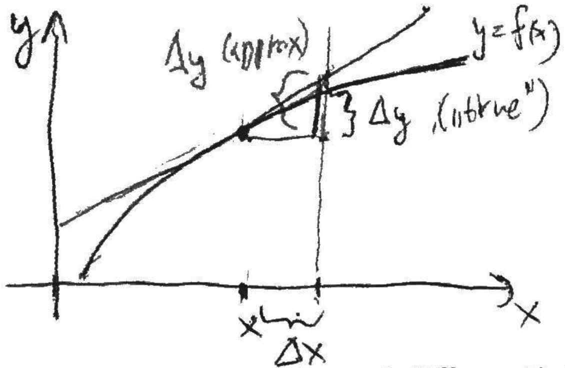
MA 16020 Lesson 21: Differentials of multivariate functions

Recall (differentials of functions of one variable): Given a function $y = f(x)$, small changes in $y = f(x)$ due to small changes of x can be approximated using the derivative:



$$\Delta y \approx (\text{slope of the tangent}) \cdot \Delta x = f'(x) \cdot \Delta x$$

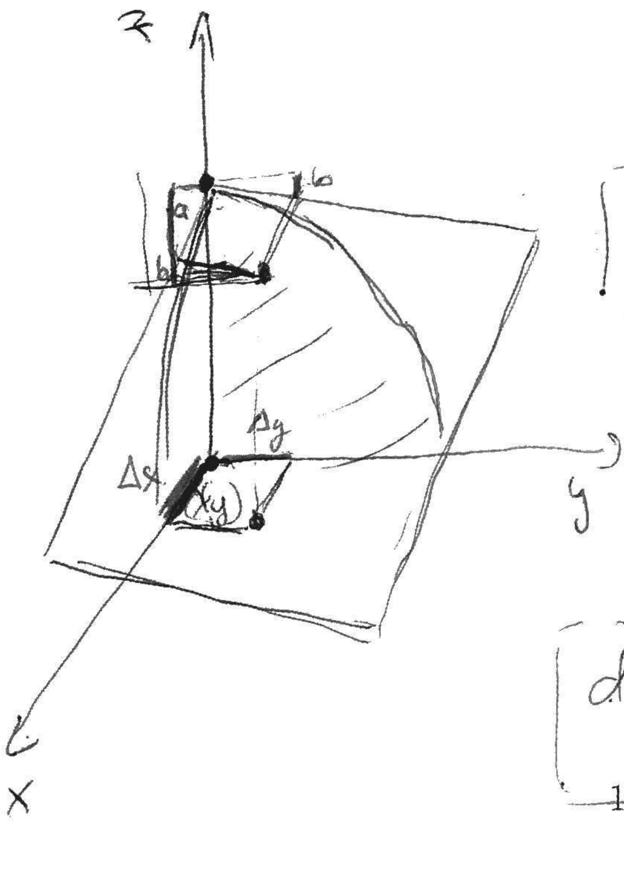
$$\boxed{\Delta y \approx f'(x) \cdot \Delta x}$$

This becomes an equality of *differentials*:

$$dy = f'(x) \cdot dx,$$

dy, dx .. differentials („infinitesimally small changes of x, y, resp.")

If $z = f(x, y)$ is a function of two variables, we can consider small changes $\Delta x, \Delta y$ in x *and* in y, and express Δz using the partial derivatives:



$$\Delta z \approx (x\text{-slope}) \cdot \Delta x + (y\text{-slope}) \cdot \Delta y$$

$$\boxed{\Delta z \approx \frac{\partial f}{\partial x}(x,y) \cdot \Delta x + \frac{\partial f}{\partial y}(x,y) \cdot \Delta y}$$

incremental approximation formula.

$$dz = \frac{\partial f}{\partial x}(x,y) \cdot dx + \frac{\partial f}{\partial y}(x,y) \cdot dy$$

equality of „differentials"

$$\boxed{dz \text{ ... total differential of } z = f(x,y)}$$

Exercise 1. Using differentials, estimate

$$\cos(\pi/4 + 0.02)\sin(\pi/4 - 0.03).$$

Want: $f\left(\frac{\pi}{4}+0.02, \frac{\pi}{4}-0.03\right)$

Know: $f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)\cdot\sin\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$

$f(x,y) = \cos(x)\cdot\sin(y)$

$\Rightarrow f\left(\frac{\pi}{4}+0.02, \frac{\pi}{4}-0.03\right) = f\left(\frac{\pi}{4},\frac{\pi}{4}\right) + \Delta z$, when $\Delta x = 0.02$, $\Delta y = -0.03$

$\frac{\partial f}{\partial x} = (-\sin(x))\cdot\sin y = -\sin x \sin y$

$\frac{\partial f}{\partial y} = \cos(x)\cdot\cos(y)$

$\underline{\Delta z} \approx \frac{\partial f}{\partial x}\left(\frac{\pi}{4},\frac{\pi}{4}\right)\cdot\Delta x + \frac{\partial f}{\partial y}\left(\frac{\pi}{4},\frac{\pi}{4}\right)\cdot\Delta y$

$= \left(-\frac{1}{2}\right)\cdot 0.02 + \left(\frac{1}{2}\right)\cdot(-0.03)$

$= \underline{-0.025}$

$\cos\left(\frac{\pi}{4}+0.02\right)\sin\left(\frac{\pi}{4}-0.03\right) \approx \frac{1}{2} + \Delta z \approx \frac{1}{2} - 0.025 = \underline{\underline{0.475}}$

Exercise 2. A cylindrical can has base of radius 4 cm and height 12 cm. If the radius is decreased by 0.5 cm, and the height is increased by 1.5 cm, use calculus to estimate the change of the volume of the can.

