MA 16020 Lesson 24: Extrema of functions of two variables II

Recall (extrema of a function of two variables):
To find local extrema of a function $z = f(x, y)$ of two varables, we need to

1. Find all the *critical points:* Points (x, y) satisfying:

$$\frac{\partial f}{\partial x} = 0 \ \& \ \frac{\partial f}{\partial y} = 0$$

2. Compute all the second-order partial derivatives of f and $D = f_{xx} \cdot f_{yy} - f_{xy}^2$

3. For a given critical point (x_0, y_0), evaluate D and f_{xx} at (x_0, y_0).

   If $D > 0$ & $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum of f.

   If $D(x_0, y_0) > 0$ & $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum of f.

   If $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

   If $D(x_0, y_0) = 0$, then the test is inconclusive for this (x_0, y_0).

Exercise 1. Find all the critical points of the function

$$f(x, y) = x^2y - 2x^2 - 3y^2 + 3y - 7 .$$

$\frac{\partial f}{\partial x} = 2xy - 4x$ $\quad$ $\frac{\partial f}{\partial y} = x^2 - 6y + 3$

$\rightsquigarrow$ $2xy - 4x = 0$ & $x^2 - 6y + 3 = 0$

$xy - 2x = 0$

$x(y-2) = 0$

$x = 0$ or $y = 2$

(A) if $x = 0$, then $0^2 - 6y + 3 = 0$

$-6y + 3 = 0$

$y = \frac{1}{2}$

(B) if $y = 2$, then $x^2 - 12 + 3 = 0$

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

$\rightarrow$ critical points are $(0, \frac{1}{2}), (3, 2), (-3, 2)$

Exercise 2. A shop provides two brands of shoes. The acquiring cost is 5 dollars per pair for the first brand and 4 dollars per pair for the second brand. If the selling prices are x dollars per pair of shoes of the first brand and y dollars per pair of shoes of the second brand, it is expected that the customers will buy approximately and $75 + y - 2x$ pairs of shoes of the first brand and $50 + x - 2y$ pairs of shoes of the second brand. Find the optimal selling prices and maximal profit.

Profit = (# pairs of brand 1 sold) · (price of brand 1) + (# pairs of brand 2 sold) · (price of brand 2) } „revenue"

− (# pairs of brand 1 sold) · 5 − (# pairs of brand 2 sold) · 4 } „costs"

$$\to P(x,y) = (75+y-2x)\cdot x + (50+x-2y)\cdot y - 5(75+y-2x) - 4(50+x-2y)$$

$$= 75x + xy - 2x^2 + 50y + xy - 2y^2 - 375 - 5y + 10x - 200 - 4x + 8y$$

$$= 81x + 53y + 2xy - 2x^2 - 2y^2 - 575$$

$$\frac{\partial P}{\partial x} = 81 + 2y - 4x \qquad \frac{\partial P}{\partial y} = 53 + 2x - 4y$$

$\to$ I: $81 + 2y - 4x = 0$ $\rightsquigarrow$ $y = \frac{4x-81}{2}$

II: $53 + 2x - 4y = 0$

$$53 + 2x - 2\,\frac{4x-81}{2} = 0$$

$$53 + 2x - 2(4x - 81) = 0$$

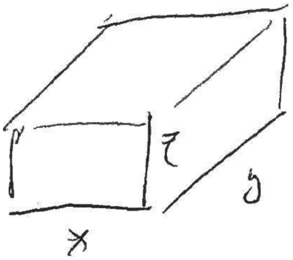
$$53 + 2x - 8x + 162 = 0$$

$$\Rightarrow -6x = -162 - 53 = -215$$

$$x = \frac{215}{6}, \quad y = \frac{4\cdot\frac{215}{6} - 81}{2} = \frac{187}{6}$$

$$\text{Profit} = P\left(\frac{215}{6}, \frac{187}{6}\right) = \frac{10213}{6} \approx 1702.16$$

Exercise 3. A rectangular box of volume 3 m^3 is to be made. The cost of material is: 25 dollars per m^2 for the bottom, 15 dollars per m^2 for the sides, and 20 dollars per m^2 for the top. Find the dimensions of the box so that the cost is minimal, and the cost of the box.



