausen category $\mathcal{A}$, we denote by $\mathcal{A}^{\mathrm{tri}}$ the associated homotopy category given by inverting the weak equivalences; note that this is a triangulated category. If $F : \mathcal{A} \to \mathcal{B}$ is a complicial exact functor between two complicial bi-Waldhausen categories such that the induced map on homotopy categories is an equivalence of categories, then the induced map on K-theory spectra is a weak equivalence. We will often consider derived functors which are *a priori* only defined on $\mathcal{A}^{tri}$. Usually, these can be lifted to functors on certain full complicial bi-Waldhausen subcategories $\mathcal{C} \subset \mathcal{A}$ such that the inclusion induces an equivalence on the associated triangulated categories. Using the formalism of homotopy morphisms, we can lift the derived functor to a morphism of K-theory spectra. A typical application is the following: Let X be a proper scheme over k, and let $\mathrm{K}(X)$ be the K-theory spectrum of perfect complexes on X. Since X is proper, we can define $R\Gamma : D^b_{\mathrm{perf}}(X) \to D^b_{\mathrm{perf}}(k)$. The above approach allows us to lift this to a homotopy morphism $R\Gamma : \mathrm{K}(X) \to \mathrm{K}(k)$, where $\mathrm{K}(X)$ is the K-theory spectrum of the category of perfect complexes on X and similarly for $\mathrm{K}(k)$. First, we may consider the (full) complicial bi-Waldhausen sub-category of flasque perfect complexes. On this subcategory, $R\Gamma$ is represented by Γ. Furthermore, the properness assumption implies that Γ preserves perfectness. We refer to the article by Thomason–Trobaugh ([14]) for more details.

We conclude this section with some remarks on K-theory spectra and determinants. We refer to ([12], section 5) for details. Given a Waldhausen category $\mathcal{A}$, any object F in $\mathcal{A}$ gives rises to a homotopy point $[F]$ of the associated K-theory spectrum $\mathrm{K}(\mathcal{A})$. In the situation of an exact category $\mathcal{E}$, this construction gives a canonical homotopy point $[F]$ of $\mathrm{K}(C^b(\mathcal{E}))$ for all $F \in \mathrm{Ob}(C^b(\mathcal{E}))$. Furthermore, to any [0,1]-connected Ω-spectrum K we can associate a canonical Picard groupoid denoted by $\Pi(\mathrm{K})$; any homotopy point of K gives rise to an object of the associated Picard groupoid. For any spectrum K, we can functorially associate a $[0,1]$ connected Ω-spectrum denoted $\mathrm{K}^{[0,1]}$ with a morphism $\mathrm{K} \to \mathrm{K}^{[0,1]}$. In the case of $\mathrm{K}(C^b(\mathcal{E}))$, we can apply the above to get an object $\mathrm{Det}(x) \in \Pi(\mathrm{K}(C^b(\mathcal{E}))^{[0,1]})$ for any homotopy point x of $\mathrm{K}(C^b(\mathcal{E}))$. Furthermore, the homotopy point construction induces a determinant functor $\mathrm{Det} : (C^b(\mathcal{E}), w) \to \Pi(K(C^b(\mathcal{E}))^{[0,1]})$, which is a universal determinant functor in the sense of Knudsen ([8]). This functor factors through the derived category $(D^b(\mathcal{E}), qis) \to \Pi(\mathrm{K}(C^b(\mathcal{E}))^{[0,1]})$ as a tensor functor. Here $D^b(\mathcal{E})$ has tensor structure coming from the additive structure. Furthermore, an identification of homotopy points gives rise to an isomorphism of the corresponding determinants. If x and y are two homotopy points of $\mathrm{K}(C^b(\mathcal{E}))$, then one has a canonical isomorphism $\cdot_{xy} : \mathrm{Det}(x) \otimes \mathrm{Det}(y) \to \mathrm{Det}(x+y)$; here $x+y$ is the homotopy sum described in 2.1.

Remark 2.3. If F and G are objects in a Waldhausen category $\mathcal{A}$, then the homotopy point $[F \oplus G]$ of the direct sum of F and G in $\mathcal{A}$ is identified with the homotopy sum $[F] + [G]$.

Let $Pic^{\mathbb{Z}}(k)$ denote the Picard groupoid of $\mathbb{Z}$-graded lines on k, whose objects are ordered pairs of one dimensional k-vector spaces and an integer n, the degree of the line. Then there is a canonical determinant functor $\det : C^b(k) \to Pic^{\mathbb{Z}}(k)$. If V is a vector space in degree zero, this functor sends V to the usual determinant line graded by the dimension of the vector space. In the particular case of a scheme X proper over k, the determinant of $R\Gamma(X, F)$ is just the usual determinant of cohomology graded by the Euler characteristic. If S is a scheme, then we can define the Picard groupoid $Pic^{\mathbb{Z}}(S)$ of $\mathbb{Z}$-graded lines on S. The grading will be a $\mathbb{Z}$-valued locally constant function on S. Then, as in the case of a field, one has a determinant functor $\det_S : C^b_{perf}(S) \to Pic^{\mathbb{Z}}(S)$. By universality, there is a canonical morphism of Picard groupoids $\mathrm{Det}_S : \Pi(K(S)) \to Pic^{\mathbb{Z}}(S)$ such that the following diagram commutes:

$$\begin{array}{ccc} C^b_{perf}(S) & \xrightarrow{\mathrm{Det}} & \Pi(K(S)) \\ & \searrow^{\det_S} & \downarrow{\scriptstyle \mathrm{Det}_S} \\ & & Pic^{\mathbb{Z}}(S). \end{array}$$

In particular, if A and B are perfect complexes, then the image of $\mathrm{Det}_S(\cdot_{[A][B]})$ is just the usual isomorphism $\det_S(A) \otimes \det_S(B) \to \det_S(A \oplus B)$.

Suppose we have an exact functor of exact categories $\mathcal{E}_1 \to \mathcal{E}_2$. Then we get an induced map $F : \mathrm{K}(\mathcal{E}_1) \to \mathrm{K}(\mathcal{E}_2)$. If $A \in \mathrm{Ob}(\mathcal{E}_1)$, then there is a canonical identification of the homotopy points $F([A])$ and $[F(A)]$. Furthermore, an identification of homotopy points gives rise to an isomorphism of the corresponding determinants. One has a similar statement for K-theory spectra of Waldhausen categories.

2.4. **Microdifferential operators.** In this section, we recall the construction of the sheaf of microdifferential operators following Laumon ([9]). We refer to ([10]) for the relevant results on filtered rings and modules.

A *filtered ring* is a pair (A, A_i) consisting of a (not necessarily commutative) ring A with an increasing filtration by subgroups A_i such that $A_iA_j \subset A_{i+j}$ and $i \in \mathbb{Z}$. The filtration is *exhaustive* if $\cup_i A_i = A$. A filtration is *separated* if $\cap_i A_i = e$. In the following, we shall assume all our filtrations are exhaustive and separated. A *filtered module* over (A, A_i) is a pair (M, M_i) consisting of an A-module M with an increasing filtration by subgroups M_i such that $A_iM_j \subset M_{i+j}$. The filtration M_i on M is said to be good if there exist $m_1, \ldots, m_r \in M$ and $k_1, \ldots, k_r \in \mathbb{Z}$ such that $M_i = \Sigma_{j=1}^r A_{i-k_j} m_j$. Again, we shall assume that the filtration on M is separated and exhaustive. Any finitely generated A-module has a good filtration.

A morphism of filtered rings $f : (A, A_i) \to (B, B_i)$ is a homomorphism of rings $f : A \to B$ such that $f(A_i) \subset B_i$. We say that f is strict if $f(A_i) = f(A) \cap B_i$. We can similarly define morphisms and strict morphisms of filtered modules.

Given a filtered ring (A, A_i), we let $gr.(A) = \oplus gr_i(A)$ where $gr_i(A) = A_i/A_{i-1}$. Thus $gr.$ is a $\mathbb{Z}$-graded ring. Any filtered module (M, M_i) gives rise to a graded $gr.(A)$-module $gr.(M) = \oplus gr_i(M)$. In the following, we shall denote by $gr(A)$ or $gr(M)$ the underlying ring or module (forgetting the grading). There is a natural ring homomorphism $\sigma : A \to gr(A)$ called the principal symbol map. Note that the kernel of the principal symbol is precisely $\cap A_i$. In particular, σ is injective if the filtration is separated.

