MA 16010 Lesson 10: Quotient rule & other trig functions

Recal: Last time we discussed **the product rule**:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)\cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

Using the product rule, one can derive the quotient rule as follows:

$$g(x)\cdot\frac{f(x)}{g(x)} = f(x) \qquad \Big/ \frac{d}{dx}\left[\dots\right]$$

$$g'(x)\cdot\frac{f(x)}{g(x)} + g(x)\cdot\underbrace{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]}_{\text{solve for this term}} = f'(x)$$

$$g(x)\cdot\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = f'(x) - g'(x)\cdot\frac{f(x)}{g(x)}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{g(x)}\left(f'(x) - g'(x)\cdot\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{g(x)^2} =$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Quotient rule: $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}}$$

Exercise: Compute $y'(x)$ when $y = \frac{x^2+3x+1}{x-4}$.

$f(x) = x^2+3x+1, \quad f'(x) = 2x+3$

$g(x) = x-4, \quad g'(x) = 1$

$$\rightarrow \cancel{\tfrac{dy}{}}\; y'(x) = \frac{(2x+3)(x-4) - 1\cdot(x^2+3x+1)}{(x-4)^2} =$$

$$= \frac{2x^2+3x-8x-12-x^2-3x-1}{(x-4)^2} = \frac{x^2-8x-13}{(x-4)^2}$$

$f(x) = 3\cos(x) - 3\sin(x)$
$g(x) = \sin(x) + \cos(x)$
$f'(x) = -3\sin x - 3\cos x$
$g'(x) = \cos x - \sin x$

Exercise: Compute $y'(\pi)$ when $y = \frac{3\cos(x) - 3\sin(x)}{\sin(x)+\cos(x)}$.

$$y'(x) = \frac{(-3\sin(x) - 3\cos x)(\sin(x) + \cos(x)) - (\cos x - \sin x)(3\cos x - 3\sin x)}{(\sin(x) + \cos(x))^2}$$

$$= \frac{-3\sin^2(x) - 3\cos x \sin x - 3\sin x \cos x - 3\cos^2 x - 3\cos^2 x + \cancel{3\sin x\cos x} + \cancel{3\sin x \cos x}}{(\sin x + \cos x)^2}$$

$-3\sin^2 x$

$= 1$

$$= \frac{-6\sin^2(x) - 6\cos^2 x}{(\sin(x) + \cos(x))^2} = \frac{-6(\overbrace{\sin^2(x) + \cos^2(x)}^{=1})}{(\sin(x) + \cos(x))^2} = \frac{-6}{(\sin x + \cos x)^2}$$

$$y'(\pi) = \frac{-6}{(\sin(\pi) + \cos(\pi))^2} = \frac{-6}{(0-1)^2} = -6$$

Exercise: Compute the derivative of $y = \frac{3x-a}{4x^2+a^2}$ where a is a constant.

$$\left[\begin{array}{ll} f(x) = 3x - a & f'(x) = 3 \\ g(x) = 4x^2 + a^2 & g'(x) = 8x \end{array}\right]$$

$$y'(x) = \frac{3 \cdot (4x^2 + a^2) - 8x \cdot (3x - a)}{(4x^2 + a^2)^2} =$$

$$= \frac{12x^2 + 3a^2 - 24x^2 + 8ax}{(4x^2 + a^2)^2} = \frac{-12x^2 + 8ax + 3a^2}{(4x^2 + a^2)^2}$$

Exercise (derivatives of the remaining trig. functions). Use the quotient rule to compute the derivatives

1. We have $\tan(x) = \frac{\sin(x)}{\cos(x)}$, therefore

$$(\tan(x))' = \frac{\cos(x)\cdot\cos(x) - (-\sin x)\cdot\sin x}{\cos^2(x)} = \frac{\overbrace{\cos^2(x)+\sin^2(x)}^{=1}}{\cos^2(x)} =$$

$$= \frac{1}{\cos^2(x)} = \underline{\underline{\sec^2(x)}}$$

2. We have $\cot(x) = \frac{\cos(x)}{\sin(x)}$, therefore

$$(\cot(x))' = \frac{(-\sin(x))\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} =$$

$$= -\frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)} = -\frac{1}{\sin^2(x)} = \underline{\underline{-\csc^2(x)}}$$

3. We have $\sec(x) = \frac{1}{\cos(x)}$, therefore

$$(\sec(x))' = \frac{0\cdot\cos(x) - (-\sin(x))\cdot 1}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)} =$$

$$= \underline{\underline{\tan(x)\cdot\sec(x)}}$$

4. We have $\csc(x) = \frac{1}{\sin(x)}$, therefore

$$(\csc(x))' = \frac{0\cdot\sin(x) - \cos(x)\cdot 1}{\sin^2(x)} = -\frac{\cos(x)}{\sin^2(x)} = -\frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)} =$$

$$= \underline{\underline{-\cot(x)\csc(x)}}$$

Summary – derivatives of trigonometric functions.

$(\sin(x))' =$ $\cos(x)$

$(\tan(x))' =$ $\sec^2(x)$

$(\sec(x))' =$ $\tan(x)\sec(x)$

$(\cos(x))' =$ $-\sin(x)$

$(\cot(x))' =$ $-\csc^2(x)$

$(\csc(x))' =$ $-\cot(x)\csc(x)$

„co-" has the added „-" sign in its derivative

Exercise: Compute the equation of the tangent line to $y = 3x^2\sec(x)$ at $x = \pi/3$.

1) compute the slope of the tangent, $y'(\frac{\pi}{3})$:

can use (A) $(\sec(x))'$ and product rule, (B) $y = \frac{3x^2}{\cos x}$ and quotient rule! The result is the same (we choose (A))

$$y'(x) = \frac{d}{dx}\left[3x^2\sec(x)\right] =$$

$$= 6x\cdot\sec(x) + 3x^2\tan(x)\sec(x)$$

$$y'\left(\frac{\pi}{3}\right) = 6\cdot\frac{\pi}{3}\cdot\sec\left(\frac{\pi}{3}\right) + 3\cdot\left(\frac{\pi}{3}\right)^2\tan\left(\frac{\pi}{3}\right)\sec\left(\frac{\pi}{3}\right)$$

$$= 2\pi\cdot\frac{1}{\cos(\pi/3)} + \frac{\pi^2}{3}\cdot\frac{\sin(\pi/3)}{\cos(\pi/3)}\cdot\frac{1}{\cos(\pi/3)} =$$

$$= 4\pi + \frac{\pi^2}{3}\cdot\sqrt{3}\cdot 2$$

$\left(\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\right)$

2) compute the tangent line:

$$t(x) = \underbrace{3\cdot\left(\frac{\pi}{3}\right)^2\sec\left(\frac{\pi}{3}\right)}_{3x^2\sec(x)\text{ for } x=\pi/3} + \left(4\pi + \frac{\pi^2}{3}\cdot\sqrt{3}\cdot 2\right)\left(x - \frac{\pi}{3}\right)$$

$$= \frac{2\pi^2}{3} + \left(4\pi + \frac{2\pi^2}{3}\cdot\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$$