MA 16010 Lesson 7: Basic rules of differentiation

Recall: The derivative of $y = f(x)$ at x is defined via limits as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(other notation for derivatives: $\frac{df}{dx}$, y', $\frac{dy}{dx}$, ...)

Today we look at practical rules for computing derivatives.

0. Constant rule: If $f(x) = c$ is a constant function, then $f'(x) = $ $\underline{0}$.

Justification:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = \underline{\underline{0}}$$

1. Power rule: We have $(x^n)' = n \cdot x^{n-1}$

Examples:

- $f(x) = x^0 = \underline{1}$: By the constant rule, $\frac{d}{dx}(1) = 0$,

  By the power rule, it should be $0 \cdot x^{-1} = 0$ ✓
- $f(x) = x^1 = x$: we expect $\frac{d}{dx}(x) = 1 \cdot x^0 = 1$

  we have $\frac{d}{dx}(x) = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$ ✓
- $f(x) = x^2$: we expect : $(x^2)' = 2x^1 = 2x$

  we have $(x^2)' = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} (2x+h) = \underline{\underline{2x}}$ ✓

Note: The rule works not only for n from non–negative integers, but for **all** exponents, including negative, rational, irrational, ... numbers.

Examples:

- $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$,
- $(x^{\frac{15}{4}})' = \frac{15}{4} x^{\frac{11}{4}}$ • $(x^{\pi})' = \pi \cdot x^{\pi - 1}$, ... ~~etc~~
- $(x^{-2})' = (-2) \cdot x^{-3}$, etc.

2. Trig and exponential functions: We have

$$(\sin(x))' = \underline{\cos(x)},$$

$$(\cos(x))' = \underline{-\sin(x)},$$

$$(e^x)' = \underline{e^x}.$$

This is true only for $y=e^x$, not for $y=a^x$ where $a \neq e$.
This is what makes the natural exponential „special" (or „natural!")

3. Sum, difference, constant multiple rules: If $f(x)$, $g(x)$ are functions and c is a constant (i.e. a number), we have:

$$(f(x)+g(x))' = \underline{f'(x)+g'(x)} \quad \text{(Sum rule)},$$

$$(f(x)-g(x))' = \underline{f'(x)-g'(x)} \quad \text{(Difference rule)},$$

$$(c \cdot f(x))' = \underline{c \cdot f'(x)} \quad \text{(Constant multiple rule)}.$$

careful: c has to be constant!

Exercise: Find $f'(x)$ when

1. $f(x) = 2x^5 - 3x^2 + 7$: $f'(x) = \frac{d}{dx}\left(2x^5 - 3x^2 + 7\right) = \frac{d}{dx}\left(2x^5\right) - \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}(7)$

$$= 2 \cdot \frac{d}{dx}\left(x^5\right) - 3 \cdot \frac{d}{dx}\left(x^2\right) + 0 = 2 \cdot \left(5 \cdot x^4\right) - 3 \cdot (2 \cdot x) + 0$$

$$= \underline{10x^4 - 6x}$$

2. $f(x) = \frac{\sqrt[3]{x^2} - 4x^{-1/5}}{\sqrt{x}} + 2\sin(x)$: we simplify the fraction first, to get powers of x:

$$f'(x) = \frac{d}{dx}\left(\frac{x^{\frac{2}{3}} - 4x^{-1/5}}{x^{\frac{1}{2}}} + 2\sin x\right) = \frac{d}{dx}\left(x^{\frac{2}{3}-\frac{1}{2}} - 4x^{-\frac{1}{5}-\frac{1}{2}} + 2\sin x\right) =$$

$$= \frac{d}{dx}\left(x^{\frac{1}{6}} - 4x^{-\frac{7}{10}} + 2\sin x\right)_2 = \frac{1}{6}x^{-\frac{5}{6}} - 4 \cdot \left(-\frac{7}{10}\right) \cdot x^{-\frac{17}{10}} + 2\cos x$$

$$= \underline{\frac{1}{6}x^{-5/6} + \frac{14}{5}x^{-\frac{17}{10}} + 2\cos x}$$

$\left.\frac{dy}{dx}\right|_{x=2}$ means derivative at $x=2$, so „$f'(2)$".

Exercise: Compute $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2}$ when $y = \frac{2}{x^3} + 7e^x + 10e^2$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^{-3} + 7e^x + 10e^2\right) = -6x^{-4} + 7e^x + 0$$

$$= -\frac{6}{x^4} + 7e^x \quad (\text{Careful: } 10e^2 \text{ is just a constant!})$$

$$\left.\frac{dy}{dx}\right|_{x=2} = -\frac{6}{2^4} + 7e^2 = -\frac{3}{8} + 7e^2$$

Exercise: Find the equation of the tangent line to the graph of $f(x) = 4 + 2\cos(x)$ at $x = \pi/3$:

We look for $t(x) = ax+b$, where

$a = f'(\frac{\pi}{3})$: $f'(x) = -2\sin(x)$

$\underline{a} = f'(\frac{\pi}{3}) = -2 \cdot \frac{\sqrt{3}}{2} = \underline{-\sqrt{3}}$

and b is such that

$t(\frac{\pi}{3}) = f(\frac{\pi}{3})$

$-\sqrt{3} \cdot \frac{\pi}{3} + b = 4 + 2\cos(\frac{\pi}{3}) = 4 + 2 \cdot \frac{1}{2} = 5$

$\rightarrow b = 5 + \frac{\sqrt{3}}{3}\pi$

$\rightarrow t(x) = -\sqrt{3}\,x + \frac{\sqrt{3}}{3}\pi + 5$