Given a filtered ring (A, A_i), the completion $(\hat{A}, \hat{A}_i)$ is defined as follows. Let

$$\hat{A}_i = \varprojlim_n A_i/A_{i-n}.$$

The inclusions $A_i \subset A_j$ for $i \leq j$ induce natural injective homomorphisms $\hat{A}_i \to \hat{A}_j$ giving rise to an inductive system; one defines $\hat{A} = \varinjlim_i A_i$. The multiplication maps $A_i \times A_j \to A_{i+j}$ induce morphisms $\hat{A}_i \times \hat{A}_j \to \hat{A}_{i+j}$. In particular, one has a ring structure on $\hat{A}$. If the canonical morphism $(A, A_i) \to (\hat{A}, \hat{A}_i)$ is an isomorphism, then we will say that (A, A_i) is complete. These notions can be generalized to filtered (A, A_i)-modules. Furthermore, by ([9], A.1.1), if (A, A_i) is complete and $gr(A)$ is noetherian, then a filtration is good if and only if $gr(M)$ is finitely generated. By ([9], A.1.1.1), if (A, A_i) is complete and $gr(A)$ is (left or right) noetherian then A and A_0 are also (left or right) noetherian. We shall assume from now on that $gr(A)$ is noetherian.

There is an alternate construction of the completion via the Rees ring. Let $A[\nu, \nu^{-1}]$ denote the ring of Laurent polynomials in ν over A graded by setting $deg(\nu) = 1$. Consider the graded subring $A. = \oplus_{i\in\mathbb{Z}} A_i \nu^i$. If (M, M_i) is a filtered (A, A_i)-module, then we can define the graded $A.$-module

$$M. = \oplus_{i\in\mathbb{Z}} M_i \nu^i \subset M \otimes_A A[\nu, \nu^{-1}].$$

The resulting functor from filtered (A, A_i)-modules with exhaustive filtration to the category of graded $A.$ modules with ν a non-zero divisor is an equivalence of categories. For each $n \geq 1$, let $A_{.,n} = A./\nu^n A.$. This gives rise to a projective system

$$\to A_{.,n+1} \to A_{.,n} \to \cdots \to A_{.,1} = gr.(A).$$

One has $\hat{A}_i = \varprojlim_n A_{i,n}$ and $\hat{A} = \varinjlim_i \varprojlim_n A_{i,n}$. One has a similar description for filtered (A, A_i)-modules.

Let A be a unital ring and $S \subset A$ a multiplicative subset. Then a localization of A with respect to S is a ring A_S with a homomorphism $\phi : A \to A_S$ such that $\phi(s)$ is invertible for all $s \in S$ and for any homomorphism $f : A \to B$ such that $f(s)$ is invertible for all $s \in S$, there exists a unique homomorphism $f' : A' \to B$ such that $f = f' \circ \phi$. If for all $s \in S$ and $a \in A$, there exists $n \in \mathbb{N}$ such that $ad(s)^n(a) = 0$, then A admits a localization with respect to S. Furthermore, the natural morphism $A \to A_S$ is flat.

Let (A, A_i) be a complete filtered ring and $S_1 \subset gr.(A)$ a homogeneous multiplicative subset. For all $n \geq 1$, let $S_n \subset A_{.,n}$ denote the inverse image of S_1 under the natural map $A_{.,n} \to A_{.,1}$. Let $S \subset A$ denote the multiplicative subset given by $s \in A$, such that $s \in A_i$ for some i with image in $gr_i A$ lying in S. Suppose from now on that $gr.(A)$ is commutative. Then each S_n is a multiplicative subset satisfying the ad condition of the previous paragraph. In particular, the localization of $A_{.,n}$ at S_n exists. Furthermore, the resulting localizations $A_{\cdot,n,S}$ form a projective system. This gives a complete filtered ring (A', A'_i) by setting

$$A'_i = \varprojlim_n A_{i,n,S} \text{ and } A' = \varinjlim_i \varprojlim_n A_{i,n,S}.$$

One has a natural homomorphism of filtered rings $\phi : (A, A_i) \to (A', A'_i)$ which induces the usual localization at the level of associated graded rings. Since we assume that $gr(A)$ is noetherian, the resulting localization is a flat morphism of filtered rings ([9], A.1.1.3). These constructions can be generalized to complete filtered modules.

Let $R_\cdot = \oplus_{i \geq 0} R_i$ be a graded commutative ring. We let R denote the underlying commutative ring. Then R_0 is a commutative ring and we may consider $X = Spec(R_0)$, and $V = Spec(R)$. Let V' denote the topological space where the underlying set is V and the topology is generated by $D(f)$ where f is a homogeneous element of R. The identity induces a continuous map $\epsilon : V \to V'$. The natural projection $V \xrightarrow{p} X$ has a canonical factorization

$$V \xrightarrow{\epsilon} V' \xrightarrow{p'} X.$$

We consider V' as a ringed space with structure sheaf given by $\epsilon_* \mathcal{O}_V$.

Suppose now that (A, A_i) is a filtered ring such that $A_i = 0$ for $i < 0$, $\cup_i A_i = A$, and $gr_\cdot(A)$ is commutative (and noetherian). It follows that (A, A_i) is complete and A_0 is commutative. If $X = Spec(A_0)$, then (A, A_i) gives rise to a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ with a filtration by quasi-coherent $\mathcal{O}_X$-submodules $\mathcal{A}_i$. We may define $\mathcal{A}_\cdot$ and $\mathcal{A}_{\cdot,n}$ as before. If we set $R_\cdot = gr_\cdot(A)$ in the previous paragraph, then we have $V = Spec(gr_\cdot(A))$. Given a basic open $D(f)$ of V', we can consider the localizations $A_{\cdot,n}(f)$ of $A_{\cdot,n}$ at the multiplicative set $S_f = \{f^m | m \in \mathbb{N}\}$. This gives a presheaf on V'. Let $\mathcal{B}'_{\cdot,n}$ denote the associated sheaf on V'. The $\mathcal{B}'_{\cdot,n}$ form a projective system equipped with a canonical isomorphism of projective systems $(\mathcal{A}_{\cdot,n})_n \to p'_*((\mathcal{B}'_{\cdot,n})_n)$. The projective system $(\mathcal{B}'_{\cdot,n})_n$ gives a sheaf of complete filtered rings $(\mathcal{B}', \mathcal{B}'_i)$ on V'. By ([9], A.3.1.2), for all homogeneous $f \in gr_\cdot(A)$, one has a canonical isomorphism:

$$(A, A_i)_{S_f} \to \Gamma(D(f), (\mathcal{B}', \mathcal{B}'_i)).$$

The *microlocalization* of $(\mathcal{A}, \mathcal{A}_i)$ is the sheaf $(\mathcal{B}, \mathcal{B}_i) := \epsilon^{-1}((\mathcal{B}', \mathcal{B}'_i))$ on V. One has canonical flat morphisms $p^{-1}(\mathcal{A}, \mathcal{A}_i) \to (\mathcal{B}, \mathcal{B}_i)$ and $p'^{-1}(\mathcal{A}, \mathcal{A}_i) \to (\mathcal{B}', \mathcal{B}'_i)$

We may apply the constructions of the previous paragraph to the case where $\mathcal{A} = \mathcal{D}_X$ is the sheaf of differential operators on a smooth affine variety X and $\mathcal{A}_i = \mathcal{D}_{X,i}$ is the filtration by order of differential operator. In this case, V is simply the cotangent bundle T^*X, and V' is the cotangent bundle with the conical topology. We shall denote the latter ringed space by T^*X^c, and $\epsilon : T^*X \to T^*X^c$ the corresponding morphism which is the identity on the underlying topological space. The resulting microlocalization, denoted $(\mathcal{E}_X, \mathcal{E}_{X,\cdot})$, is the sheaf of micro-differential operators on the cotangent bundle T^*X. We shall denote by $\mathcal{E}_X^c$ the corresponding sheaf on T^*X^c. Since all the constructions above are functorial, we can extend these constructions to an arbitrary smooth variety X.

3. K-theory of graded rings

Let R be a $\mathbb{Z}$-graded (or $\mathbb{Z} \times \mathbb{Z}$-graded) regular[2] ring. Denote by $\mathrm{K}^g(R)$ the K-theory spectrum of the category of finitely generated graded R-modules. In the following we shall consider left modules and the corresponding K-theory spectra. All the categories in this section will be exact and we shall play with Quillen K-theory spectra. These can be replaced by the corresponding Waldhausen versions (which we shall do in the next section).

Remark 3.1. Let $\mathrm{K}^{g'}(R)$ denote the K-theory spectrum of graded R-modules which are (graded) projective and finitely generated. Then one has a canonical homotopy morphism $\mathrm{K}^{g'}(R) \to \mathrm{K}^g(R)$ which is induced by the inclusion. This morphism is a homotopy equivalence by regularity. We have chosen fibrant-cofibrant models for our K-theory spectra and so this morphism has a canonical inverse (as a homotopy morphism). We shall use this to identify these two spectra.

