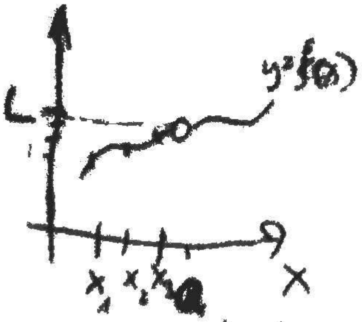
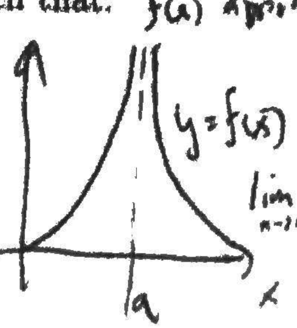
# MA 16020 Lesson 15: Improper integrals

**Recall (limits)**: The limit of the function $f(x)$ as $x$ approaches $a$, $\lim_{x\to a} f(x)$, is a value $L$ such that: $f(x)$ approaches $L$ as $x$ approaches $a$



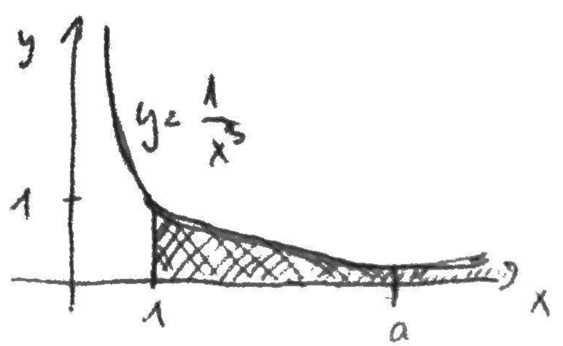

Recall: $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

$\lim_{x\to a} f(x) g(x) = (\lim f(x))(\lim g(x))$

$\lim_{x\to a} f(x)/g(x) = \lim f(x) / \lim g(x)$, assuming the RHS makes sense.

An *improper integral* is a definite integral $\int_a^b f(x)\mathrm{d}x$ such that the integrand $f(x)$ is defined on $(a,b)$, but not necessarily at $a$ or $b$.

**Example.** Evaluate the integral




$$\int_1^\infty \frac{\mathrm{d}x}{x^3}.$$

Area of the „truncated region" $= \int_1^a \frac{dx}{x^3} =$

$= \int_1^a x^{-3} dx = \left[-\frac{1}{2} x^{-2}\right]_1^a = -\frac{1}{2} a^{-2} + \frac{1}{2} \cdot 1^{-2} =$

$= \frac{1}{2} - \frac{1}{2a^2}$

$$\int_1^\infty \frac{dx}{x^3} = \lim_{a\to\infty} \int_1^a \frac{dx}{x^3} = \lim_{a\to\infty} \left(\frac{1}{2} - \underbrace{\frac{1}{2a^2}}_{\to 0}\right) = \frac{1}{2}$$

**Key idea:** The integral $\int_a^b f(x)\mathrm{d}x$ can be computed as

$\lim_{c\to b_-} \int_a^c f(x) dx$ and/or $\lim_{c\to a_+} \int_c^b f(x) dx$



( ! )
a c b

**Exercise 1.** Evaluate the integral

$$\int_8^\infty \frac{5\mathrm{d}x}{x(\ln(x))^3}. = \lim_{a\to\infty} \int_8^a \frac{5\,dx}{x(\ln(x))^3}$$

$$\int_8^a \frac{5\,dx}{x(\ln x)^3} = \left| \begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \\ x = 8 \leadsto u = \ln 8 \\ x = a \leadsto u = \ln a \end{array} \right| = \int_{\ln(8)}^{\ln(a)} \frac{5}{u^3}\,du = \left[-\frac{5}{2}u^{-2}\right]_{\ln(8)}^{\ln(a)} =$$

$$= -\frac{5}{2(\ln(a))^2} + \frac{5}{2(\ln(8))^2}$$

$$\Rightarrow \int_8^\infty \frac{5\,dx}{x(\ln x)^3} = \lim_{a\to\infty}\Big(\underbrace{-\frac{5}{2(\ln(a))^2}}_{\to 0} + \frac{5}{2\ln(8)^2}\Big) = \underline{\underline{\frac{5}{2\ln(8)^2}}} \approx 0.578$$

