MA 16010 Lesson 36: Exponential Decay

Recall: The solution to the equation $y' = ky$ is: $y = C \cdot e^{kt}$, C a constant

when $k > 0$, we speak of: exponential growth

Today: We consider the case $k < 0$. Then we speak of: exponential decay

[ex: $k=-1$ .. $y = Ce^{-t} = \frac{C}{e^t}$]

typical situation: Matter that decays "randomly"

→ the more of the stuff you have, the more of it decays in a given interval of time.

Example: The amount $A(t)$ of a radioactive isotope (that decays over time) obeys the equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -0.0002A$$

(where t s time in years). How long does it take for an initial amount $A(0)$ of the isotope to be reduced to half?

$A(t) = C \cdot e^{-0.0002t}$, where $A(0) = C \cdot e^0 = C$ is the initial amount

we want: time t such that

$$A(t) = \frac{1}{2} A(0) = \frac{1}{2} C$$

$$\cancel{C} \cdot e^{-0.0002t} = \frac{1}{2} \cancel{C}$$

$$e^{-0.0002t} = 1/2 \quad / \ln(\ldots)$$

$$-0.0002t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{\ln(1/2)}{-0.0002} \approx$$

≈ 3465.7 years

The time that we obtained in the previous problem as the (aptly named)

half-life of the isotope

("half-time", "half-life of decay"...)

Exercise: The radioactive isotope ^{226}Ra has a half–life of approximately 1599 years. There are 210g of ^{226}Ra now. How much of ^{226}Ra is left after ~~1~~5000 years?

$A(t)$ ... amount of radium after t years ... $A(t) = C \cdot e^{kt}$

where $A(0) = C \cdot e^0 = C = 210 \leadsto A(t) = 210\, e^{kt}$

to find k:

$A(1599) = \frac{1}{2} A(0) = \frac{210}{2} = 105$

$210\, e^{k \cdot 1599} = 105$

$e^{k\, 1599} = \frac{105}{210} = \frac{1}{2}$

$k \cdot 1599 = \ln\left(\frac{1}{2}\right)$

$k = \frac{\ln(1/2)}{1599}$

$\to A(t) = 210 \cdot e^{\frac{\ln(\frac{1}{2})}{1599} \cdot t}$

$\left(= 210 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1599}} \right)$

$A(5000) = 210\, e^{\frac{\ln(\frac{1}{2})}{1599} \cdot 5000}$

≈ 24 g

Exercise: A drug in a patient's body has half–life of 7 hours. If a patient takes a dose of 500 mg at 9:00 am, how much of the drug remains in his system at 9:00 am the next day?

$A(t)$ = amount of the drug t hours after 9:00 am

$A(t) = C \cdot e^{kt}$, $C = A(0) = 500$ $\to A(t) = 500\, e^{kt}$

To find k:

$A(7) = 250$

$500\, e^{k7} = 250$

$e^{k7} = \frac{1}{2}$

$k7 = \ln\left(\frac{1}{2}\right)$

$k = \frac{\ln(\frac{1}{2})}{7}$

$A(t) = 500 \cdot e^{\frac{\ln(\frac{1}{2})}{7} \cdot t}$

$\left(= 500 \left(\frac{1}{2}\right)^{\frac{t}{7}} \right)$

$A(24) = 500 \cdot e^{\frac{\ln(\frac{1}{2})}{7} \cdot 24}$

≈ 46.44 mg

Carbon dating.

The isotope ^{14}C (Carbon-14) is created in the atmosphere due to cosmic rays. Plants incorporate it during photosynthesis, and as a result, living organisms naturally contain ^{14}C. Once the plant or animal dies, the concentration of ^{14}C starts decaying. The half–life of ^{14}C is 5,730 years.

Exercise: An ancient mammal bone contains 2 mg of ^{14}C. Based on the size of the bone, we estimate that the bone contained 250 g of ^{14}C when the mammal was alive. Approximately how long ago did the animal die?

$A(t)$ = amount of ^{14}C in the bone t years after death

$$A(t) = C \cdot e^{kt}, \qquad \left.\begin{array}{l} A(0) = 250 \\ A(0) = C \cdot e^0 = C \end{array}\right\} C = 250, \quad A(t) = 250\, e^{kt}$$

To find k:

$$A(5730) = \tfrac{1}{2} \cdot 250 = 125$$

$$250 \cdot e^{k \cdot 5730} = 125$$

$$e^{k \cdot 5730} = \frac{1}{2}$$

$$k \cdot 5730 = \ln\left(\tfrac{1}{2}\right)$$

$$k = \frac{\ln(1/2)}{5730} \rightarrow A(t) = 250 \cdot e^{\frac{\ln(1/2)}{5730} \cdot t}$$

Want t such that

$$A(t) = 0.002 \text{ g}$$

$$250\, e^{\frac{\ln(1/2)}{5730} t} = 0.002$$

$$e^{\frac{\ln(1/2)}{5730} t} = \frac{0.002}{250} = 0.000008$$

$$\frac{\ln(\frac{1}{2})}{5730} t = \ln(0.000008)$$

$$t = \frac{\ln(0.000008)}{\ln(1/2)} \cdot 5730$$

$$\approx 97\,017.89 \text{ years}$$