MATH 16020 Lesson 1B: Integration by Substitution II

Spring 2021

Warm-up. Evaluate $\int \frac{1}{\sqrt{3x+4}}dx$ via the appropriate substitution.

Let $u = 3x+4 \Rightarrow du = 3dx \Rightarrow dx = \frac{du}{3}$. Thus:

$$\int \frac{1}{\sqrt{3x+4}}dx = \int \frac{1}{\sqrt{u}}\frac{du}{3} = \int \frac{1}{3}u^{-1/2}du = \frac{2}{3}\sqrt{u} + C = \boxed{\frac{2}{3}\sqrt{3x+4} + C}$$

Example 1. Evaluate $\int_4^{15} \frac{1}{\sqrt{3x+4}}dx$

Definite integral of same function. Can apply FTC from above antiderivative, (i.e., $\int_4^{15} \frac{1}{\sqrt{3x+4}}dx = \left[\frac{2}{3}\sqrt{3x+4}\right]_4^{15} = \cdots$), but let's try another trick: Change bounds with the u-sub.:

$x = 4 \Rightarrow u = 3(4)+4 = 16$

$x = 15 \Rightarrow u = 3(15)+4 = 49$

$$\Rightarrow \int_4^{15} \frac{1}{\sqrt{3x+4}}dx = \int_{16}^{49} \frac{1}{\sqrt{u}}du = \left[\frac{2}{3}\sqrt{u}\right]_{u=16}^{u=49}$$

$$= \frac{2}{3}\sqrt{49} - \frac{2}{3}\sqrt{16}$$

$$= \frac{2}{3}(7) - \frac{2}{3}(4) = 2$$

Example 2. Suppose a strain of bacteria initially has 20 bacteria present and the number of bacteria $N(t)$ at time t (in seconds) has a rate that is modeled by:

$$N'(t) = \frac{t}{\sqrt{3t+4}}$$

How many bacteria are present 3 seconds later? Round to the nearest number of bacteria.

$N'(t) = \frac{t}{\sqrt{3t+4}}$ ← Given

$N(0) = 20$

Find: $N(3)$.

Can use an IVP, but let's use FTC:

$$N(3) - N(0) = \int_0^3 N'(t)\,dt \Rightarrow \boxed{N(3) = N(0) + \boxed{\int_0^3 \frac{t}{\sqrt{3t+4}}\,dt}}$$

$$\int_0^3 \frac{t}{\sqrt{3t+4}}\,dt = \int_4^{13} \frac{t}{\sqrt{u}} \frac{du}{3} = \int_4^{13} \frac{u-4}{9\sqrt{u}}\,du$$

$$= \frac{1}{9}\int_4^{13} u^{1/2} - 4u^{-1/2}\,du$$

$$= \frac{1}{9}\left[\frac{2}{3}u^{3/2} - 8u^{1/2}\right]_4^{13}$$

$$= \frac{1}{9}\left(\frac{2}{3}(13^{3/2}) - 8(13^{1/2})\right) - \frac{1}{9}\left(\frac{2}{3}(4^{3/2}) - 8(4^{1/2})\right)$$

$$\approx 1.45 \approx 1$$

$u = 3t+4 \longleftrightarrow \frac{u-4}{3}$

$du = 3dt$

$\Rightarrow dt = \frac{du}{3}$

$t = 3 \Rightarrow u = 3(3)+4 = 13$

$t = 0 \Rightarrow u = 3(0)+4 = 4$

$$\Rightarrow N(3) = 20 + 1 = 21$$

Example 3. If the area of the region under the curve

$$y = (10x+5)(x^2+x)^4$$

and bounded by $x = 0$, $y = 0$, and $x = a$ is 32, and $a > 0$, what is a?

Area

$$32 = \int_0^a (10x+5)(x^2+x)^4 dx = \int_0^{a^2+a} \cancel{(10x+5)}^{5} u^4 \frac{du}{\cancel{2x+1}} = \int_0^{a^2+a} 5u^4 du$$

Solve for a

$u = x^2+x$
$du = 2x+1 dx$
$\Rightarrow dx = \frac{du}{2x+1}$
$\Rightarrow \begin{cases} x=0 \Rightarrow u=0 \\ x=a \Rightarrow u=a^2+a \end{cases}$

$$= [u^5]_0^{a^2+a}$$
$$= (a^2+a)^5 = 32$$
$$\Rightarrow a^2+a = 2$$
$$\Rightarrow a^2+a-2=0$$
$$\Rightarrow (a+2)(a-1)=0$$
$$\Rightarrow a=1, \cancel{-2} \Rightarrow a=1$$

Example 4. The velocity $v(t)$ of a particle moving along the t-axis is given by:

$$v(t) = -3t\sin(t^2)$$

If the particle starts at 11, find the position $s(t)$ at time t.

Recall; $v(t) = s'(t)$, first find $s(t) = \int v(t)dt = \int -3t\sin(t^2)dt$

$u = t^2$
$du = 2t\,dt$
$\Rightarrow dt = \frac{du}{2t}$

$$= \int -3\cancel{t}\sin(u)\frac{du}{2\cancel{t}}$$
$$= \int -\frac{3}{2}\sin(u)du = \frac{3}{2}\cos(u) + C = \boxed{\frac{3}{2}\cos(t^2) + C}$$

$$\Rightarrow s(0) = 11 \Rightarrow s(0) = \frac{3}{2}\cos(0^2) + C = \frac{3}{2} + C = 11 \Rightarrow C = \frac{19}{2}$$

$$\Rightarrow \boxed{s(t) = \frac{3}{2}\cos(t^2) + \frac{19}{2}}$$