3.1. **Homotopy Invariance for graded K-theory.** In the following, let R be a $\mathbb{Z}$-graded regular noetherian ring with $S = R[x]$. We consider S as a $\mathbb{Z} \times \mathbb{Z}$-graded ring with grading given by $S_{n,m} = R_n x^m$. Similarly, we consider $S[y]$ and $S[y, y^{-1}]$ as $\mathbb{Z} \times \mathbb{Z}$-graded rings setting $deg(y) = (-1, 1)$.

Let $\mathrm{Mod}^{\mathrm{gr}}(\mathrm{R})$ denote the category of finitely generated graded R-modules. Given a graded R-module M, let $M(-1)$ denote the graded module whose underlying module is the same as M, but the grading is shifted by one. In particular, $M(-1)_n = M_{n-1}$. Then one has a natural shifting of degree functor

$$t : \mathrm{Mod}^{\mathrm{gr}}(\mathrm{R}) \to \mathrm{Mod}^{\mathrm{gr}}(\mathrm{R})$$

where $t(M) := M(-1)$. In ([15]) Van den Bergh shows the existence of a long exact sequence in graded K-theory

$$\cdots \to \mathrm{K}_i^g(R) \xrightarrow{t} \mathrm{K}_i^g(R) \xrightarrow{ff} \mathrm{K}_i(R) \to \cdots$$

where ff is the morphism induced by the forgetful functor.

The main goal of this section is to prove a spectrum analog of the above theorem. The proof is the same once translated in the right language. We include it here for the sake of completeness.

[2]A graded regular ring is a noetherian ring such that every graded (left) module has finite projective dimension.

Theorem 3.2. *The functors t and ff induce a homotopy cofibre sequence of spectra:*

$$\mathrm{K}^g(R) \to \mathrm{K}^g(R) \to \mathrm{K}(R).$$

Proposition 3.3. *One has a homotpy cofibre sequence of* K*-theory spectra:*

$$\mathrm{K}^g(S) \to \mathrm{K}^g(S[y]) \to \mathrm{K}^g(S[y, y^{-1}]).$$

Proof. This follows from Quillen's localization theorem applied to the category$\mathcal{A} = \{S[y]-$graded modules$\}$ and $\mathcal{B} = \{S[y] -$ graded modules where y acts nilpotently$\}$. The quotient $\mathcal{A}/\mathcal{B}$ is naturally identified with the category of finitely generated graded $S[y, y^{-1}]$-modules. Therefore, the localization theorem gives a homotopy cofibre sequence:

$$\mathrm{K}^g(\mathcal{B}) \to \mathrm{K}^g(S[y]) \to \mathrm{K}^g(S[y, y^{-1}]).$$

Furthermore, the resolution theorem shows that the natural morphism $K^g(S[y]/yS[y]) \to K(\mathcal{B})$ is a homotopy equivalence. Finally, identifying $S[y]/yS[y]$ with S gives the desired cofibre sequence. □

Remark 3.4. Let f denote the morphism $\mathrm{K}^g(S) \to \mathrm{K}^g(S[y])$ from the previous lemma. Then f is simply induced by the functor which sends a S-module M to M considered as a $S[y]$-module with trivial y action. We can also consider the $S[y]$-module $M[y] := S[y] \otimes_S M$. One has the following exact sequence relating these two functors from S-modules to $S[y]$-modules:

$$0 \to M[y](1, -1) \to M[y] \to M \to 0.$$

If we let F denote the functor which sends M to $M[y](1, -1)$ and G denote the functor which sends M to $M[y]$, then the above exact sequence gives a homotopy between $f : \mathrm{K}^g(S) \to \mathrm{K}^g(S[y])$ and $G - F$. Furthermore, if

$$H : \{\text{graded} - S[y] - \text{modules}\} \to \{\text{graded} - S[y] - \text{modules}\}$$

denotes the functor which sends M to $M(1, -1)$, then $F = H \circ G$. Thus, we have a homotopy identification $f \cong (Id - H) \circ G$

The following lemma, due to Van den Bergh, is a doubly graded analog of the proposition in section 3 of Quillen ([13]).

Lemma 3.5. *Let C be a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ graded ring. Let $\mathrm{K}^g(C)$ be* K*-theory spectrum of finitely generated ($\mathbb{Z} \times \mathbb{Z}$) graded projective C-modules. Then $\mathrm{K}^g(C)$ is a $\mathbb{Z}[t, t^{-1}]$-module where t acts by sending M to $M(0, -1)$ and t^{-1} acts by sending M to $M(0, 1)$. One has a $\mathbb{Z}[t, t^{-1}]$-module isomorphism:*

$$\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(C_{\text{-},0}) \to \mathrm{K}^g_i(C).$$

Proof. Let $P \in \bar{P} := \{\mathbb{Z} \times \mathbb{Z} -$ graded projective$-$C$-$modules$\}$. Let F_kP be the C-submodule of P generated by $P_{-,n} = \oplus_i P_{i,n}$ for $n \leq k$. Let $\bar{P}_q$ be the full subcategory of $P \in \bar{P}$ consisting of P such that $F_{-q-1}P = 0$ and $F_qP = P$. One has $T : \bar{P} \to \{\text{graded projective} - C_{-,0} - \text{modules}\}$ where P goes to $C_{-,0} \otimes_C P$. Then P is non-canonically isomorphic to $C \otimes_{C_{-,0}} T(P) = \sqcup_n C(0, -n) \otimes_{C_{-,0}} T(P)_{-,n}$. Hence, $P \to F_kP$ is an exact functor and $F_nP/F_{n-1}P \to C(0,-n) \otimes_{C_{-,0}} T(P)_{-,n}$ is an isomorphism. One has $Id = \oplus_{-q \leq i \leq q} F_i/F_{i-1} : \bar{P}_q \to \bar{P}_q$ gives an isomorphism $\sqcup_{-q \leq i \leq q} t^i \otimes K^g_j(C_{-,0}) \to K^g_j(\bar{P}_q)$. Since, $\cup_q \bar{P}_q = P$, the result follows. □

Note that for our applications the ring in question will always be regular noetherian and so we can apply the previous lemma to K-theory spectra of finitely generated modules as well. In particular, we may apply Lemma 3.5 to $C = S[y]$ or $C = S$ where $S = R[x]$ with the grading given above. This gives rise to isomorphisms:

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(S[y]_{-,0}) \to \mathrm{K}^g_i(S[y]),$$
$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(S_{-,0}) \to \mathrm{K}^g_i(S).$$

Furthermore, $S[y]_{-,0} = R = S_{-,0}$. In particular, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(R) & \longrightarrow & \mathrm{K}^g_i(S) \\ \downarrow & & \downarrow f \\ \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(R) & \longrightarrow & \mathrm{K}^g_i(S[y]) \end{array}$$

Here the horizontal arrows are isomorphisms and the left vertical is defined by the commutativity of the diagram. The explicit description of f given in Remark 3.4 shows that the left vertical is given by the composition:

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(R) \to \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(R) \to \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{K}^g_i(R),$$

where the first arrow is induced by sending a $\mathbb{Z}$-graded R-module M to $M(1)$, and the second map is given by multiplication by $(1-t)$. It follows that f is injective and the cokernel is given by $\mathrm{K}^g_i(S[y,y^{-1}])$ (Proposition 3.3). Since the cokernel of the left column is $\mathrm{K}^g_i(R)$ one has an induced isomorphism $\mathrm{K}^g_i(R) \to \mathrm{K}^g_i(S[y,y^{-1}])$ where the map is induced by extension of scalars. In particular, we have the following corollary.

Corollary 3.6. *The natural morphism induced by extensions of scalars*

$$\mathrm{K}^g(R) \to \mathrm{K}^g(S[y,y^{-1}])$$

is a weak homotopy equivalence.

Lemma 3.7. *(*[15]*) Let S be a $\mathbb{Z} \times \mathbb{Z}$-graded ring. Let S' be the associated $\mathbb{Z}$-graded module with grading $S'_n = \sum_j S_{n-j,j}$. Then one has an equivalence of categories:*

$$\{S[x,x^{-1}] - \textit{graded modules}, deg(x) = (-1,1)\} \to \{S' - \textit{graded modules}\}.$$

The functor is induced by sending M to $M/(x-1)M$.