Volume of the can $= V(r,h) = \pi\cdot r^2\cdot h$

$r = 4$ cm $\quad \Delta r = -0.5$ cm

$h = 12$ cm $\quad \Delta h = 1.5$ cm

Want: estimate for ΔV

$\frac{\partial V}{\partial r} = 2\pi r h, \quad \frac{\partial V}{\partial h} = \pi r^2$

$\Delta V \approx \frac{\partial V}{\partial r}(4,12)\cdot\Delta r + \frac{\partial V}{\partial h}(4,12)\cdot\Delta h =$

$= (2\cdot\pi\cdot 4\cdot 12)\cdot(-0.5) + (\pi\cdot 4^2)\cdot(1.5) = -24\pi \approx -75.398\ \text{cm}^3$

volume will decrease by approx. $75.398\ \text{cm}^3$

(actual decrease $= 83.645\ \text{cm}^3$)

Exercise 3. A police radar gun measured that a car traveled the distance 52 meters in 2.5 seconds. Assuming that the maximum error in measurement of distance is 0.2 meters and the maximum error in measurement of the time is 0.1 seconds, estimate the maximum error in calculating the speed of the car.

Speed $= v = \dfrac{s}{t}$ s ... distance traveled
t ... time

$\rightarrow v(s,t) = \dfrac{s}{t}$,   Measured distance $= 52\,m \pm 0{,}2\,m$
time $= 2{,}5\,s \pm 0{,}1\,s$

$\rightarrow \Delta s = \pm 0{,}2\,m$
$\Delta t = \pm 0{,}1\,s$

Want: maximum error of v
= maximal possible value of Δv

$$\frac{\partial v}{\partial s} = \frac{1}{t}, \qquad \frac{\partial v}{\partial t} = -\frac{s}{t^2}$$

$$\Delta v \approx \frac{\partial v}{\partial s}(52, 2.5)\cdot \Delta s + \frac{\partial v}{\partial t}(52, 2.5)\cdot \Delta t =$$

$$= \frac{1}{2.5}\cdot(\pm 0{,}2) + \left(-\frac{52}{(2{,}5)^2}\right)\cdot(\pm 0{,}1)$$

To get Δv maximal possible, have to choose the signs "$\pm$" so that both summands are positive

$\rightarrow$ in this case we choose $\Delta s = +0.2$, $\Delta t = -0{,}1$

$$\rightsquigarrow \Delta_{max} v = \frac{0{,}2}{2.5} + \frac{5.2}{(2.5)^2} = 0{,}912\ m/s$$

$(\approx 3{,}3\ km/h)$

Exercise 4. The pressure (in Pa) of a certain gas in a container is given by the equation

$$P = 160\frac{T}{V},$$

where T is the temperature of the gas (in $°K$) and V is the volume of the container (in m^3). The volume of the container is measured to be $4m^3$, with the maximum error of measurement $0.2m^3$, and the temperature of the gas is measured to be $310°K$, with the maximum error of measurement $4°K$. Find the relative percentage error in computing the pressure of the gas.

$V = 4 \pm 0.2 \ m^3$ $\quad \Delta V = \pm 0.2$ $\quad P = P(T,V) = 160\frac{T}{V}$

$T = 310 \pm 4 \ °K$ $\quad \Delta T = \pm 4$

1) We compute the maximum absolute error as before:

$$\frac{\partial P}{\partial T} = \frac{160}{V}, \quad \frac{\partial P}{\partial V} = -\frac{160T}{V^2}$$

$$\Delta P = \frac{\partial P}{\partial T}(310,4)\cdot \Delta T + \frac{\partial P}{\partial V}(310,4)\cdot \Delta V =$$

$$= \left(\frac{160}{4}\right)\cdot(\pm 4) + \left(-\frac{160\cdot 310}{4^2}\right)\cdot(\pm 0.2)$$

$$\underline{\Delta_{max} P} = \frac{160}{4}\cdot 4 + \frac{160\cdot 310}{42}\cdot 0.2 = 160 + 620 = \underline{780 \ Pa}$$

2) Relative error $= \dfrac{\text{absolute error}}{\text{the absolute quantity}}$

(Relative percentage error = relative error $\cdot 100\%$)

$$\text{Relative error} = \frac{780}{P(310,4)} = \frac{780}{160\cdot\frac{310}{4}} = \frac{780}{12400} = 0.0629$$

Relative percentage error $\approx 6.29\ \%$