MA 16010 Lesson 29: Area and Riemann Sums

Sigma notation. We use "Σ" to write sums of bunch of terms succintly.

terms to be added

For example, $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

index i goes from 1 to 4

Exercise: Evaluate

$$\sum_{i=2}^{5} (-1)^i (i-1) = (-1)^2(2-1) + (-1)^3(3-1) + (-1)^4(4-1) + (-1)^5(5-1)$$

$$= (-1)^2 \cdot 1 + (-1)^3 \cdot 2 + (-1)^4 \cdot 3 + (-1)^5 \cdot 4 = 1 - 2 + 3 - 4 = -2$$

$$\sum_{i=0}^{4} \frac{\sqrt{i}}{i+1} = \frac{\sqrt{0}}{0+1} + \frac{\sqrt{1}}{1+1} + \frac{\sqrt{2}}{2+1} + \frac{\sqrt{3}}{3+1} + \frac{\sqrt{4}}{4+1} =$$

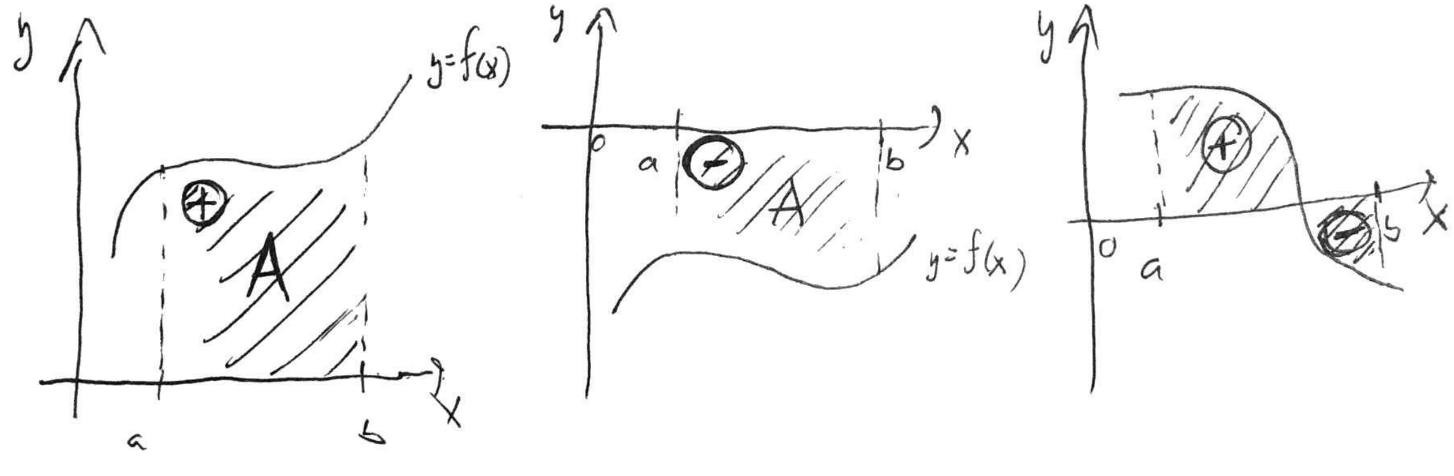
$$= 0 + \frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} + \frac{2}{5} = \frac{9}{10} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} \left(\approx 1.804\right)$$