Proposition 3.8. *The natural inclusion $R \to R[x]$ induces a weak homotopy equivalence of spectra:*

$$\mathrm{K}^g(R) \to \mathrm{K}^g(R[x]).$$

Proof. The morphism $\mathrm{K}^g(R) \to \mathrm{K}^g(R[x])$ factors as

$$\mathrm{K}^g(R) \to \mathrm{K}^g(S[y, y^{-1}]) \to \mathrm{K}^g(R[x])$$

where the second morphism is induced by sending M to $M/(y-1)M$. By Corollary 3.6 the first morphism is a weak homotopy equivalence. On the other hand, the previous lemma gives that the second morphism is also a weak homotopy equivalence. $\square$

Proof. (Theorem 3.2)
Proceeding as in the proof of Proposition 3.3, one can show that there is a homotopy cofibre sequence:

$$\mathrm{K}^g(R) \to \mathrm{K}^g(R[x]) \to \mathrm{K}^g(R[x, x^{-1}]).$$

The first map $\tilde{f}$ is given by sending M to itself with trivial x action. Then the exact sequence

$$0 \to M[x](-1) \to M[x] \to M \to 0,$$

where the first map is multiplication by x, gives a commutative diagram of homotopy morphisms of spectra:

$$\begin{array}{ccc} \mathrm{K}^g(R) & \xrightarrow{\tilde{f}} & \mathrm{K}^g(R[x]) \\ & \searrow^{1-t} & \uparrow \\ & & \mathrm{K}^g(R) \end{array}$$

Since the vertical arrow is a homotopy equivalence by the previous proposition, we can deduce a homotopy cofibre sequence:

$$\mathrm{K}^g(R) \to \mathrm{K}^g(R) \to \mathrm{K}^g(R[x, x^{-1}])$$

where the first morphism is given by $1-t$. One also has a commutative diagram:

$$\begin{array}{ccc} \mathrm{K}^g(R[x]) & \longrightarrow & \mathrm{K}^g(R[x, x^{-1}]) \\ & \searrow^{ff} & \downarrow \\ & & \mathrm{K}(R) \end{array}$$

Here ff is given by the forgetful functor and the right vertical is induced by sending M to $M/(x-1)M$. The vertical map is a weak homotopy equivalence since the functor from graded $R[x, x^{-1}]$-modules to R-modules induced by sending M to $M/(x-1)M$ is an equivalence of categories. Therefore, one has a homotopy cofibre sequence

$$\mathrm{K}^g(R) \xrightarrow{1-t} \mathrm{K}^g(R) \xrightarrow{ff} \mathrm{K}(R)\ .$$

$\square$

3.2. **Construction of the microlocalization for K-theory of rings.** Let (A, F) be a complete $\mathbb{Z}$-filtered ring. We shall assume $B = gr_F(A)$ is noetherian and graded-regular. Let $\tilde{A} = \sum F_n z^n \subset A[z, z^{-1}]$. It follows that A and $\tilde{A}$ are also noetherian and regular (graded-regular). In this section, we apply the results of the previous section to construct a morphism of spectra

$$\mathrm{K}(A) \to \mathrm{K}(B)$$

and study some of its properties.

We consider $\tilde{A} \subset A[z, z^{-1}]$ as a graded subring with z in degree one. Note that the Ore localization of $\tilde{A}$ at the set $S = \{1, z, z^2, \ldots\}$ is isomorphic to $A[z, z^{-1}]$. The following lemma is due to Quillen in the case of positively filtered rings ([13], proof of Thm. 7). The proof in the case of $\mathbb{Z}$-filtered rings as above is exactly the same.

Lemma 3.9. *Let (A, F) be as above. Then one has a homotopy cofibre sequence of spectra:*

$$\mathrm{K}^g(B) \to \mathrm{K}^g(\tilde{A}) \to \mathrm{K}(A).$$

Proof. Let $\mathcal{A}$ denote the category of finitely generated graded $\tilde{A}$-modules, and $\mathcal{B} \subset \mathcal{A}$ denote the full subcategory of modules with nilpotent z action. Then Quillen's localization theorem gives a fibre sequence:

$$\mathrm{K}(\mathcal{B}) \to \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A}/\mathcal{B}).$$

The devissage theorem shows that the natural morphism $\mathrm{K}^g(B) \to \mathrm{K}(\mathcal{B})$ is a weak homotopy equivalence. Furthermore, the quotient category $\mathcal{C} = \mathcal{A}/\mathcal{B}$ is naturally equivalent to the category of finitely generated graded $\tilde{A}_S$-modules, where $S = \{1, z, z^2, \ldots\}$. The latter category is equivalent to the category of finitely generated A-modules. □

Theorem 3.10. *One has a natural homotopy morphism of spectra $gr_A : \mathrm{K}(A) \to \mathrm{K}(B)$.*

Proof. Applying Theorem 3.2 to the $\mathbb{Z}$-graded ring B, gives a natural homotopy cofibre sequence

$$\mathrm{K}^g(B) \xrightarrow{1-t} \mathrm{K}^g(B) \longrightarrow \mathrm{K}(B)\ .$$

The natural quotient map $\tilde{A} \to \tilde{A}/z\tilde{A} = B$ induces a morphism of K-theory spectra

$$g : \mathrm{K}(B) \to \mathrm{K}(\tilde{A}).$$

On the other hand, one has an exact sequence of graded rings

$$0 \to \tilde{A}(-1) \to \tilde{A} \to B \to 0,$$

where the first map is given by multiplication by z. In particular, B is of (graded) Tor-dimension 1 over $\tilde{A}$ and, for M a finitely generated B-module, $\mathrm{Tor}_0^{\tilde{A}}(B, M) = M$ and $\mathrm{Tor}_1^{\tilde{A}}(B, M) = M(-1)$. It follows that if $g : \mathrm{K}^g(\tilde{A}) \to \mathrm{K}^g(B)$ denotes the morphism induced by extension of scalars, then the above exact sequence (i.e. resolution of B as

$\tilde{A}$-module) gives a homotopy between $g \circ i$ and $1 - t : \mathrm{K}^g(B) \to \mathrm{K}^g(B)$. One has a commutative diagram of homotopy morphisms:

$$\begin{array}{ccccc} \mathrm{K}^g(B) & \xrightarrow{i} & \mathrm{K}^g(\tilde{A}) & \longrightarrow & \mathrm{K}(A) \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle g} & & \\ \mathrm{K}^g(B) & \xrightarrow{1-t} & \mathrm{K}^g(B) & \xrightarrow{ff} & \mathrm{K}(B) \end{array}$$

We've seen that the bottom row is a cofibre sequence (Theroem 3.2), and the top row is also a cofibre sequence by Lemma 3.9. It follows that there is a canonical homotopy morphism $gr : \mathrm{K}(A) \to \mathrm{K}(B)$ making the above diagram commute. The choice of $\mathrm{K}(A) \to \mathrm{K}(B)$ is canonical upto a choice of homotopy making the first diagram commute. However, we have fixed such a homotopy (determined by the exact sequence $0 \to \tilde{A}(-1) \to \tilde{A} \to B \to 0$). □

Let (A, F) be a complete $\mathbb{Z}$-filtered ring as before and suppose $B = gr_F(A)$ is noetherian and graded-regular. Suppose (A', F) is another $\mathbb{Z}$-filtered ring with $B' = gr_F(A')$ satisfying the same assumptions as (A, F). Let $(A, F) \stackrel{f}{\to} (A', F')$ be a morphism of filtered rings and $B \stackrel{gr(f)}{\to} B'$ and $\tilde{A} \stackrel{\tilde{F}}{\to} \tilde{A}'$ the induced morphisms. We shall assume that A' (resp. B', $\tilde{A}'$) is flat as a right module over A (resp. B, $\tilde{A}$). In this case, extension of scalars induces natural morphisms $\mathrm{K}(A) \stackrel{f^*}{\to} \mathrm{K}(A')$, $\mathrm{K}(B) \stackrel{gr(f)^*}{\to} \mathrm{K}(B')$, and $\mathrm{K}(\tilde{A}) \stackrel{\tilde{f}^*}{\to} \mathrm{K}(\tilde{A}')$.

Remark 3.11. We shall mainly be interested $gr(A)$ is commutative, and where (A', F') is an appropriate micolocalization of (A, F). In this case, the flatness condition will follow from general results on flatness of localization.

The morphism constructed in the previous theorem is essentially a microlocalization morphism at the level of K-theory spectra of rings. The following proposition will allows us to glue these morphisms to obtain a microlocalization morphism for K-theory spectra of sheaves of microdifferential operators.

