MATH 16020 Lesson 2: Integration by Substitution III

Spring 2021

Example 1. Suppose the height of an alien plant increases at the rate:

$$H'(t) = \frac{1}{\sqrt{t}\sqrt[3]{(1+\sqrt{t})}} \text{ cm/hour}$$

for t in hours since 6:00 AM. How tall does the plant grow from 7:00AM to 3:00PM? Round answer to 3 decimal places. ← Change in height

$\Updownarrow$ $t=1$, $\Updownarrow$ $t=9$

$$\Rightarrow H(9)-H(1) = \int_1^9 \frac{1}{\sqrt{t}\sqrt[3]{1+\sqrt{t}}}\,dt = \int_2^4 \frac{1}{\sqrt{t}\sqrt[3]{u}}\,2\sqrt{t}\,dt = \int_2^4 \frac{2}{\sqrt[3]{u}}\,du = \left[\frac{3}{2}\cdot 2u^{2/3}\right]_2^4$$

$$\boxed{\begin{array}{l} u = 1+\sqrt{t} \Rightarrow du = \frac{dt}{2\sqrt{t}} \Rightarrow dt = 2\sqrt{t}\,dt \\ t=1 \Rightarrow u = 1+\sqrt{1} = 2 \\ t=9 \Rightarrow u = 1+\sqrt{9} = 4 \end{array}}$$

$$= \left[3u^{2/3}\right]_2^4$$

$$= 3(4^{2/3}) - 3(2^{2/3})$$

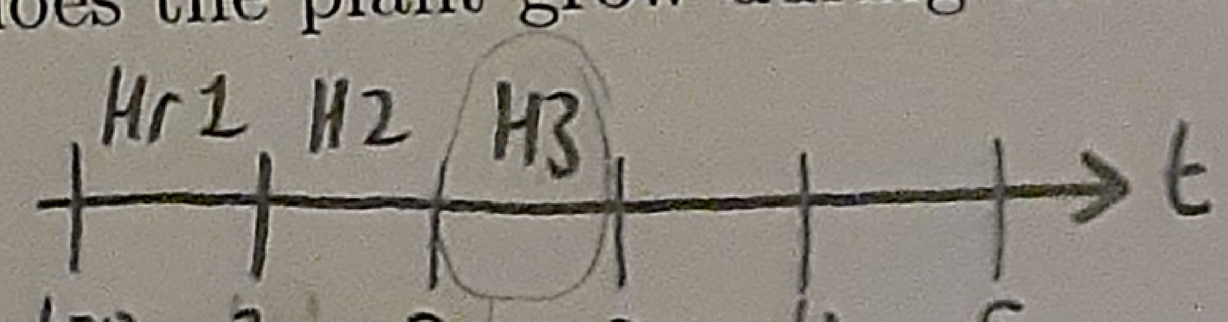
$$\approx \boxed{2.797 \text{ cm}}$$

Example 2. Suppose now this plant grows at the rate:

$$H'(t) = \frac{1}{\sqrt{t}\sqrt[3]{(1+\sqrt{t})}} \text{ cm/hour}$$

← Same function inside integral.

t hours after it was planted. How tall does the plant grow during the third hour? Round answer to 3 decimal places.



$$\Rightarrow \text{Find } \int_2^3 \frac{1}{\sqrt{t}\sqrt[3]{1+\sqrt{t}}}\,dt = \int_{1+\sqrt{2}}^{1+\sqrt{3}} 2u^{-1/3}\,du = \left[3u^{2/3}\right]_{1+\sqrt{2}}^{1+\sqrt{3}}$$

Only change from Ex. 1 is bounds,

$$\boxed{\begin{array}{l} t=2 \Rightarrow u = 1+\sqrt{2} \\ t=3 \Rightarrow u = 1+\sqrt{3} \end{array}}$$

$$= 3(1+\sqrt{3})^{2/3} - 3(1+\sqrt{2})^{2/3}$$

$$\approx \boxed{0.464 \text{ cm}}$$

Example 3. Suppose as a particle slows down, its velocity is:

$$v(t) = 2e^{1-t} - 1 \text{ cm/s}$$

If the particle starts slowing down at time $t = 0$ seconds, find the distance it takes for the particle to stop.