**Exercise 2.** Evaluate the integral

$$\int_4^\infty \frac{\mathrm{d}x}{\sqrt{x-3}} = \lim_{a\to\infty} \int_4^a \frac{dx}{\sqrt{x-3}},$$

$$\int_4^a \frac{dx}{\sqrt{x-3}} = \left| \begin{array}{l} u = x-3 \\ du = dx \\ x = 4 \leadsto u = 1 \\ x = a \leadsto u = a-3 \end{array} \right| = \int_1^{a-3} \frac{du}{\sqrt{u}} = \int_1^{a-3} u^{-\frac{1}{2}}\,du = \left[2u^{\frac{1}{2}}\right]_1^{a-3} =$$

$$= \underline{2\sqrt{a-3} - 2}, \quad \text{so}$$

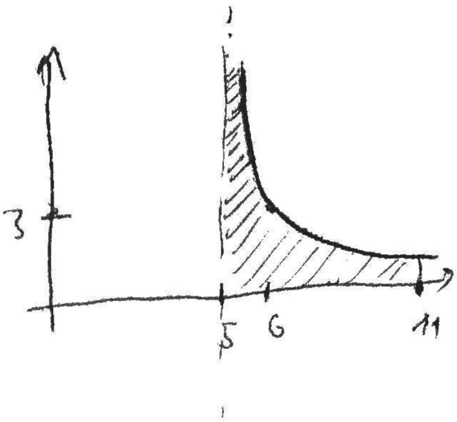
$$\int_4^\infty \frac{dx}{\sqrt{x-3}} = \lim_{a\to\infty}\big(\underbrace{2\sqrt{a-3}}_{\to\infty} - 2\big) = \infty$$

$$\to \underline{\underline{\int_4^\infty \frac{dx}{\sqrt{x-3}} = \infty}}$$

(„the integral diverges")

**Exercise 3.** Evaluate the integral

$$\int_5^{11} \frac{3\mathrm{d}x}{\sqrt[3]{x-5}}.$$




$$\int_5^{11}\frac{3dx}{\sqrt[3]{x-5}} = \lim_{a\to 5_+}\int_a^{11}\frac{3dx}{\sqrt[3]{x-5}}.$$

$$\int_a^{11}\frac{3dx}{\sqrt[3]{x-5}} = \left| \begin{array}{l} u=x-5 \\ du=dx \\ x=a \rightsquigarrow u=a-5 \\ x=11 \rightsquigarrow u=11-5=6 \end{array}\right| = \int_{a-5}^{6} \frac{3du}{\sqrt[3]{u}} = \int_{a-5}^{6} 3u^{-\frac{1}{3}}du = \left[\frac{9}{2} u^{\frac{2}{3}}\right]_{a-5}^{6} =$$

$$= \frac{9}{2}\cdot 6^{\frac{2}{3}} - \frac{9}{2}(a-5)^{\frac{2}{3}}$$

$$\to \int_5^{11}\frac{3dx}{\sqrt[3]{x-5}} = \lim_{a\to 5_+}\Big(\frac{9}{2}\cdot 6^{\frac{2}{3}} - \underbrace{\frac{9}{2}(a-5)^{\frac{2}{3}}}_{\to 0 \text{ as } a\to 5}\Big) = \frac{9}{2}\cdot 6^{\frac{2}{3}} \approx 14.859$$

**Exercise 4.** Evaluate the integral

$$\int_4^{\infty} \frac{3e^{-\sqrt{x}}}{2\sqrt{x}}\mathrm{d}x = \lim_{a\to\infty}\int_4^{a} \frac{3e^{-\sqrt{x}}}{2\sqrt{x}}dx$$

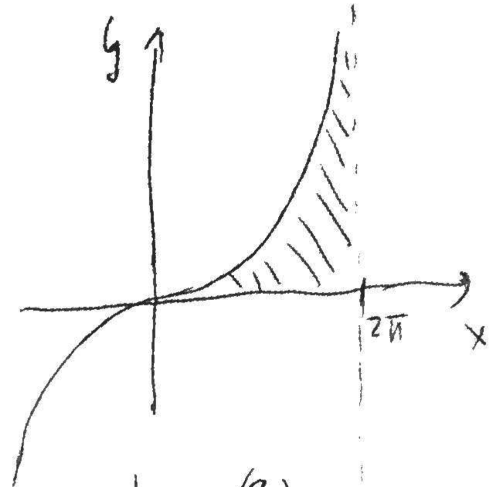
$$\int_4^{a} \frac{3e^{-\sqrt{x}}}{2\sqrt{x}}dx = \left| \begin{array}{l} u=-\sqrt{x} \\ du=-\frac{1}{2}x^{-\frac{1}{2}}dx=-\frac{1}{2\sqrt{x}}dx \\ x=a \rightsquigarrow u=-\sqrt{a} \\ x=4 \rightsquigarrow u=-\sqrt{4}=-2 \end{array}\right| = \int_{-2}^{-\sqrt{a}} -3\cdot e^{u}\,du =$$