Proposition 3.12. *Let A and A' be as above. The following diagram of homotopy morphisms is naturally homotopically commutative:*

$$\begin{array}{ccc} \mathrm{K}(A) & \xrightarrow{gr_A} & \mathrm{K}(B) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle gr(f)^*} \\ \mathrm{K}(A') & \xrightarrow{gr_{A'}} & \mathrm{K}(B'). \end{array}$$

Proof. We first note that the squares in the diagram

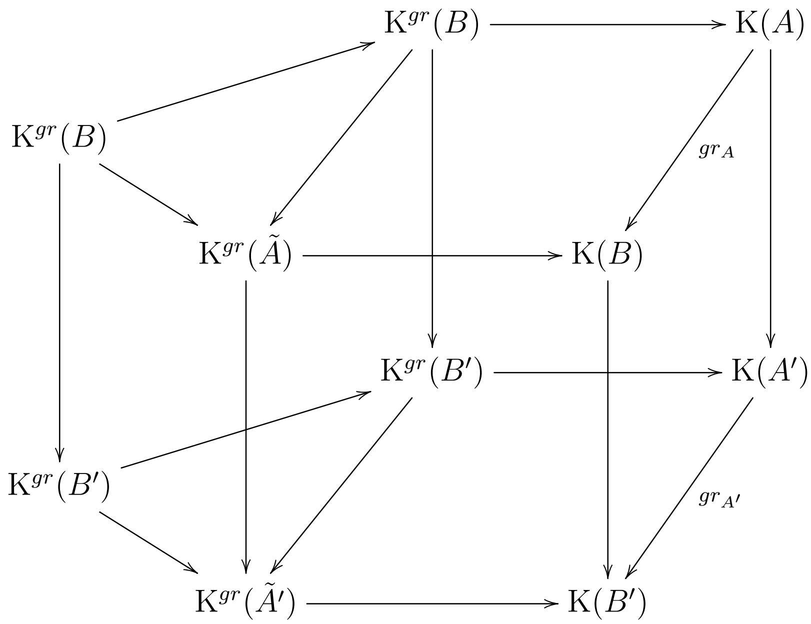
$$\begin{array}{ccccc}
\mathrm{K}^{gr}(B) & \longrightarrow & \mathrm{K}^{gr}(\tilde{A}) & \longrightarrow & \mathrm{K}(A) \\
{\scriptstyle gr(f)^*}\downarrow & (a) & \downarrow{\scriptstyle \tilde{f}^*} & (b) & \downarrow{\scriptstyle f^*} \\
\mathrm{K}^{gr}(B') & \longrightarrow & \mathrm{K}^{gr}(\tilde{A}') & \longrightarrow & \mathrm{K}(A')
\end{array}$$

are naturally homotopy commutative. Let us first prove that square (a) is homotopy commutative. The top horizontal arrow in (a) is induced by push forward along the quotient map $i^A : \tilde{A} \to \tilde{A}/z\tilde{A} = B$. Denote the induced morphism on K-theory spectra by i^A_* and similarly for $i^{A'}_*$. We need to show that the homotopy morphisms $\tilde{f}^* \circ i^A_*$ and $i^{A'}_* \circ gr(f)^*$ are naturally identified. At the level of categories, the first composition sends a graded B-module M to $M \otimes_{\tilde{A}} \tilde{A}'$ and the second sends M to $M \otimes_B B'$ considered as a $\tilde{A}'$-module via $i^{A'}$. On the other hand, these two functors are naturally isomorphic since one has natural isomorphisms $M \otimes_B B' \cong M \otimes_B (B \otimes_{\tilde{A}} \tilde{A}') \cong M \otimes_{\tilde{A}} \tilde{A}'$. In the case of square (b), both horizontal arrows are induced by pull back via the quotient map $\tilde{A} \to \tilde{A}/z - 1\tilde{A} = A$ and $\tilde{A}' \to \tilde{A}'/z - 1\tilde{A}' = A'$. In this case, the result follows from the usual associativity of tensor products.

One also has a naturally homotopy commutative diagram:

$$\begin{array}{ccccc}
\mathrm{K}^{gr}(B) & \xrightarrow{1-t} & \mathrm{K}^{gr}(B) & \longrightarrow & \mathrm{K}(B) \\
\downarrow & (c) & \downarrow & (d) & \downarrow \\
\mathrm{K}^{gr}(B') & \xrightarrow{1-t} & \mathrm{K}^{gr}(B') & \longrightarrow & \mathrm{K}(B')
\end{array}$$

Since t and 1 commute with base extension, it follows that (c) is naturally homotopy commutative. Similarly (d) also commutes since the horizontal map in (d) are induced by the forgetful functor. It follows that one has a diagram



in which everything commutes except for the rightmost vertical square and all horizontal arrows are homotopy cofibre sequences. It follows that one has a natural identification of the homotopy morphisms $gr(f)^* \circ gr_A$ and $gr_{A'} \circ f^*$.

□

If (A, F) is, in addition, a positively filtered ring, then an alternate construction of gr_A follows from the following theorem of Quillen.

Theorem 3.13 (Quillen)**.** *Suppose (A, F) be a $\mathbb{Z}$ filtered ring as above with $F_n = 0$ for all $n < 0$. Then if B is of finite Tor dimension as a $F_0(A)$-module and $F_0(A)$ is finite Tor dimension as a B-module, the natural homotopy morphism induced by the inclusion $K(F_0(A)) \to K(A)$ is a weak equivalence, and therefore, has an inverse as a homotopy morphism.*

Corollary 3.14. *Let (A, F) be as in the previous theorem. Then one has a natural homotopy morphism $gr_A^Q : K(A) \to K(B)$.*

Proof. By the previous proposition one has $K(F_0(A)) \to K(A)$. Taking its inverse as a homotopy morphism and then composing with $K(F_0(A)) \to K(B)$ gives the required map. Note that in this case the latter map is also weak equivalence since B has finite Tor dimension over $F_0(A)$. □

We have the following compatibility between gr_A and gr_A^Q defined in the previous section.

Proposition 3.15. *For (A, F) as in Quillen's theorem above the two homotopy morphisms gr_A^Q and gr_A are naturally identified as homotopy morphisms.*

Proof. This follows from the fact that one has a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}^g(B) & \longrightarrow & \mathrm{K}^g(\tilde{A}) & \xrightarrow{j_A^*} & \mathrm{K}(A) \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle gr_A^Q} \\ \mathrm{K}^g(B) & \longrightarrow & \mathrm{K}^g(B) & \xrightarrow{ff} & \mathrm{K}(B) \end{array}$$

since a choice of homotopy identification for the left square determines the right vertical arrow up to unique homotopy identification. But, we have fixed a homotopy identification for the left square. To see that the diagram is commutative we need only show that the right square is commutative. In particular, we need to construct a natural homotopy identification between the homotopy morphisms $gr_A^Q \circ j_A^*$ and $ff \circ g$. One has a factorization $gr_A^Q : \mathrm{K}(A) \to \mathrm{K}(F_0(A)) \to \mathrm{K}(B)$. The first arrow is the canonical homotopy inverse of the natural extension of scalars $\mathrm{K}(F_0(A)) \to \mathrm{K}(A)$. It follows that it is sufficient to note that the following diagrams are commutative:

$$\begin{array}{ccc} \mathrm{K}^g(\tilde{A}) & \longrightarrow & \mathrm{K}(A) \\ & \nwarrow & \uparrow \\ & & \mathrm{K}(F_0(A)) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{K}^g(\tilde{A}) & \longleftarrow & \mathrm{K}(F_0(A)) \\ \downarrow & & \downarrow \\ \mathrm{K}^g(B) & \longrightarrow & \mathrm{K}(B). \end{array}$$

For the first diagram all the morphisms are given by extension of scalars and so there is a natural homotopy identification induced by associativity of tensor product. The commutativity of the second diagram follows by a similar argument since all the morphisms are again given by extension of scalars. ☐

3.3. **Microlocalization for K-theory of sheaves.** In this section, we globalize the K-theory microlocalization morphism to sheaves of rings. As before, let X denote a smooth variety over a field k of characteristic zero. Recall $\mathcal{E}_X^c$ is the sheaf of microdifferential operators on T^*X^c, where T^*X^c denotes the cotangent bundle with the conical topology. We consider the latter as a ringed space with structure sheaf given by $\mathcal{O}_{T^*X^c} := \epsilon_* \mathcal{O}_{T^*X}$, where $\epsilon : T^*X \to T^*X^c$ is the natural morphism of ringed spaces given by the identity on the underlying topological spaces. We refer to section 2.4 for the details. The associated graded of $\mathcal{E}_X^c$ is canonically isomorphic to the structure sheaf $\mathcal{O}_{T^*X^c}$. The main goal of this section is to globalize the constructions of the previous sections to get natural homotopy morphisms of K-theory spectra:

$$\mathrm{K}(\mathcal{E}_{X^c}|_V) \to \mathrm{K}(\mathcal{O}_{T^*X^c}|_V)$$

where V is a conic open subset, which are functorial in V. In particular, we shall construct such a morphism at the level of presheaves of spectra. First, we begin with an analogous construction for the sheaf of differential operators.

