MA 16010 Lesson 32: Fundamental Theorem of Calculus

Recall: If $y = f(x)$ is a function, we consider

1) the **definite integral** $\int_a^b f(x)\,dx$: (signed) area below $y = f(x)$ and above $[a,b]$

– the result is a number

2) the **indefinite integral** $\int f(x)\,dx$: $= F(x)\,(+C)$ such that $F'(x) = f(x)$

– the result is a function

Fundamental Theorem of Calculus relates the two integrals:

If ($f(x)$ is continuous on $[a, b]$ and) $\int f(x)\,dx = F(x) + C$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \left(= \left[F(x) \right]_a^b = F(x)\Big|_a^b \right)$$

⇝ it gives a practical method to compute definite integrals.

Example. Let us compute $\int_1^3 (2x^3 + 3)\,dx$:

1) find the antiderivative / indef. integral:

$$\int (2x^3+3)\,dx = 2 \cdot \frac{x^4}{4} + 3x + C = \frac{x^4}{2} + 3x + C$$

2) Now apply FTC:

$$\int_1^3 (2x^3+3)\,dx = \left[\frac{x^4}{2} + 3x \right]_1^3 = \left(\frac{3^4}{2} + 9 \right) - \left(\frac{1^4}{2} + 3 \right)$$

$$= \frac{81}{2} + 9 - \frac{1}{2} - 3 = \frac{80}{2} + 6 = 40 + 6 = \underline{\underline{46}}$$

Exercise: Compute the following definite integrals.

(a) $\int_1^4 \frac{x^2+\sqrt[3]{x^2}}{\sqrt[3]{x}}\,dx$:

$$\int \frac{x^2+\sqrt[3]{x^2}}{\sqrt[3]{x}}\,dx = \int \frac{x^2+x^{2/3}}{x^{1/3}}\,dx = \int \left(x^{5/3}+x^{1/3}\right)dx$$

$$= \frac{3}{8}x^{8/3} + \frac{3}{4}x^{4/3} + C$$

$$\leadsto \int_1^4 \frac{x^2+\sqrt[3]{x^2}}{\sqrt[3]{x}}\,dx = \left[\frac{3}{8}x^{8/3}+\frac{3}{4}x^{4/3}\right]_1^4 = \frac{3}{8}\cdot 4^{8/3}+\frac{3}{4}\cdot 4^{4/3}-\frac{3}{8}\cdot 1^{8/3}-\frac{3}{4}\cdot 1^{4/3}$$

$$= \underline{\underline{\frac{3}{8}4^{8/3}+\frac{3}{4}4^{4/3}-\frac{3}{8}-\frac{3}{4}}} \left(\approx 18.756\right)$$

(b) $\int_0^5 (3e^x - 8)\,dx$:

$$\int_0^5 (3e^x-8)\,dx = \left[3e^x - 8x\right]_0^5 = (3e^5-8\cdot 5)-(3e^0-8\cdot 0)$$

$$= 3e^5 - 40 - 3 = \underline{\underline{3e^5-43}} \left(\approx 402.239\right)$$

Recall: $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{(-1)}(+C) = -\frac{1}{x}$,

~~BUT~~ while $\int \frac{1}{x}\,dx = \ln|x|(+C)$!

(c) $\int_2^3 \frac{x+1}{x^2}\,dx$:

$$\int_2^3 \frac{x+1}{x^2}\,dx = \int_2^3 \left(\frac{1}{x}+\frac{1}{x^2}\right)dx = \left[\ln|x| - \frac{1}{x}\right]_2^3 =$$

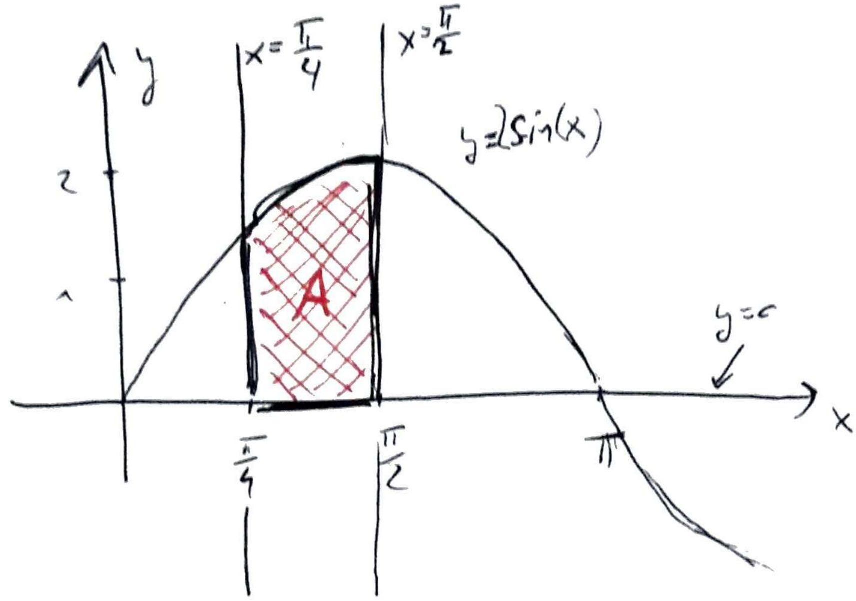
$$= \left(\ln(3)-\frac{1}{3}\right)-\left(\ln(2)-\frac{1}{2}\right) = \ln(3)-\ln(2)-\frac{1}{3}+\frac{1}{2} =$$

$$= \underline{\underline{\ln(3)-\ln(2)+\frac{1}{6}}} \left(\approx 0.572\right)$$

Exercise: Find the area of the region enclosed by the curves given by the equations

$$y = 2\sin(x), \qquad y = 0, \qquad x = \frac{\pi}{4}, \qquad x = \frac{\pi}{2}.$$

Sketch:



$$\text{Area} = \int_{\frac{\pi}{4}}^{\pi/2} 2\cdot\sin(x)\,dx = \Big[-2\cos(x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -2\cdot\cos\left(\frac{\pi}{2}\right) - \left(-2\cdot\cos\left(\frac{\pi}{4}\right)\right)$$

$$= -2\cdot 0 + 2\cdot\cos\left(\frac{\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2} = \underline{\underline{\sqrt{2}}}$$