$$= \left[-3e^{u}\right]_{-2}^{-\sqrt{a}} = -3e^{-\sqrt{a}} + 3e^{-2}$$

$$\rightsquigarrow \int_4^{\infty} \frac{3e^{-\sqrt{x}}}{2\sqrt{x}}dx = \lim_{a\to\infty}\left(-3e^{-\sqrt{a}} + 3e^{-2}\right) = \lim_{a\to\infty}\Big(-\underbrace{\frac{3}{e^{\sqrt{a}}}}_{\to 0} + 3e^{-2}\Big) =$$

$$= 3e^{-2} \approx 0.406$$

**Exercise 5.** Evaluate the integral




$$\int_0^{2\pi} \tan\left(\frac{\theta}{4}\right) d\theta\,.$$

$$\int_0^{2\pi} \tan\left(\frac{\theta}{4}\right) d\theta = \lim_{a\to 2\pi_-} \int_0^{a} \tan\left(\frac{\theta}{4}\right) d\theta$$

$$\int_0^{a} \tan\left(\frac{\theta}{4}\right) d\theta = \int_0^{a} \frac{\sin(\theta/4)}{\cos(\theta/4)}\, d\theta = \left| \begin{array}{l} u = \cos\left(\frac{\theta}{4}\right) \\ du = -\sin\left(\frac{\theta}{4}\right)\cdot \frac{1}{4}\cdot d\theta \\ \theta = 0 \leadsto u = \cos(0) = 1 \\ \theta = a \leadsto u = \cos\left(\frac{a}{4}\right) \end{array} \right| = \int_1^{\cos\left(\frac{a}{4}\right)} -\frac{4}{u}\, du =$$

$$= \Big[ -4\ln|u| \Big]_1^{\cos\left(\frac{a}{4}\right)} = -4\ln\left|\cos\left(\frac{a}{4}\right)\right| + \underbrace{4\ln(1)}_{=0} = -4\ln\left|\cos\left(\frac{a}{4}\right)\right|$$

$$\leadsto \int_0^{2\pi} \tan\left(\frac{\theta}{4}\right) d\theta = \lim_{a\to 2\pi_-} -4\ln\left|\cos\left(\frac{a}{4}\right)\right| = \underline{\underline{\infty}} \quad \left[ \begin{array}{l} \cos\left(\frac{a}{4}\right) \to \cos\left(\frac{2\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0, \\ \ln\left|\cos\left(\frac{a}{4}\right)\right| \to \text{"}\ln(0)\text{"} = -\infty \\ -4\ln\left|\cos\left(\frac{a}{4}\right)\right| = \text{"}-4\cdot\ln(0)\text{"} = \infty \end{array} \right] \text{ diverges}$$

**Exercise 6.** Evaluate the integral

$$\int_2^{\infty} \frac{dx}{x\ln(2x^3)}\,. = \lim_{a\to\infty} \int_2^{a} \frac{dx}{x\ln(2x^3)}$$

$$\int_2^{a} \frac{dx}{x\ln(2x^3)} = \left| \begin{array}{l} u = \ln(2x^3) \\ du = \frac{1}{2x^3}\cdot 6x^2\cdot dx = \frac{3}{x}\,dx \\ x = a \leadsto u = \ln(2a^3) \\ x = 2 \leadsto u = \ln(16) \end{array} \right| = \int_{\ln(16)}^{\ln(2a^3)} \frac{1}{3}\cdot\frac{1}{u}\, du =$$

$$= \left[ \frac{1}{3}\ln|u| \right]_{\ln(16)}^{\ln(2a^3)} = \frac{1}{3}\ln\left|\ln(2a^3)\right| - \frac{1}{3}\ln(\ln(16))$$

$$\int_2^{\infty} \frac{dx}{x\ln(2x^3)} = \lim_{a\to\infty} \underbrace{\frac{1}{3}\ln\left|\ln(2a^3)\right|}_{} - \frac{1}{3}\ln(\ln(16)) = \infty, \text{ integral diverges}$$

$\ln(2a^3) \to \infty$ with $a\to\infty$,
$\ln(\ln(2a^3)) \to \infty$ with $\ln(2a^3)\to\infty$