Let $\mathcal{K}_{\mathcal{D}_X}$ denote the presheaf of spectra on X given by assigning to $U \subset X$ the K-theory spectrum $\mathrm{K}(\mathcal{D}_U)$. Recall that the natural morphism

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(\mathrm{C}^b(\mathcal{D}_X))$$

is a weak equivalence, where $\mathrm{C}^b(\mathcal{D}_X)$ is the Waldhausen category of perfect complexes of $\mathcal{D}_X$-modules. Since X is smooth, we could also take instead the category of bounded complexes of coherent $\mathcal{D}_X$-modules. Since we assume all our spectra are fibrant-cofibrant, we may canonically invert this morphism as a homotopy morphism. We shall use this to identify the two spectra. Note that we have the same result if we use instead the bounded derived category of all modules with coherent cohomology. If $Z \subset X$, then let $\mathrm{K}_Z(\mathcal{D}_X)$ denote the K-theory spectrum of the category of perfect complexes of $\mathcal{D}_X$-modules with supports in Z. Again, we may consider (bounded) complexes with coherent cohomology instead (or equivalently, bounded complexes of coherent $\mathcal{D}_X$-modules).

Proposition 3.16. *The presheaf of spectra $\mathcal{K}_{\mathcal{D}_X}$ satisfies descent in the Zariski topology on X. In particular, it is fibrant for the local injective model structure.*

Proof. Recall, we must show that the given sheaf satisfies the Mayer-Vietoris property. On the other hand, for this it is enough to show that excision holds. In particular, it is enough to show that if $Z \subset U \subset X$, where U is open in X and Z is closed, then one has a weak equivalence:

$$\mathrm{K}_Z(\mathcal{D}_X) \to \mathrm{K}_Z(\mathcal{D}_U).$$

Let $\mathrm{D}^b_{qc,Z}(\mathcal{D}_X)$ denote the bounded derived category of quasi-coherent $\mathcal{D}_X$-modules with support in Z. Note that the compact objects in this category are precisely the perfect complexes. Therefore, it is enough to show that the natural restriction map

$$\mathrm{D}^b_{qc,Z}(\mathcal{D}_X) \to \mathrm{D}^b_{qc,Z}(\mathcal{D}_U)$$

is an equivalence of categories. For this note that the $\mathcal{D}$-module push-forward induces a morphism

$$Rj_* : \mathrm{D}^b_{qc,Z}(\mathcal{D}_U) \to \mathrm{D}^b_{qc,Z}(\mathcal{D}_X).$$

To see this, use the fact that for open immersions $\mathcal{D}$-module push forward is given by the usual push-forward of quasi-coherent sheaves and base change. Finally, note that the unit (resp. counit) $1 \to Rj_* \circ j^*$ (resp. $j^* \circ Rj_* \to 1$) is an isomorphism when restricted to category of complexes supported on Z.

□

Let $\mathcal{K}_{T^*X}$ denote the presheaf of spectra in the Zariski topology on T^*X whose sections over an open U are given by $\mathrm{K}(U)$. Recall that this presheaf satisfies descent in the Zariski topology on T^*X. In particular, it is fibrant for the local injective model structure. In the following, for a Zariski open U, we let U^{aff} denote the set of affine opens in U. If $\mathcal{F}$ is a presheaf of spectra on X, we let $\mathcal{F}^{aff}$ denote the presheaf of spectra whose sections over an open U are given by $\mathrm{holim}_{V\subset U^{aff}}\mathcal{F}(V)$. If $\mathcal{F}$ is fibrant, then by the remarks at the end of section 2.2, the natural morphism $\mathcal{F}\to\mathcal{F}^{aff}$ is a global weak equivalence.

Theorem 3.17. *There is a natural homotopy morphism $gr:\mathcal{K}_{\mathcal{D}_X}\to p_*\mathcal{K}_{T^*X}$ of presheaves of spectra on X. In particular, one has an induced homotopy morphism on global sections:*

$$\mathrm{K}(\mathcal{D}_X)\to\mathrm{K}(T^*X).$$

Proof. By the previous proposition and the remarks above, the natural morphism $\mathcal{K}_{\mathcal{D}_X}\to\mathcal{K}^{aff}_{\mathcal{D}_X}$ is a global weak equivalence. Similarly, the natural morphism $p_*\mathcal{K}_{T^*X}\to(p_*\mathcal{K}_{T^*X})^{aff}$ is a global weak equivalence. Therefore, it is enough to construct a morphism

$$gr:\mathcal{K}^{aff}_{\mathcal{D}_X}\to(p_*\mathcal{K}_{T^*X})^{aff}.$$

In particular, we need to construct (functorial in U) morphisms

$$gr_U:\mathrm{holim}_{V\subset U^{aff}}\mathrm{K}(\mathcal{D}_V)\to\mathrm{holim}_{V\subset U^{aff}}\mathrm{K}(\mathcal{O}_{T^*V}).$$

If V is affine, then taking global sections induces a weak equivalence

$$\mathrm{K}(\mathcal{D}_V)\to\mathrm{K}(\Gamma(V,\mathcal{D}_V))$$

and similarly for $\mathrm{K}(T^*V)$. Furthermore, the natural morphism

$$\mathrm{K}(\Gamma(T^*V,\mathcal{O}_{T^*V})\to\mathrm{K}(\Gamma(V,gr(\mathcal{D}_V)))$$

is a weak equivalence. The latter is simply $\mathrm{K}(gr(\Gamma(V,\mathcal{D}_V)))$. Therefore, it suffices to construct functorial in V morphisms

$$\mathrm{K}(\Gamma(V,\mathcal{D}_V))\to\mathrm{K}(gr(\Gamma(V,\mathcal{D}_V))).$$

In this situation, we have already constructed a morphism gr given by Theorem 3.10. The functoriality then follows from Proposition 3.12 □

Remark 3.18. Since $\mathcal{D}_X$ is positively filtered and X is smooth, we could also have used the morphism gr^Q from Corollary 3.14 instead in the proposition above. As a result, we can also construct a morphism $gr^Q:\mathcal{K}_{\mathcal{D}_X}\to p_*\mathcal{K}_{T^*X}$. However, by Proposition 3.15, this gr^Q is naturally homotopic to the one in the theorem above.

Let $\mathcal{K}_{\mathcal{E}^c_X}$ denote the presheaf of spectra on T^*X^c whose sections over an open conic V are given by $\mathrm{K}(\mathcal{E}^c_X|_V)$. Similarly, let $\mathcal{K}_{T^*X^c}$ denote the presheaf of spectra on T^*X^c whose sections on an open conic V are given by $\mathrm{K}(\mathcal{O}_{T^*X^c}|_V)$.

Theorem 3.19. *Let X be as above. Then there is a natural homotopy morphism of presheaves of spectra on T^*X^c*

$$gr:\mathcal{K}_{\mathcal{E}^c_X}\to\mathcal{K}_{T^*X^c}.$$

In particular, for all open conics $V \subset V'$, one has a commutative diagram:

$$\begin{array}{ccc} \mathrm{K}(\mathcal{E}^c_X|_{V'}) & \xrightarrow{gr} & \mathrm{K}(T^*X^c|_{V'}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{E}^c_X|_V) & \xrightarrow{gr} & \mathrm{K}(T^*X^c|_V). \end{array}$$

The vertical morphisms are the natural restriction morphisms.

Proof. Suppose first that X is affine. For a conic open $V \subset T^*X^c$, let V^{aff} denote the set of $D(f) \subset V$ where f is homogeneous. If $\mathcal{F}$ is a presheaf of spectra on T^*X^c, then let $\mathcal{F}^{aff}$ denote the presheaf whose sections on an open conic V are given by $\mathrm{holim}_{U \subset V^{aff}} \mathcal{F}(U)$. Now we have canonical morphisms $\mathcal{K}_{\mathcal{E}^c_X} \to \mathcal{K}^{aff}_{\mathcal{E}^c_X}$ and $\mathcal{K}_{T^*X^c} \to \mathcal{K}^{aff}_{T^*X^c}$. Furthermore, $\mathcal{K}_{T^*X^c}$ satisfies descent. It follows that the latter morphism is a global weak equivalence and therefore we may canonically invert it as a homotopy morphism. In particular, it suffices to construct a morphism of presheaves of spectra

$$\mathcal{K}^{aff}_{\mathcal{E}^c_X} \to \mathcal{K}^{aff}_{T^*X^c}.$$

For each $U \subset V^{aff}$ we have the following natural morphisms:

$$\mathrm{K}(\mathcal{E}^c_X|_U) \to \mathrm{K}(\Gamma(U, \mathcal{E}^c_X|_U)) \xrightarrow{gr} \mathrm{K}(gr(\Gamma(U, \mathcal{E}^c_X|_U))) \to \mathrm{K}(\Gamma(U, \mathcal{O}_{T^*X})) \leftarrow \mathrm{K}(\mathcal{O}_{T^*X}|_U) \to \mathrm{K}(\mathcal{O}_{T^*X^c}|_U).$$