Exercise: Use the Σ–notation to write down the sum

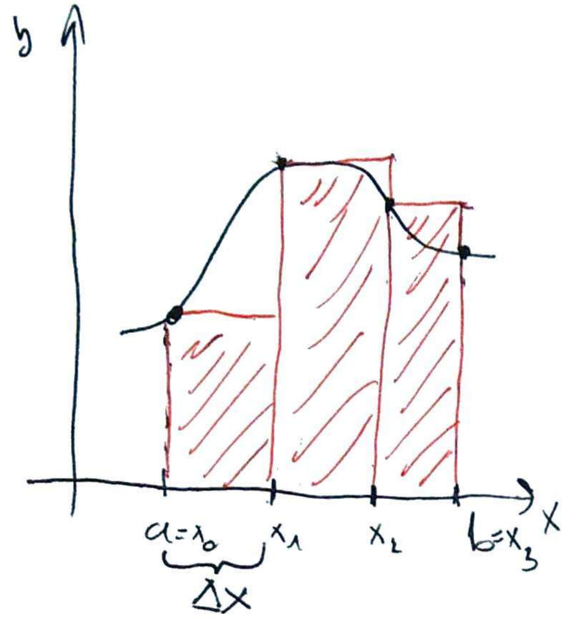
$$\underbrace{(\sqrt{3}-2)^2}_{i=1} + \underbrace{(\sqrt{4}-3)^2}_{i=2} + \underbrace{(\sqrt{5}-4)^2}_{i=3} + \cdots + \underbrace{(\sqrt{n+2}-n-1)^2}_{i=n}$$

$$= \sum_{i=1}^{n} \left(\sqrt{i+2} - (i+1)\right)^2$$

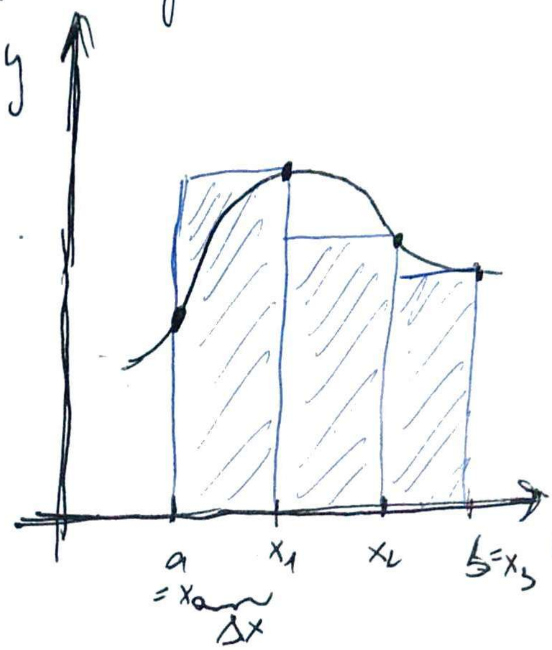
Area under the curve. For a function $y = f(x)$, we want to compute/ /estimate the **(signed) area under the curve** over a given interval $[a, b]$:



To approximate the area, we use **Riemann sums** = approximations of the area by the area of thin ~~rec~~ rectangles:



Left Riemann sums



Right Riemann sums

Let's say we use n such rectangles ($n =$ 3 in the picture above).

The base of each one has **length** $\Delta x = \frac{b-a}{n}$

The **height** of each rectangle is:

For the **left Riemann sums**, it is the y-value of the left endpoint. $(f(x_i))$

For the **right Riemann sums**, it is the y-value of the right endpoint $(f(x_i))$

The **area of one rectangle** is therefore $f(x_i) \cdot \Delta x = f(x_i) \cdot \frac{b-a}{n}$,

and the approximation of the **overall area** therefore is:

$L_n = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \dots + f(x_{n-1}) \cdot \Delta x$

$$\boxed{L_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x = \sum_{i=0}^{n-1} f(x_i) \cdot \frac{b-a}{n}}$$

(Left R.S.)

$R_n = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \dots + f(x_n) \cdot \Delta x$

