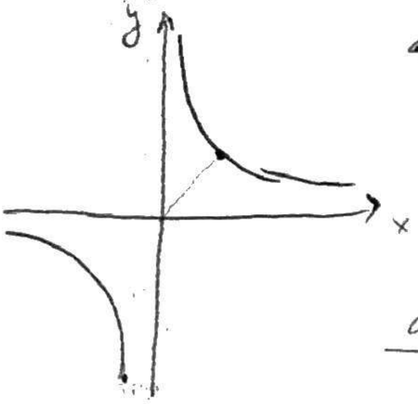
# MA 16020 Lesson 25: Lagrange multipliers I

**Constrained min, max:** Given a function $z = f(x,y)$ of two variables, sometimes it is is desirable to find the mionimum or maximum of $f$ only among certain subset of the $xy$-plane, namely among all the points $(x, y)$ satisfying certain equation (*constraint*)

$$g(x,y) = C$$

**Example:** Find the closest point of the curve $xy = 1$ to the origin.




distance of a point $(x,y)$ from the origin $= \sqrt{x^2+y^2}$

→ want to minimize the function $\sqrt{x^2+y^2}$ subject to the condition $xy = 1$

→ alternatively, we can minimize the square of the distance: minimize the function $\underbrace{x^2+y^2}_{f(x,y)}$ subject to $\underbrace{xy}_{g(x,y)} = 1$.

To solve problems of this sort, we again want to find critical points, for which we use a special version of first derivative test called *method of Lagrange multipliers.*

When trying to minimize/maximize the value of the function $z = f(x, y)$ subject to the constraint $g(x,y) = C$, the critical points are given as points $(x,y)$ that are solutions to the system of the equations:

$$(*) \quad \begin{aligned} f_x &= \lambda \cdot g_x, \\ f_y &= \lambda \cdot g_y \end{aligned} \quad \left(\text{or } (f_x, f_y) = \lambda \cdot (g_x, g_y)\right)$$

$\lambda$ ... new variable, usually called Lagrange multiplier

as well as the original constraint:

$$g(x,y) = C$$

typical strategy (not always): Solve both equations (*) for $\lambda$, then make the RHS's equal, and use $g(x,y) = C$.

**Exercise 1.** Finishing Example from previous page, i.e. find the closest point of the curve $xy = 1$ to the origin.

Minimize the function $f(x,y) = x^2+y^2$ subject to $xy = 1$

Need to solve

$$\boxed{\begin{aligned} 2x &= \lambda \cdot y \\ 2y &= \lambda \cdot x \\ xy &= 1 \end{aligned}}$$

$\rightsquigarrow \lambda = \frac{2x}{y}$ (or $y=0$)

$\rightsquigarrow \lambda = \frac{2y}{x}$ (or $x=0$)

$\frac{2x}{y} = \frac{2y}{x}$

$2x^2 = 2y^2$

$x^2 = y^2$

$x = \pm\sqrt{y^2} = \pm|y|$

$x = \pm y$

Evaluate at crit. pts:

$f(1,1) = f(-1,-1) = 2$

(A) $x = y$

$x^2 = 1$

$x = \pm 1$

$y = \frac{1}{x}$

$\Rightarrow$ points $\boxed{(1,1) \quad (-1,-1)}$

(B) $x = -y$

$\cancel{x}(-y)\cdot y = 1$

$-y^2 = 1$

$y^2 = -1$

no solution

(C) $y = 0$

$x \cdot 0 = 1$

no solution

(D) $x = 0$

$0 \cdot y = 1$

no solution

**Exercise 2.** Find the maximum of the function $f(x, y) = 8x^2 - 2y$ subject to the constraint $x^2 + y^2 = 4$..

$$\boxed{\begin{aligned} 16x &= \lambda \cdot 2x \\ -2 &= \lambda \cdot 2y \\ x^2+y^2 &= 4 \end{aligned}}$$

$\rightsquigarrow \lambda = \frac{16x}{2x} = 8$ (or $x=0$)

$\rightsquigarrow \lambda = \frac{-2}{2y} = -\frac{1}{y}$ (or $y=0$)

$8 = -\frac{1}{y}$

$-8y = 1$

$y = -\frac{1}{8}$

(A) $y = -\frac{1}{8}$

$x^2 + \left(-\frac{1}{8}\right)^2 = 4$

$x^2 + \frac{1}{64} = 4$

$x^2 = 4 - \frac{1}{64} = \frac{255}{64}$

$x = \pm\sqrt{\frac{255}{64}} = \pm\frac{\sqrt{255}}{8}$

$\rightsquigarrow$ points $\left(\pm\frac{\sqrt{255}}{8}, -\frac{1}{8}\right)$

(B) $y = 0$

then $-2 = \lambda \cdot 2 \cdot 0 = 0$

no solution

(C) $x = 0$

$0^2 + y^2 = 4$

$y^2 = 4$

$y = \pm 2$

$\rightarrow$ points $(0, \pm 2)$

Evaluate f at crit. points

$f\left(\pm\frac{\sqrt{255}}{8}, -\frac{1}{8}\right) = 8 \cdot \frac{255}{64} + \frac{2}{8}$