Here gr is the morphism constructed in the previous section, and the first arrow is given by taking global sections. Since $U = D(f)$, for some homogeneous f, $gr(\Gamma(U, \mathcal{E}^c_X|_U)) = \Gamma(U, \mathcal{O}_{T^*X})$. The arrow facing left is given by taking global sections and, since U is affine, this morphism is a weak equivalence. In particular, we can invert this morphism canonically as a homotopy morphism. The rightmost arrow is given by push forward along ϵ. Note that this is exact since everything is affine. Taking homotopy limit over U gives the desired morphism. Furthermore, it is clear that everything is compatible under restriction.
If X is not affine, let X^{aff} denote the set of affine opens in X. Then one can construct the required morphisms as follows. For each conic open U consider the following diagram:

$$\mathcal{K}_{\mathcal{E}^c_X}(U) \to \mathrm{holim}_{V \subset X^{aff}} \mathcal{K}_{\mathcal{E}^c_V}(U|_V) \to \mathrm{holim}_{V \subset X^{aff}} \mathcal{K}_{T^*V^c}(U|_V) \leftarrow \mathcal{K}_{T^*X^c}(U).$$

The first arrow is given by restriction, the second is the affine construction above, and the left facing arrow is also given by restriction. Finally, again by descent, the left facing arrow is a weak equivalence.

□

Corollary 3.20. *One has a natural commutative diagram of pre-sheaves (on T^*X^c):*

$$\begin{array}{ccc} \mathcal{K}_{\mathcal{E}^c_X} & \xrightarrow{gr} & \mathcal{K}_{T^*X^c} \\ \uparrow & & \uparrow \\ \mathrm{K}(\mathcal{D}_X) & \xrightarrow{gr} & \mathrm{K}(T^*X) \end{array}$$

Here $\mathrm{K}(\mathcal{D}_X)$ *and* $\mathrm{K}(T^*X)$ *are considered as constant presheaves.*

Proof. First note that it is enough to construct the vertical map on global sections. In particular, we need to construct morphisms

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(\mathcal{E}_X^c)$$

and

$$\mathrm{K}(T^*X) \to \mathrm{K}(T^*X^c).$$

Since the natural morphism $p'^{-1}\mathcal{D}_X \to \mathcal{E}_X^c$ is flat, we may define

$$\mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(\mathcal{E}_X^c)$$

by simply pulling back and extending of scalars. In the affine case, it is simply given by localization. The morphism

$$\mathrm{K}(T^*X) \to \mathrm{K}(T^*X^c)$$

is given by push-forward along ϵ. Since gr is constructed by first passing to basic affines and global sections on basic affines (for both $\mathcal{D}_X$ and $\mathcal{E}_X^c$), we can assume X is affine. We are reduced to showing that the following diagram commutes, where $D(f)$ is a basic affine associated to a homogeneous f in T^*X:

$$\begin{array}{ccc} \mathrm{K}(\Gamma(D(f), \mathcal{E}_X^c)) & \longrightarrow & \mathrm{K}(\Gamma(D(f), \mathcal{O}_{T^*X})) \\ \uparrow & & \uparrow \\ \mathrm{K}(\Gamma(X, \mathcal{D}_X)) & \longrightarrow & \mathrm{K}(\Gamma(T^*X, \mathcal{O}_{T^*X})) \end{array}$$

Since the vertical morphisms are given by localization and, hence flat, the commutativity is given by Proposition 3.12. □

The natural morphism $\epsilon^{-1}\mathcal{O}_{T^*X^c} \to \mathcal{O}_{T^*X}$ is flat. In particular, the functor which sends a coherent $\mathcal{O}_{T^*X^c}$-module $\mathcal{M}$ to the pull back $\epsilon^*\mathcal{M} := \mathcal{O}_{T^*X} \otimes_{\epsilon^{-1}\mathcal{O}_{T^*X^c}} \mathcal{M}$ induces a morphism $\mathrm{K}(T^*X^c) \to \mathrm{K}(T^*X)$. Furthermore, this is compatible under restriction to open conics. In particular, one has a commutative diagram:

$$\begin{array}{ccc} \mathrm{K}(T^*X^c|_V) & \xrightarrow{\epsilon^*} & \mathrm{K}(T^*X|_V) \\ \uparrow & & \uparrow \\ \mathrm{K}(T^*X^c) & \xrightarrow{\epsilon^*} & \mathrm{K}(T^*X) \end{array}$$

where the vertical maps are induced by restriction. Furthermore, note that the following diagram

$$\begin{array}{ccc} \mathrm{K}(T^*X^c) & \xrightarrow{\epsilon^*} & \mathrm{K}(T^*X) \\ \uparrow{\scriptstyle \epsilon_*} & \nearrow{\scriptstyle Id} & \\ \mathrm{K}(T^*X) & & \end{array}$$

commutative. To see this, note that for a coherent $\mathcal{O}_{T^*X}$-module $\mathcal{M}$, the adjunction $\epsilon^*\epsilon_*(\mathcal{M}) \to \mathcal{M}$ is an isomorphism.

Corollary 3.21. *For every open conic $V \subset T^*X$, there is a natural commutative diagram:*

$$\begin{array}{ccc} \mathcal{K}(\mathcal{E}^c_X|_V) & \xrightarrow{gr} & \mathrm{K}(V) \\ \uparrow & & \uparrow \\ \mathrm{K}(\mathcal{D}_X) & \xrightarrow{gr} & \mathrm{K}(T^*X) \end{array}$$

Proof. This is a direct consequence of the previous corollary and the previous remarks. □

Let $S \subset T^*X$ denote a closed conic with complement open conic V. In this situation, we can consider the K-theory spectrum $\mathrm{K}_S(\mathcal{E}_{X^c})$ of coherent modules $\mathcal{M}$ such that the pull back $\epsilon^*\mathcal{M}$ has support in S. One can similarly define the K-theory spectrum $\mathrm{K}_S(\mathcal{D}_X)$.

Theorem 3.22. *Let S and X be as above. Then there is a natural homotopy morphism:*

$$\mathcal{E}_S : \mathrm{K}_S(\mathcal{E}_{X^c}) \to \mathrm{K}_S(T^*X).$$

Proof. First note that, by the remarks above, we have a natural commutative diagram of fibrant-cofibrant spectra:

$$\begin{array}{ccc} \mathrm{K}(\mathcal{E}_{X^c}|_V) & \xrightarrow{\epsilon^*\circ gr} & \mathrm{K}(\mathcal{O}_{T^*X|_V}) \\ \uparrow & & \uparrow \\ \mathrm{K}(\mathcal{E}_{X^c}) & \xrightarrow{\epsilon^*\circ gr} & \mathrm{K}(\mathcal{O}_{T^*X}) \\ \uparrow & & \uparrow \\ \mathrm{K}_S(\mathcal{E}_{X^c}) & & \mathrm{K}_S(\mathcal{O}_{T^*X}) \end{array}$$

where V is the open complement of S. Furthermore, the composition

$$\mathrm{K}_S(\mathcal{E}_{X^c}) \to \mathrm{K}(\mathcal{E}_{X^c}|_V)$$

is canonically homotopic to zero. It follows that the composition

$$\mathrm{K}_S(\mathcal{E}_{X^c}) \to \mathrm{K}(\mathcal{E}_{X^c}) \to \mathrm{K}(\mathcal{O}_{T^*X}) \to \mathrm{K}(\mathcal{O}_{T^*X|_V})$$

is canonically homotopic to zero. Since the right vertical in the above commutative diagram is a homotopy fibre sequence, it follows that one has a natural morphism:

$$\mathrm{K}_S(\mathcal{E}_{X^c}) \to \mathrm{K}_S(\mathcal{O}_{T^*X})$$

making the diagram commute. □

The pushforward ϵ_* induces a natural morphism

$$\mathrm{K}_S(\mathcal{E}_X) \to \mathrm{K}_S(\mathcal{E}_{X^c}).$$

Furthermore, pull back by p induces a natural morphism

$$\mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(\mathcal{E}_X)$$

and pull back by p' induces a natural morphism

$$\mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(\mathcal{E}_{X^c})$$

Lemma 3.23. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{K}_S(\mathcal{D}_X) & \xrightarrow{p'^{-1}} & \mathrm{K}_S(\mathcal{E}_{X^c}) \\ & \searrow^{p^{-1}} & \uparrow{\scriptstyle \epsilon_*} \\ & & \mathrm{K}_S(\mathcal{E}_X) \end{array}$$