Know: Volume $= xyz = 3$ $\leadsto$ $\boxed{z = \frac{3}{xy}}$

$$C = \text{Cost of the box} = \underbrace{25xy}_{\text{bottom}} + \underbrace{20xy}_{\text{top}} + \underbrace{15(2xz + 2yz)}_{\text{sides}} =$$

$$= 25xy + 20xy + 15\left(2\cdot x\cdot \frac{3}{xy} + 2\cdot y\cdot \frac{3}{xy}\right)$$

$$= 45xy + \frac{90}{y} + \frac{90}{x}$$

$$\frac{\partial C}{\partial x} = 45y - \frac{90}{x^2} \qquad \frac{\partial C}{\partial y} = 45x - \frac{90}{y^2}$$

$\leadsto$ $45y - \frac{90}{x^2} = 0$ & $45x - \frac{90}{y^2} = 0$

$45y = \frac{90}{x^2}$ $/ : 45$

$y = \frac{2}{x^2}$

plug in

$45x - \frac{90}{\left(\frac{2}{x^2}\right)^2} = 0$

$45x - \frac{90x^4}{4} = 0$

$x - \frac{1}{2}x^4 = 0$

$x\left(1 - \frac{1}{2}x^3\right) = 0$

$x = 0$ or $\frac{1}{2}x^3 = 1$

$\boxed{\text{but } y = \frac{2}{x^2} \text{ is not defined!}}$

$x^3 = 2$

$x = \sqrt[3]{2}$

$y = \frac{2}{(\sqrt[3]{2})^2} = \sqrt[3]{2}$

$\to$ ideal dimensions are $x = \sqrt[3]{2}$
$y = \sqrt[3]{2}$
$z = \frac{3}{xy} = \frac{3}{(\sqrt[3]{2})^2}$,

cost $= C(\sqrt[3]{2}, \sqrt[3]{2}) =$
$= 45\cdot(\sqrt[3]{2})^2 + 90/\sqrt[3]{2} + 90/\sqrt[3]{2} \approx$
≈ 214.299 dollars

Exercise 3. If a certain strain of bacteria is fed by x grams of nutrient A, y grams of nutrient B and z grams of nutrient C, it will ultimately produce x^3y^3z grams of a desired chemical. The cost of the nutrients are: 15 dollars per gram for nutrient A, 5 dollars per gram of nutrient B and 2 dollars per gram of nutrient C. What is the minimal cost to produce 500 grams of the desired chemical?

Know: $x^3y^3z = 500 \Rightarrow \boxed{z = \frac{500}{x^3y^3}}$

$$C = \text{Cost} = 15x + 5y + 2\cdot z = 15x + 5y + 2\cdot\frac{500}{x^3y^3} =$$

$$= 15x + 5y + \frac{1000}{x^3y^3}$$

$$\frac{\partial C}{\partial x} = 15 - 3\cdot\frac{1000}{x^4y^3} = 15 - \frac{3000}{x^4y^3}, \quad \frac{\partial C}{\partial y} = 5 - \frac{3000}{x^3y^4}$$

$$\rightsquigarrow 15 - \frac{3000}{x^4y^3} = 0 \quad \& \quad 5 - \frac{3000}{x^3y^4} = 0$$

$15x^4y^3 - 3000 = 0$

$15x^4y^3 = 3000$

$5x^3y^4 - 3000 = 0$

$5x^3y^4 = 3000$

$15x^4y^3 (= 3000 =) = 5x^3y^4 \quad /:5$

$3x^4y^3 = x^3y^4$,

$3x^4y^3 - x^3y^4 = 0$

$x^3y^3(3x - y) = 0$

$\Rightarrow$ either $x = 0$, or $y = 0$, (may disregard these cases)

or $3x - y = 0$

$y = 3x$

Plug this back into

$15x^4y^3 = 3000$:

$15\cdot x^4\cdot(3x)^3 = 3000$

$27\cdot 15\cdot x^7 = 3000$

$x = \sqrt[7]{\frac{3000}{27\cdot 15}} = \sqrt[7]{\frac{200}{27}}$

$y = 3x = 3\sqrt[7]{\frac{200}{27}}$

Minimal cost $= C\left(\sqrt[7]{\frac{200}{27}}, 3\sqrt[7]{\frac{200}{27}}\right) \approx$

≈ 46.5914