$= \frac{255+2}{8} = \frac{257}{8}$

$f(0,2) = 8\cdot 0 - 2\cdot 2 = -4$

$f(0,-2) = 8 \cdot 0 - 2(-2) = +4$

max $= \frac{257}{8}$ at $\left(\pm\frac{\sqrt{255}}{8}, -\frac{1}{8}\right)$

**Exercise 3.** Find the point(s) $(x, y)$ where the function $f(x, y) = \ln(3xy^2)$ attains maximal value, subject to constraint $7x^2 + y^2 = 21$.

$$\frac{\partial f}{\partial x} = \frac{1}{3xy^2} \cdot 3y^2 = \frac{1}{x} \qquad \frac{\partial g}{\partial x} = 14x$$

$$\frac{\partial f}{\partial y} = \frac{1}{3xy^2} \cdot 6xy = \frac{2}{y} \qquad \frac{\partial g}{\partial y} = 2y$$

$$\leadsto \boxed{\begin{array}{l} \frac{1}{x} = \lambda \cdot 14x \\ \frac{2}{y} = \lambda \cdot 2y \\ 7x^2 + y^2 = 21 \end{array}}$$

$\leadsto$ $\lambda = \frac{1}{14x^2}$ ($x=0$ cannot happen)

$\leadsto$ $\lambda = \frac{2}{2y^2} = \frac{1}{y^2}$ ($y=0$ cannot happen)

$\frac{1}{14x^2} = \frac{1}{y^2}$

$\underline{y^2 = 14x^2}$

$y^2 = 14x^2$, $7x^2 + y^2 = 21$

$\leadsto$ plug in $14x^2$ for $y^2$:

$7x^2 + 14x^2 = 21$

$21x^2 = 21$

$x^2 = 1$

$\underline{x = \pm 1}$

$7 \cdot 1 + y^2 = 21$

$y^2 = 14$

$\underline{y = \pm\sqrt{14}}$

$\leadsto$ critical points

$(1, \sqrt{14})$, $(-1, \sqrt{14})$, $(1, -\sqrt{14})$, $(-1, -\sqrt{14})$

Evaluate f at the critical points:

for $(-1, \pm\sqrt{14})$: $f(-1, \pm\sqrt{14})$ is not defined!

for $(1, \pm\sqrt{14})$:

$f(1, \pm\sqrt{14}) = \ln(3 \cdot 1 \cdot 14) =$

$= \ln(42) \approx 3.738$

$\leadsto$ max attained at the points

$(1, \pm\sqrt{14})$

**Exercise 4.** Find the minimal value of the function $f(x,y) = y^3 e^{x^2}$ subject to the constraint $10x^2 - 3y = 8$.

$$\frac{\partial f}{\partial x} = y^3 \cdot e^{x^2} \cdot 2x \qquad \frac{\partial g}{\partial x} = 20x$$

$$\frac{\partial f}{\partial y} = 3y^2 \cdot e^{x^2} \qquad \frac{\partial g}{\partial y} = -3$$

$\leadsto$ system of equations

$$\boxed{\begin{aligned} y^3 e^{x^2} \cdot 2x &= 20x \cdot \lambda \\ 3y^2 e^{x^2} &= -3\lambda \\ 10x^2 - 3y &= 8 \end{aligned}}$$

$\leadsto$ ($x=0$, or) $\lambda = \frac{1}{10} y^3 \cdot e^{x^2}$

$\leadsto$ $\lambda = -y^2 e^{x^2}$

$\Rightarrow$ $\frac{1}{10} y^3 e^{x^2} = -y^2 e^{x^2}$

$e^{x^2}$ is always $\neq 0$ $\Rightarrow$ it's safe to cancel out

$\leadsto$ $\frac{1}{10} y^3 = -y^2$

$\Rightarrow$ $y^3 = -10y^2$

$y^3 + 10y^2 = 0$

$y^2(y+10) = 0$

$\rightarrow$ $y=0$, or $y=-10$

$x=0$: then $0 - 3y = 8$

$y = -\frac{8}{3}$

$\rightarrow$ point $\left(0, -\frac{8}{3}\right)$

$y=0$: then $10x^2 = 8$

$x^2 = \frac{8}{10} = \frac{4}{5}$

$x = \pm\sqrt{\frac{4}{5}} = \pm\frac{2}{\sqrt{5}}$

$\leadsto$ points $\left(\frac{2}{\sqrt{5}}, 0\right), \left(-\frac{2}{\sqrt{5}}, 0\right)$

$y=-10$

then $10x^2 + 30 = 8$

$10x^2 = -22$

$x^2 = -\frac{22}{10}$ no solution

To find min, evaluate:

$f\left(\pm\frac{2}{\sqrt{5}}, 0\right) = 0^3 \cdot e^{\frac{4}{5}} = 0$

$f\left(0, -\frac{8}{3}\right) = \left(-\frac{8}{3}\right)^3 \cdot e^0 = \left(-\frac{8}{3}\right)^3 =$

$= -\frac{512}{27}$ min