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with singular support in S. Then the morphism

$$\mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(\mathcal{E}_{X^c})$$

is induced by sending $\mathcal{M}$ to $\mathcal{E}_{X^c} \otimes_{p'^{-1}\mathcal{D}_X} p'^{-1}\mathcal{M}$. The composition

$$\mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(\mathcal{E}_X) \to \mathrm{K}_S(\mathcal{E}_{X^c})$$

is induced by sending $\mathcal{M}$ to $\epsilon_*(\mathcal{E}_X \otimes_{p^{-1}\mathcal{D}_X} p^{-1}\mathcal{M})$. The result follows by noting that the adjunction

$$\mathcal{N} \to \epsilon_*\epsilon^{-1}\mathcal{N}$$

is an isomorphism for all $\mathcal{N}$. □

As a result of the lemma and the previous theorem we have the natural homotopy morphisms:

$$\mathcal{E}_S : \mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X)$$

and

$$\mathcal{E}_S : \mathrm{K}_S(\mathcal{E}_X) \to \mathrm{K}_S(T^*X)$$

Furthermore, these fit into a commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_S(\mathcal{D}_X) & \xrightarrow{\mathcal{E}_S} & \mathrm{K}_S(T^*X) \\ & \searrow & \uparrow{\scriptstyle \mathcal{E}_S} \\ & & \mathrm{K}_S(\mathcal{E}_X) \end{array}$$

Theorem 3.24. *Let $S' \subset S$ where S' and S are closed conic subsets. Then the following diagram is naturally homotopically commutative:*

$$\begin{array}{ccc} \mathrm{K}_S(\mathcal{E}_{X^c}) & \xrightarrow{\mathcal{E}_S} & \mathrm{K}_S(T^*X) \\ \uparrow & & \uparrow \\ \mathrm{K}_{S'}(\mathcal{E}_{X^c}) & \xrightarrow{\mathcal{E}_{S'}} & \mathrm{K}_{S'}(T^*X) \end{array}$$

One has a similar statement for the corresponding spectra of $\mathcal{D}_X$-modules with supports.

Proof. This follows from the construction of $\mathcal{E}_S$ given in Theorem 3.22. The point is that the gr is natural in conic opens V. □

3.4. **Epsilon factors for $\mathcal{D}_X$-modules.** In this section, we show how to use the results of the previous section to construct a theory of epsilon factors for complexes of $\mathcal{D}_X$-modules (resp. $\mathcal{E}_X$-modules). Once we have microlocalization morphisms $\mathcal{E}_S$ as in the previous section, the construction of epsilon factors follows the same basic recipe as in [12]. In this section, all we shall work with the K-theory spectra of the Waldhausen category of bounded perfect complexes. For instance, $\mathrm{K}(\mathcal{D}_X)$ will denote the K-theory spectrum of perfect complexes of $\mathcal{D}_X$-modules. Similarly, for $\mathrm{K}(\mathcal{E}_X)$, $\mathrm{K}(X)$, etc. Since X is smooth, these are canonically weak equivalent to the corresponding Quillen K-theory spectra of locally free or coherent modules. In particular, all the construction of the previous sections apply to these spectra (by taking canonical inverses as homotopy morphisms).
In the following, we fix a closed conic $S \subset T^*X$. Let $U \subset X$ denote an open subset, $Y = X \setminus U$, and ν be a 1-form on U such that $\nu(U) \cap S = \emptyset$. Furthermore, we shall assume that X is projective. By ([12], Lemma 3.2.1), this gives rise to a commutative diagram:

$$\begin{array}{ccc}
\mathrm{K}(V) & \xrightarrow{\nu^*} & \mathrm{K}(U) \\
\uparrow & & \uparrow \\
\mathrm{K}(T^*X) & \xrightarrow{(\pi^*)^{-1}} & \mathrm{K}(X) \\
\uparrow & & \uparrow \\
\mathrm{K}_S(T^*X) & \xrightarrow{\mathcal{E}_\nu} & \mathrm{K}(Y)
\end{array}$$

In particular, we have microlocalization morphisms:

$$\mathrm{K}_S(\mathcal{D}_X) \overset{\mathcal{E}_S}{\to} \mathrm{K}_S(T^*X) \overset{\mathcal{E}_\nu}{\to} \mathrm{K}(Y).$$

Let $\mathcal{E}_{\nu,S}$ denote the composition. Similarly, we have a morphism

$$\mathrm{K}_S(\mathcal{E}_X) \to \mathrm{K}(Y).$$

By abuse of notation, we shall also denote the latter morphism by $\mathcal{E}_{\nu,S}$.

This gives rise to a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{K}(\mathcal{D}_X) & \longrightarrow & \mathrm{K}(T^*X) & \longrightarrow & \mathrm{K}(X) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{K}_S(\mathcal{D}_X) & \longrightarrow & \mathrm{K}_S(T^*X) & \longrightarrow & \mathrm{K}(Y)
\end{array}$$

By ([12]), the composition

$$\mathrm{K}(\mathcal{D}_X) \xrightarrow{gr} \mathrm{K}(T^*X) \xrightarrow{(\pi^*)^{-1}} \mathrm{K}(X) \xrightarrow{R\Gamma} \mathrm{K}(k)$$

is homotopic to $R\Gamma_{dr} : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$.

If $\mathcal{M}$ is a perfect complex of $\mathcal{D}_X$-modules with singular support in S, then it gives rise to a homotopy point $[\mathcal{M}]$ of $\mathrm{K}_S(\mathcal{D}_X)$. The above remarks now give the following:

Theorem 3.25. *Let X, $U \subset X$, $\mathcal{M}$, $S \subset T^*X$ and ν be as above. Then*

1: *The homotopy points $[R\Gamma_{dr}(X, \mathcal{M})]$ and $[R\Gamma(\mathcal{E}_{\nu,Y}(\mathcal{M}))]$ are naturally identified. In particular, passing to determinants gives a natural isomorphism (in $Pic^{\mathbb{Z}}(k)$):*

$$\det(R\Gamma_{dR}(X, \mathcal{M})) \to \varepsilon_{\nu,S}(\mathcal{M}),$$

where $\varepsilon_{\nu,S}(\mathcal{M}) := \mathrm{Det}([R\Gamma(\mathcal{E}_{\nu,Y}(\mathcal{M}))]$.

2: *Suppose $\nu = \mu$ on an open neighborhood U' of Y. Then $\mathcal{E}_{\nu,Y}(\mathcal{F})$ and $\mathcal{E}_{\mu,Y}(\mathcal{F})$ are naturally identified.*

3: *Let $\mathcal{G}, \mathcal{F} \in D^b_S(\mathcal{D}_X)$, and ν be as above. Suppose $\mathcal{F}|_{U'} = \mathcal{G}|_{U'}$ for some open $U' \subset X$ such that $Y \subset U'$. Then $\mathcal{E}_{\nu,Y}(\mathcal{F})$ and $\mathcal{E}_{\nu,Y}(\mathcal{G})$ are naturally identified.*

Remark 3.26. 1: If Y is the finite disjoint union of Y_i, then $\mathrm{K}(Y) = \prod \mathrm{K}(Y_i)$. In particular, one has homotopy points $\mathcal{E}_{\nu,Y_i}(\mathcal{F})$ of $\mathrm{K}(Y_i)$ and a canonical identification $\mathcal{E}_{\nu,Y}(\mathcal{F}) = \sum_i \mathcal{E}_{\nu,Y_i}(\mathcal{F})$.
2: If k' is a finite extension of k, then $\varepsilon_{\nu,Y}(\mathcal{F}) \otimes_k k' = \varepsilon_{\nu_{k'},Y_{k'}}(\mathcal{F}_{k'})$.

The same method as in section 5 of ([12]) can be used to show that the previous theorem gives rise to a theory of de Rham epsilon factors for curves (in the sense of [3]). We expect that all three theories of epsilon factors for curves are naturally isomorphic. We refer to loc. cit. for details.

Furthermore, we expect that the microlocalization morphisms $\mathcal{E}_S$ constructed here is naturally identified with the construction of ([12]) via filtered $\mathcal{D}_X$-modules. It is easy to see (although we do not give a proof here), that the two microlocalization morphisms are the same at the level of homotopy groups.

Finally, note that the homotopy point $\mathcal{E}_{\nu,S}(\mathcal{M})$ only depends on the homotopy point $[\mathcal{E}_X \otimes_{p^{-1}\mathcal{D}_X} p^{-1}\mathcal{M}]$. In particular, one wonders if there is a theory of epsilon factors for general $\mathcal{E}_X$-modules. The crucial thing is to check that the composition

$$\mathrm{K}(\mathcal{E}_X) \to \mathrm{K}(T^*X) \to \mathrm{K}(X) \to \mathrm{K}(k)$$

is homotopic to $R\Gamma_{dr} : \mathrm{K}(\mathcal{E}_X) \to \mathrm{K}(k)$. However, the author did not check this statement.

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