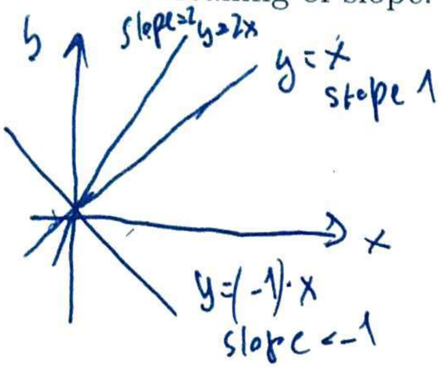
# MA 16010 Lesson 6: The Derivative

**Recall (slopes of linear functions).**

The *slope* of a linear function $f(x) = ax + b$ is the number ___$a$___.

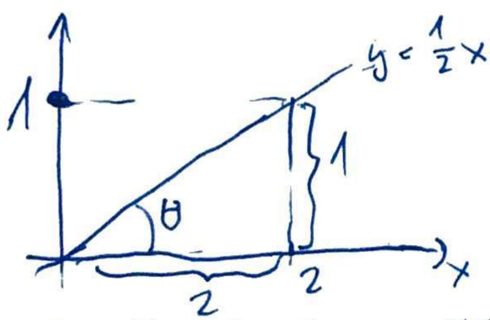
Meaning of slope: 1) the bigger the slope is, the faster the lin. function grows

2) the smaller (negative) the slope is, the faster the lin. function declines




Meaning of slope geometrically: ~~the~~ slope of a line = the tangent of the angle between the line and the x-axis

Ex:

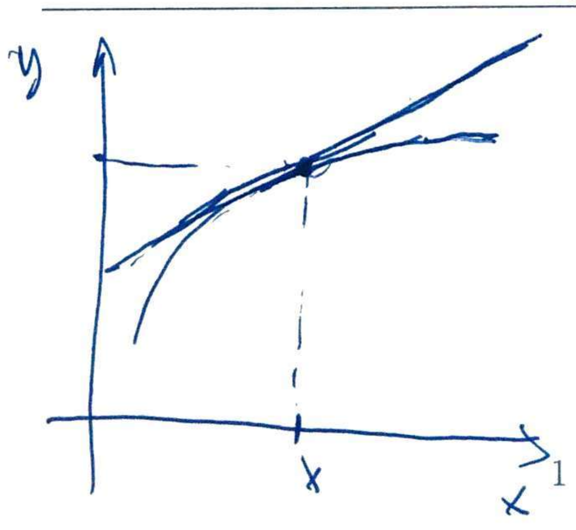


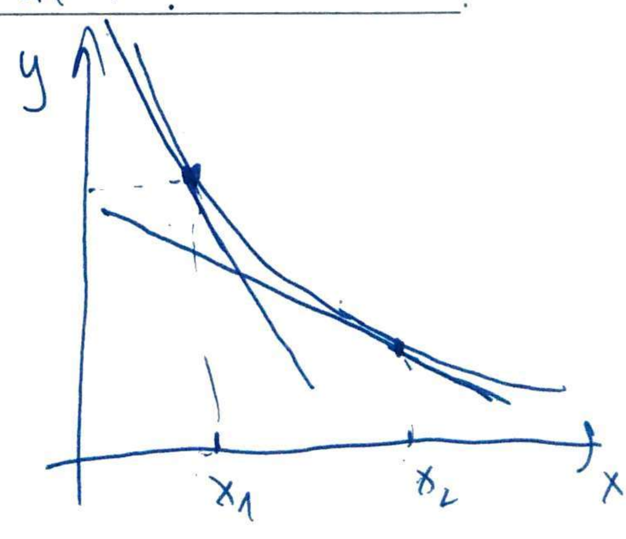

$\tan\theta = \frac{1}{2}$

**The derivative.** For a function $y = f(x)$, we want to be able to:

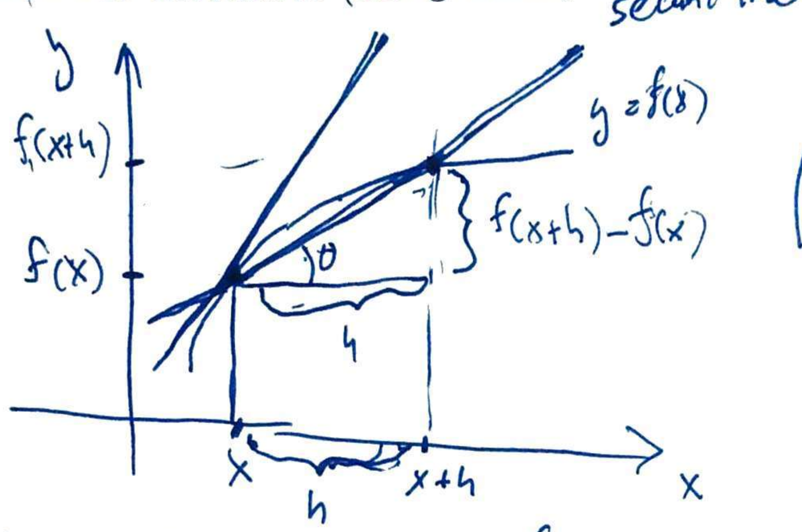
- Find the tangent line too its graph at a given point $x$,
- In particular, find the slope of tangent line: This is called