Assume $s(0)=0$. ← L-C wants you to

First find when particle stops by solving $v(t)=0$

$\Rightarrow 2e^{1-t}-1=0 \Rightarrow e^{1-t}=\frac{1}{2} \Rightarrow 1-t=\ln(\frac{1}{2})$

$\Rightarrow \boxed{t=1-\ln(\frac{1}{2})}$

Given $s'(t)=2e^{1-t}-1$, Find $s(1-\ln(\frac{1}{2}))$

$s(0)=0$

$\Rightarrow s(1-\ln(\frac{1}{2}))-s(0)=\int_0^{1-\ln(1/2)} 2e^{1-t}-1\,dt = \int_1^{\ln(1/2)} 1-2e^u\,du = \left[u-2e^u\right]_1^{\ln(1/2)}$

$u=1-t$
$du=-dt \Rightarrow dt=-du$
$\Rightarrow t=1-\ln(\frac{1}{2}) \Rightarrow u=\ln(\frac{1}{2})$
$\Rightarrow t=0 \Rightarrow u=1$

$=(\ln(\frac{1}{2})-2e^{\ln(1/2)})-(1-2e^1)$

$=\ln(\frac{1}{2})-2(\frac{1}{2})-1+2e^1$

≈ 2.743

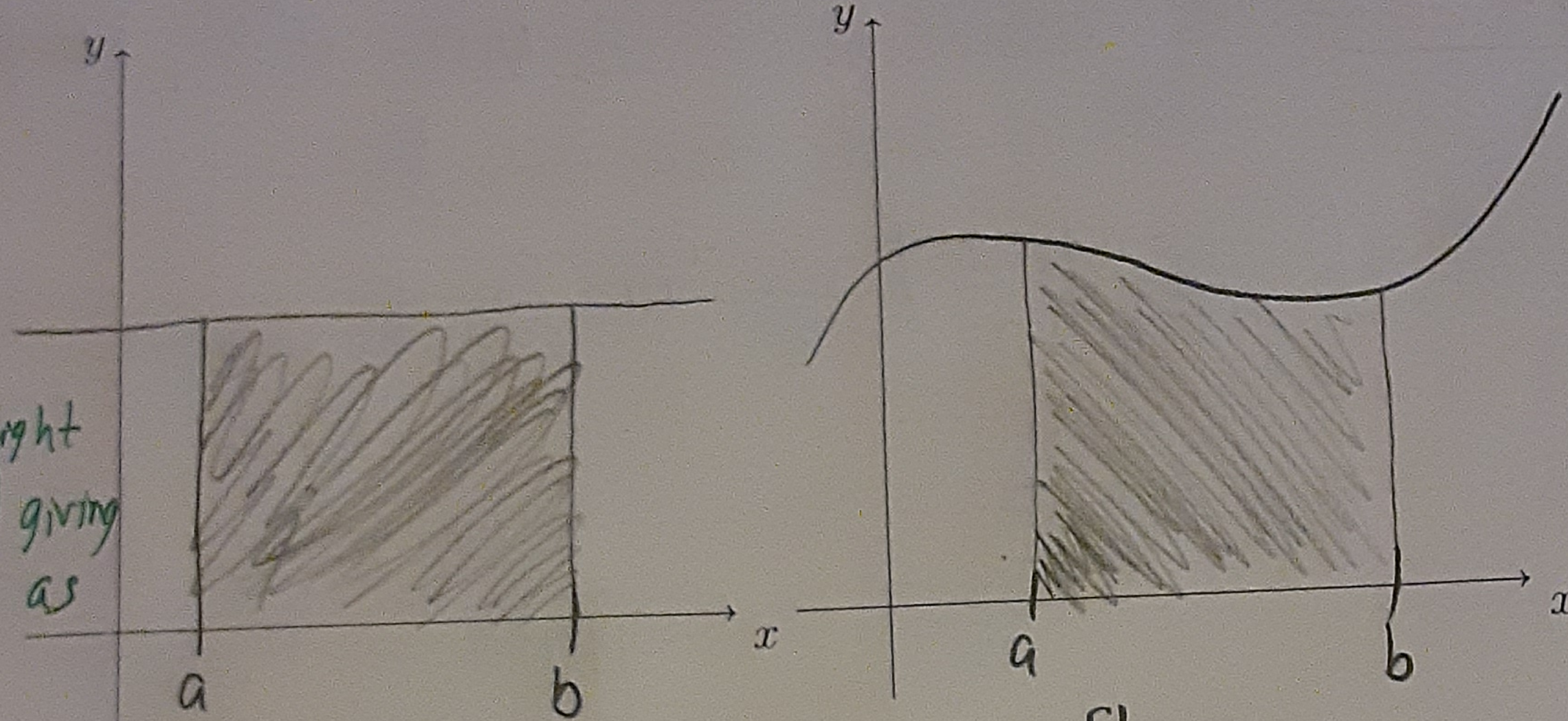
$\Rightarrow s(1-\ln(\frac{1}{2}))-\cancel{s(0)}^{0}=\boxed{s(1-\ln(\frac{1}{2}))\approx 2.743 \text{ cm}}$

