MA 16010 Lesson 16: Related rates II

Recall: In a **problem with related rates**, we are given several functions $x = x(t), y = y(t), \ldots$ of another variable t, related by an equation, e.g.

$$xy = x^2 + y\,.$$

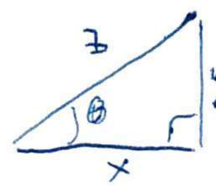
After taking the derivative with respect to t (using the chain rule for all variables,) we obtain an equation that relates the derivatives:

$$\frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt} = 2x\cdot\frac{dx}{dt} + \frac{dy}{dt}$$

Today: We consider a bit more complicated related rates problems. Some of them will involve more than two variables.

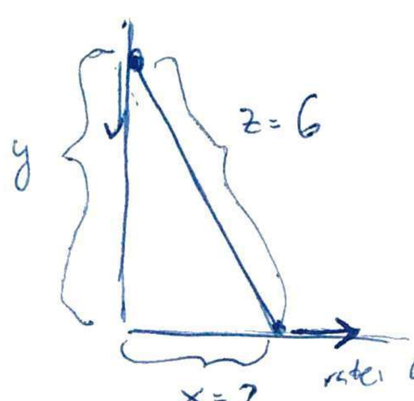
Recall: Pythagorean theorem.

$$\boxed{x^2+y^2=z^2}$$



(Also recall: $\tan\theta = \frac{y}{x}$)

Exercise: A ladder of length 6 ft leaning on a wall is sliding down. Currently, the base of the ladder is 2 ft away from the wall and is sliding away from the wall at the rate 0.5 ft/sec. At what rate is the top of the ladder sliding down?



want: $\frac{dy}{dt} = ?$

1) find y:

$x^2+y^2=36$, $x=2$ initially

$\rightarrow 4+y^2=36 \rightarrow y^2=32 \Rightarrow y=\sqrt{32}$ ft

$y = 4\sqrt{2}$ ft

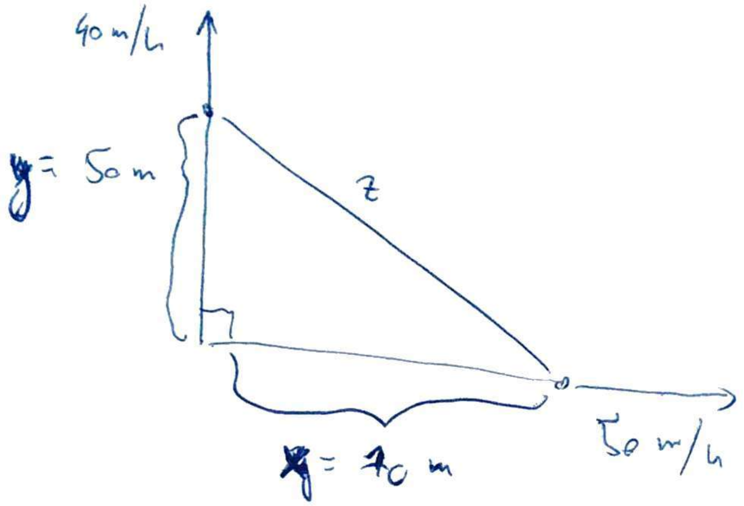
2) we have $x^2+y^2=36 \;\Big|\; \frac{d}{dt}$

$$\underbrace{2\cdot x}_{2}\cdot\underbrace{\frac{dx}{dt}}_{0.5} + 2\cdot\underbrace{y}_{4\sqrt{2}}\cdot\frac{dy}{dt} = 0$$

$$2 + 8\sqrt{2}\,\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{2}{8\sqrt{2}} = -\frac{\sqrt{2}}{8}\text{ ft/sec} \approx -0.177 \text{ ft/sec}$$

Exercise: Two cars leave a common point at the same time. One of the cars travels north, the other travels east. After some time, the first car traveled 50 miles and is currently traveling at the speed 40 mph. The second car traveled 70 miles and is currently going at the speed 50 mph. At what rate does the distance between the cars grow at this moment?



1) Find initial z:

$z^2 = 50^2 + 70^2$

$z^2 = 2500 + 4900 = 7400$

$z = \sqrt{7400} = 10\sqrt{74}$ miles

Want to find $\frac{dz}{dt}$

2) we have $z^2 = x^2 + y^2 \quad \Big| \frac{d}{dt}$

$$2z \cdot \frac{dz}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

Know: $x = 70$, $\frac{dx}{dt} = 50$, $y = 50$, $\frac{dy}{dt} = 40$, $z = 10\sqrt{74}$