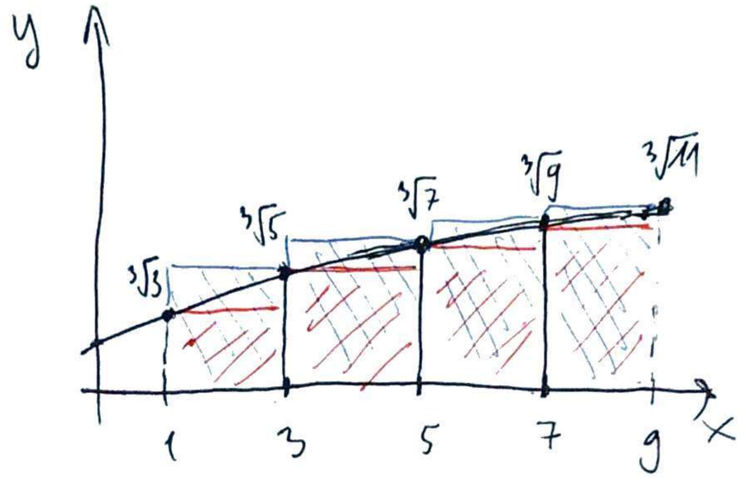
$$\boxed{R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{n}}$$

(Right R.S.)

Exercise: Use the left and right Riemann sums with 4 rectangles to estimate the (signed) area under the curve of

$$y = \sqrt[3]{x+2}$$

on the interval $[1, 9]$. (Round your answers to two decimal places.)



$$\Delta x = 2 \left(= \frac{9-1}{4}\right)$$

Left R.S.:

$$L_4 = 2\cdot\sqrt[3]{3} + 2\cdot\sqrt[3]{5} + 2\cdot\sqrt[3]{7} + 2\cdot\sqrt[3]{9} \approx \underline{\underline{14.29}}$$

Right R.S.:

$$R_4 = 2\cdot\sqrt[3]{5} + 2\cdot\sqrt[3]{7} + 2\cdot\sqrt[3]{9} + 2\cdot\sqrt[3]{11} \approx \underline{\underline{15.85}}$$

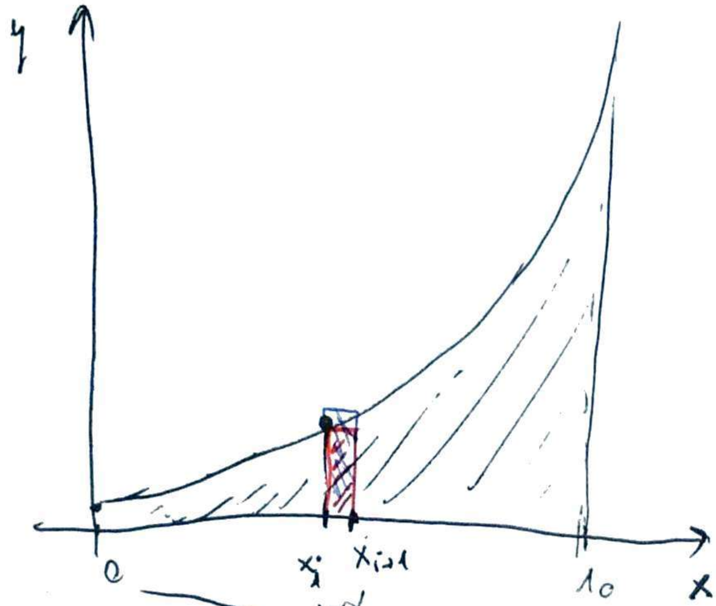
(For comparison: True area is ≈ 15.1

Error is $\approx 5.4\%$)

Exercise: Use the left and right Riemann sums with 100 rectangles to estimate the (signed) area under the curve of

$$y = e^x + 1$$

on the interval $[0, 10]$. (Write answers with the sigma notation.)



$$\Delta x = \frac{b-a}{n} = \frac{10-0}{100} = \frac{1}{10}$$

$x_0 = 0$

$x_1 = \frac{1}{10}$

$x_2 = \frac{2}{10}$

$\vdots$

$x_i = \frac{i}{10}$

Left R.S:

$$L_{100} = \sum_{i=0}^{99} \left(e^{\frac{i}{10}} + 1\right) \cdot \frac{1}{10}$$

Right R.S:

$$R_{100} = \sum_{i=1}^{100} \left(e^{i/10} + 1\right) \cdot \frac{1}{10}$$

Out of curiosity:

$L_{100} \approx 20952.544$

$R_{100} \approx 23155.091$

True area ≈ 22035.466

Error $\approx 4.9\%$