Definition. For $f(x)$ defined on $[a, b]$, the **average value** of $f(x)$ on $[a, b]$ is:

$$f_{AVG} = \frac{\boxed{\int_a^b f(x)\,dx}}{b-a} \leftarrow \text{Area under } f(x) \text{ over } [a,b].$$

f_{AVG}

Constant height over $[a,b]$ giving same area as $\int_a^b f(x)\,dx$



$f_{AVG}(b-a)$ $\qquad$ $\int_a^b f(x)\,dx$

SAME

Example 4. Find the average value of $f(x) = 6x^2 + 2$ over $[1,3]$.

$$f_{AVG} = \frac{\int_1^3 6x^2+2\,dx}{3-1} = \frac{1}{2}\int_1^3 6x^2+2\,dx = \int_1^3 3x^2+1\,dx$$

$$= \left[x^3+x\right]_1^3$$

$$= (3^3+3)-(1^3+1) = 28$$

Example 5. Suppose another alien plant is shrinking at the rate of:

$$H'(t) = -100e^{-5t} \text{ cm/min}, \quad H(0) = 300$$

If the plant has an initial recorded height of 300 cm, find the average height of the plant 4 minutes after this initial recording. Round answer to 3 decimal places.

CAREFUL!! We need avg. height $H(t)$, NOT $H'(t)$, so find $H(t)$ first.
(when in doubt, check units!)

$$H(t) = \int H'(t)\,dt = \int -100e^{-5t}\,dt = \int \frac{-100}{-5} e^{u}\,du = \int 20e^{u}\,du = 20e^{u} + C$$

$u = -5t$
$du = -5dt$
$\Rightarrow dt = \frac{du}{-5}$

$$= 20e^{-5t} + C = H(t)$$

$$\Rightarrow H(0) = 20 + C = 300 \Rightarrow C = 280$$

$$\Rightarrow \boxed{H(t) = 20e^{-5t} + 280}$$

Now find avg. height over 1^{st} 4 min.

$$H_{AVG} = \frac{\int_0^4 20e^{-5t} + 280\,dt}{4-0} = \int_0^4 5e^{-5t} + 70\,dt = \left[-e^{-5t} + 70t\right]_0^4$$

$$= \left[-e^{-20} + 280\right] - \left[-e^{0} + 0\right]$$

$$= -e^{-20} + 281$$

$$\approx \boxed{281.000 \text{ cm}}$$

e^{-20} really small!!