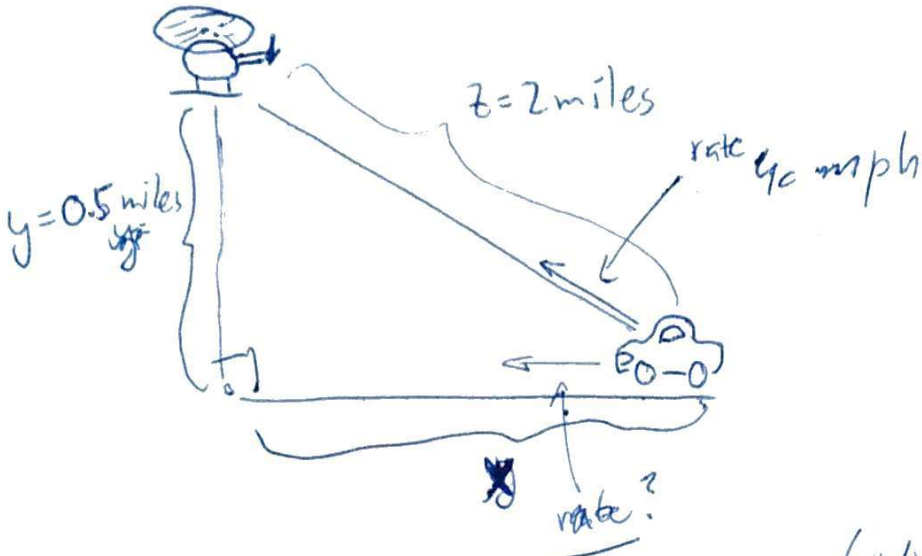
⇝ plug in:

$$20\sqrt{74} \cdot \frac{dz}{dt} = 2 \cdot 70 \cdot 50 + 2 \cdot 50 \cdot 40$$

$$2\cancel{0}\sqrt{74}\,\frac{dz}{dt} = 7000 + 4000 = 1100\cancel{0}$$

$$\frac{dz}{dt} = \frac{1100}{2\sqrt{74}} \text{ mph} \approx 63.94 \text{ mph}$$

Exercise: A police helicopter is hovering at an altitude of 0.5 mile above a straight road. Using radar, the pilot determines that a car on the road (traveling towards the helicopter) is at a distance of exactly 2 miles from the helicopter, and this distance is decreasing at the rate of 40 mph. What is the speed of the car?



1) Find x:

$$(0.5)^2 + x^2 = z^2$$

$$x^2 = 4 - \left(\tfrac{1}{2}\right)^2 = \tfrac{15}{4}$$

$$x = \sqrt{\tfrac{15}{4}} = \tfrac{\sqrt{15}}{2}$$

2) have $x^2 + y^2 = z^2 \quad \Big| \frac{d}{dt}$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

(What is $\frac{dy}{dt}$?

Helicopter hovers at fixed altitude $\to$ y does not change

$\to \frac{dy}{dt} = 0$)

have $x = \frac{\sqrt{15}}{2}$, $y = \frac{1}{2}$, $z = 2$, $\frac{dz}{dt} = -40$, $\frac{dy}{dt} = 0$

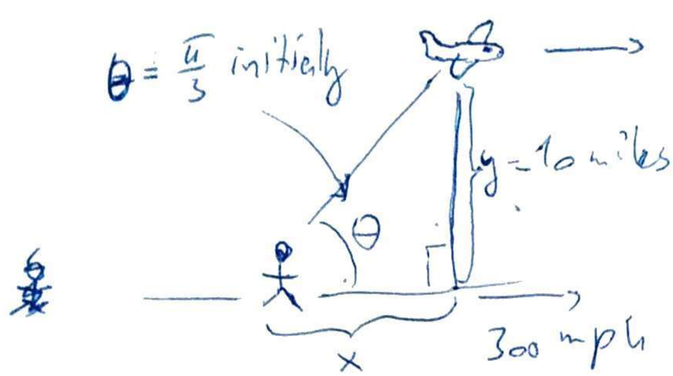
$\Rightarrow$ $$\sqrt{15}\cdot\frac{dx}{dt} + 1\cdot 0 = 2\cdot 2\cdot(-40)$$

$$\sqrt{15}\cdot\frac{dx}{dt} = -160$$

$$\frac{dx}{dt} = -\frac{160}{\sqrt{15}} \text{ mph} \approx -41.31 \text{ mph}$$

($\to$ speed of the car $\approx$ 41.31 mph)

Exercise: We are watching a plane at the altitude 10 miles that is flying away at the speed 300 mph. At what rate does the angle of elevation change at the moment when it is equal $\pi/3$ (rad)?



$\tan\theta = \frac{y}{x}$

1) find x:

$$\underbrace{\tan\left(\frac{\pi}{3}\right)}_{\sqrt{3}} = \frac{10}{x}$$

$$\sqrt{3}\cdot x = 10$$

$$x = \frac{10}{\sqrt{3}} \text{ miles}$$

2) $\tan\theta = \frac{y}{x} \quad \Big| \frac{d}{dt}$

Know: $x = \frac{10}{\sqrt{3}}$, $y = 10$, $\frac{dx}{dt} = 300$, $\frac{dy}{dt} = 0$, $\theta = \frac{\pi}{3}$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{\frac{dy}{dt}\cdot x - \frac{dx}{dt}\cdot y}{x^2}$$

$$\underbrace{\sec^2\left(\frac{\pi}{3}\right)}\cdot \frac{d\theta}{dt} = \frac{0 - 300\cdot 10}{\left(\frac{10}{\sqrt{3}}\right)^2} = -\frac{3000}{\left(\frac{100}{3}\right)} = -90$$

$$= \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

$$4\,\frac{d\theta}{dt} = -90$$

$$\frac{d\theta}{dt} = -\frac{90}{4} = -22.5 \text{ rad/hour}$$