___the derivative of f at x.___



**How to find the derivative (using limits).**




(sometimes $h$ is denoted $\Delta x$)

slope of the secant line = $\dfrac{f(x+h) - f(x)}{h}$

As $h$ gets smaller and smaller, the secant line approaches the tangent line. Therefore

$$\left(\begin{array}{c}\text{the slope of the} \\ \underline{\text{tangent line}}\end{array}\right) = \lim_{h \to 0} \left(\begin{array}{c}\text{slope of the} \\ \text{secant line}\end{array}\right)$$

**Definition.** The derivative of $f(x)$ at $x$ is defined as

$$f'(x) \left( = \frac{df}{dx}(x) \right) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Example (derivative from definition step by step):**

Compute the slope of the tangent line of $f(x) = 5x^2 - 2x + 8$ at general $x$:

- $f(x+h) = 5(x+h)^2 - 2(x+h) + 8 = 5(x^2 + 2xh + h^2) - 2x - 2h + 8$
  $= 5x^2 + 10xh + 5h^2 - 2x - 2h + 8$
- $f(x+h) - f(x) = \cancel{5x^2} + 10xh + 5h^2 \cancel{-2x} - 2h \cancel{+8} - (\cancel{5x^2} \cancel{-2x} \cancel{+8}) = 10xh + 5h^2 - 2h$
- $\frac{f(x+h)-f(x)}{h} = \dfrac{10xh + 5h^2 - 2h}{h} = 10x + 5h - 2$
- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (10x + 5h - 2) = 10x - 2$

$\boxed{\text{Ex}}$ $f'(3) = 10 \cdot 3 - 2 = 28$ slope of tangent line to $f(x)$ at $x = 3$ is $28$

**Example:**

Compute $f'(x)$ for $f(x) = \frac{3}{4x+1}$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{4(x+h)+1} - \frac{3}{4x+1}}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{3(4x+1) - 3(4(x+h)+1)}{(4(x+h)+1)(4x+1)}}{h} = \lim_{h\to 0} \frac{\frac{\cancel{12x} + \cancel{3} - \cancel{12x} - 12h - \cancel{3}}{(4(x+h)+1)(4x+1)}}{h}$$

$$= \lim_{h\to 0} \frac{-12\cancel{h}}{(4(x+h)+1)(4x+1)\cancel{h}} = \lim_{h\to 0} \frac{-12}{(4(x+h)+1)(4x+1)}$$

$$= \frac{-12}{(4x+1)^2}$$

**Example:**

Find $f'(3)$ when $f(x) = x^2 + 7$:

Either compute $f'(x)$ in general, and plug in $x=3$

or: compute $f'(3)$ directly:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h \to 0} \frac{\left((3+h)^2+7\right) - (3^2+7)}{h} = \lim_{h \to 0} \frac{\cancel{9}+6h+h^2+\cancel{7}-\cancel{9}-\cancel{7}}{h}$$

$$= \lim_{h \to 0} \frac{6h+h^2}{h} = \lim_{h \to 0} (6+h) = 6$$

**Example:**

Find the equation of the tangent line to the graph of $f(x) = \frac{3}{x^2+1}$ at $x = 2$:

1) we find the slope of the tangent:

$$f'(2) = \lim_{h \to 0} \frac{\frac{3}{(2+h)^2+1} - \frac{3}{2^2+1}}{h} = \lim_{h \to 0} \frac{\frac{15 - 3\left((2+h)^2+1\right)}{\left((2+h)^2+1\right)\cdot 5}}{h} =$$

$$= \lim_{h \to 0} \frac{15-3-3(4+4h+h^2)}{h\cdot 5\cdot\left((2+h)^2+1\right)} = \lim_{h \to 0} \frac{15-3-12-12h-3h^2}{h\cdot 5\left((2+h)^2+1\right)} =$$

$$= \lim_{h \to 0} \frac{(-12-3h)\cancel{h}}{\cancel{h}\cdot 5\cdot\left((2+h)^2+1\right)} = \frac{-12}{25} \quad (\approx -\tfrac{1}{2}, \text{ but not quite})$$

2) tangent line has equation $t(x) = -\frac{12}{25}x + b$

→ need to find $b$.

know: $t(2) = f(2) = \frac{3}{5}$

→ ~~$-\frac{12}{25}x + b = \frac{3}{5}$~~

→ $-\frac{12}{25}\cdot 2 + b = \frac{3}{5}$

$$b = \frac{3}{5} + \frac{24}{25} = \frac{15+24}{25} = \frac{39}{25}$$

$$t(x) = -\frac{12}{25}x + \frac{39}{25}$$