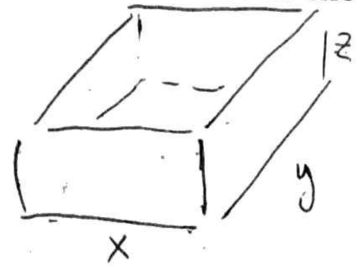
MA 16010 Lesson 26: Optimization III

Exercise: An open box is to be made such that its volume is $20\,\mathrm{m}^3$. The sides are to be made of wood and its base is to be made of metal. Additionally, the length of the box should be twice its width.

If the cost of wood is $5\,\$/\mathrm{m}^2$ and the cost of metal is $8\,\$/\mathrm{m}^2$, find the minimal cost of the box.



Want: minimize cost

$$C = 8\cdot xy + 5(2yz + 2xz)$$
$$= 8xy + 10yz + 10xz$$

Constraints: $\left.\begin{array}{l} y = 2x \\ xyz = 20 \end{array}\right\}$ $x\cdot 2x\cdot z = 20$

$$2x^2 z = 20$$
$$z = \frac{20}{2x^2} = \frac{10}{x^2}$$

$$C = 8\cdot xy + 10yz + 10xz = 8x(2x) + 10(2x)\frac{10}{x^2} + 10\cdot x\cdot\frac{10}{x^2}$$
$$= 16x^2 + \frac{200}{x} + \frac{100}{x} = 16x^2 + \frac{300}{x}$$ ("objective function")

$$C' = 32x - \frac{300}{x^2}$$

$$32x - \frac{300}{x^2} = 0$$
$$32x = \frac{300}{x^2}$$
$$32x^3 = 300$$
$$x^3 = \frac{300}{32} = \frac{150}{16} = \frac{75}{8}$$
$$x = \sqrt[3]{\frac{75}{8}} = \frac{1}{2}\sqrt[3]{75}$$

Minimal cost is $C = 16\cdot\left(\frac{1}{2}\sqrt[3]{75}\right)^2 + \frac{300}{\frac{1}{2}\sqrt[3]{75}} = 4\left(\sqrt[3]{75}\right)^2 + \frac{600}{\sqrt[3]{75}}$ $(\approx 213.41\ \$)$

Exercise: A product can be sold for the price $\$\,p$, and at this price, q units of the product will be sold where $q = 3000 - 200p$. The cost of producing one unit is $\$\,10$.

(a) Maximize revenue. Revenue = (# units sold)(price of each unit)

$$R = (3000 - 200p)p = 3000p - 200p^2$$

$$R' = 3000 - 400p$$

$$3000 - 400p = 0$$

$$400p = 3000$$

$$4p = 30$$

$$p = \frac{30}{4} = 7.5\ \$$$

Max. revenue is $R = (3000 - 200 \cdot (7.5)) \cdot (7.5) = 11\,250\ \$$

(b) Maximize profit.

Profit = (revenue) − (production cost)

$$P = (3000 - 200p)p - 10 \cdot (3000 - 200p)$$

$$= 3000p - 200p^2 - 30000 + 2000p$$

$$= 5000p - 200p^2 - 30\,000$$

$$P' = 5000 - 400p$$

$$5000 - 400p = 0$$

$$400p = 5000$$

$$4p = 50$$

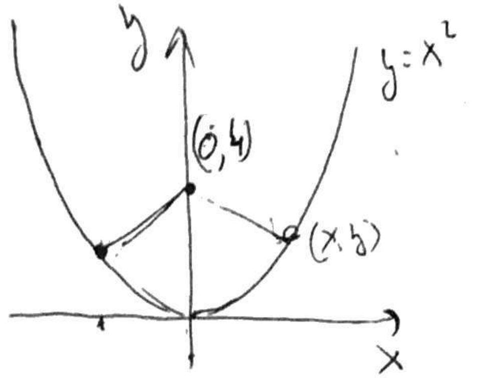
$$p = \frac{50}{4} = 12.5$$

Maximal profit is

$$P = 5000 \cdot (12.5) - 200 \cdot (12.5)^2 - 30000$$

$$= 1250\ \$$$

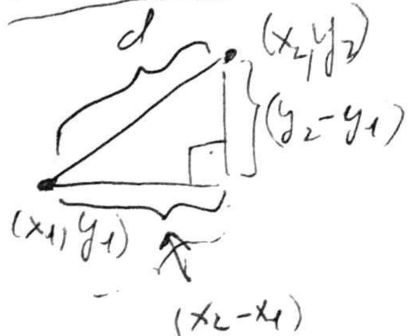
Exercise: Find the point (x, y) lying on the parabola $y = x^2$ that is closest to the point $(0, 4)$.



<u>Recall:</u>

The distance between points (x_1,y_1) and (x_2,y_2) is given by

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$



$$d^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

<u>Want:</u> Minimize $d=\sqrt{(x-0)^2+(y-4)^2}=$

$$=\sqrt{x^2+(y-4)^2}$$

<u>Constraints:</u> $y=x^2$

$$\rightsquigarrow d=\sqrt{x^2+(x^2-4)^2}=\left(x^2+(x^2-4)^2\right)^{\frac{1}{2}}$$

$$d'=\frac{1}{2}\left(x^2+(x^2-4)^2\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left[x^2+(x^2-4)^2\right]=$$

$$=\frac{1}{2}\left(x^2+(x^2-4)^2\right)^{-\frac{1}{2}}\left(2x+2(x^2-4)\cdot 2x\right)$$

$$=\frac{2x+4(x^2-4)x}{2\sqrt{x^2+(x^2-4)^2}}=\frac{x+2(x^2-4)x}{\sqrt{x^2+(x^2-4)^2}}=\frac{x+2x^3-8x}{\sqrt{x^2+(x^2-4)^2}}$$

$$=\frac{2x^3-7x}{\sqrt{x^2+(x^2-4)^2}}$$

$$\frac{2x^3 - 7x}{\sqrt{x^2 + (x^2-4)^2}} = 0$$

$$2x^3 - 7x = 0$$

$$x(2x^2 - 7) = 0$$

$x = 0$

$$2x^2 - 7 = 0$$
$$2x^2 = 7$$
$$x^2 = \frac{7}{2}$$
$$x = \pm\sqrt{\frac{7}{2}}$$

to find which one gives min distance, plug into $d = \sqrt{x^2 + (x^2-4)^2}$

$x = 0$ .. $d = 4$

$x = \pm\sqrt{\frac{7}{2}}$

$$d = \sqrt{\frac{7}{2} + \left(\frac{7}{2} - 4\right)^2} = $$
$$= \sqrt{\frac{7}{2} + \frac{1}{4}} < 4$$

$=$

→ Points at min distance are

$$(x, y) = \left(\pm\sqrt{\frac{7}{2}}, \frac{7}{2}\right)$$