MA 16020 Lesson 30: Systems of linear equations

A *linear equation* in two (three) variables is an equation of the form:

$ax + by = c$ where a, b, c, d are numbers

$(ax + by + cz = d)$

We will typically consider *systems* of linear equations.

Example. Find a solution (x, y, z) to the system of linear equations

$$
\begin{aligned}
\text{I} \quad 3x + 2y + 3z &= 7,\\
\text{II} \quad 4x - 3y - z &= -2,\\
\text{III} \quad x + y + z &= 3.
\end{aligned}
$$

Solution 1 ("standard"; sketch):

$x + y + z = 3$

$\rightsquigarrow \boxed{z = 3 - x - y}$

Plug into I, II:

$3x + 2y + 3(3 - x - y) = 7$

$4x - 3y - (3 - x - y) = -2$

~~$0x + 2y + 9$~~

$0x - y + 9 = 7$

$4x - 3y - 3 + x + y = 2$

$-y + 9 = 7$

$5x - 2y = 1$

... and so on

Solution 2 ("elimination method"): We add/subtract equations from one another to eliminate variables from them.

$(\text{I}) - 3(\text{III})$

$(3x + 2y + 3z) - 3(x + y + z) = 7 - 3 \cdot 3$

$0x - y + 0z = -2$

$-y = -2$

$\boxed{y = 2}$

$(\text{II}) - 4(\text{III})$:

$(4x - 3y - z) - 4(x + y + z) = -2 - 12 = -14$

$0x - 7y - 5z = -14$

$-14 - 5z = -14$

$-5z = 0$

$\boxed{z = 0}$

Plug $z = 0, y = 2$ into $x + y + z = 3$

$x + 2 + 0 = 3$

$\boxed{x = 1}$

solution $\boxed{(x, y, z) = (1, 2, 0)}$

We classify a system of linear equations based on its solutions as follows:

(A) Consistent independent: The system has a solution, and the solution is unique.

Example: $3x+2y+3z=7$ (previous example)

$4x-3y-z=-2$

$x+y+z=3$

(B) Consistent dependent: The system has a solution, but not unique

→ the solution will depend on one (or more) parameters.

Example:

I. $x+y+z=1$

II. $x-y+3z=3$

(III. $2x+0y+4z=2$)

(I − III): $0x+2y-2z=-2$

$2y-2z=-2$

$y-z=-1$

$\boxed{y=z-1}$

(I + II)

$2x+0y+4z=4$

$x+2z=2$

$\boxed{x=2-2z}$

$z=t$ ... $x=2-2t$

$y=t-1$

$(2-2t, t-1, t)$ is a solution, for arbitrary value of t

(C) Inonsistent:

The system does not have any solutions.

Example:

I $3x+y+z=1$

II $x-y+z=2$

III $x+y=0$

II − III

$0x-2y+z=2$

(A) $\boxed{-2y+z=2}$

I − 3(III):

$0x-2y+z=1$

(B) $\boxed{-2y+z=1}$

(A) − (B)

$0=2-1=1$

$0=1$ X

Example. Solve the system of linear equations

$$
\begin{aligned}
4x + 4y + 2z &= 2,\\
3x - 2y + z &= 0,\\
x + 4y + z &= 1.
\end{aligned}
$$

To make the work with the equation more efficient, we record all the relevant coefficients in the *augmented matrix* for the system:

$$
\begin{aligned}
4x + 4y + 2z &= 2\\
3\cdot x - 2y + 1\cdot z &= 0\\
1\cdot x + 4\cdot y + 1\cdot z &= 1
\end{aligned}
\quad \rightsquigarrow \quad
\left[\begin{array}{ccc|c} 4 & 4 & 2 & 2\\ 3 & -2 & 1 & 0\\ 1 & 4 & 1 & 1 \end{array}\right]
$$

(„x" „y" „z" = numbers)

The relevant operations for the elimination method become the following *row operations* on the matrix:

1. swap ~~rows~~ ith and jth row, „$R_i \leftrightarrow R_j$"
2. Multiply ith row by a nonzero constant a, „$aR_i \to R_i$"
3. add a b-multiple of ith row to the jth row, „$R_j + bR_i \to R_j$"

Using the row operations, we perform *Gaussian elimination*: the goal is to obtain a matrix of the form(s) (called *row echelon form*):

$$
\left[\begin{array}{ccc|c} 1 & * & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 1 & * \end{array}\right],
\left[\begin{array}{ccc|c} 1 & * & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 0 & * \end{array}\right],
\left[\begin{array}{ccc|c} 1 & * & * & *\\ 0 & 0 & 1 & *\\ 0 & 0 & 0 & * \end{array}\right],
$$

$$
\left[\begin{array}{ccc|c} 1 & * & * & *\\ 0 & 0 & 0 & *\\ 0 & 0 & 0 & 0 \end{array}\right],
\left[\begin{array}{ccc|c} 0 & 1 & * & *\\ 0 & 0 & 1 & *\\ 0 & 0 & 0 & * \end{array}\right],
\left[\begin{array}{ccc|c} 0 & 1 & * & *\\ 0 & 0 & 0 & *\\ 0 & 0 & 0 & 0 \end{array}\right],
$$

$$
\left[\begin{array}{ccc|c} 0 & 0 & 1 & *\\ 0 & 0 & 0 & *\\ 0 & 0 & 0 & 0 \end{array}\right]
$$

(number of initial 0's in rows is increasing, first nonzero entry in a row is 1)

Let us now solve the problem using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 2 \\ 3 & -2 & 1 & 0 \\ 1 & 4 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 3 & -2 & 1 & 0 \\ 4 & 4 & 2 & 2 \end{array}\right] \xrightarrow{(-3)R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & -14 & -2 & -3 \\ 4 & 4 & 2 & 2 \end{array}\right] \xrightarrow{-4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & -14 & -2 & -3 \\ 0 & -12 & -2 & -2 \end{array}\right] \xrightarrow{-R_3 + R_2 \to R_2}$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & -12 & -2 & -2 \end{array}\right] \xrightarrow{-6R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{\text{divide } R_2, R_3 \text{ by } -2} \left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\boxed{z = -2} \qquad x + 4y + z = 1$$

$$\boxed{y = \frac{1}{2}} \qquad x + 2 - 2 = 1$$

$$\boxed{x = 1}$$

Exercise (if time permits). The dog nutrition from brand A contains 15 g of protein and 210 g of carbohydrates per can, while the food from brand B contains 20 g of protein and 150 g of carbohydrates per can. If the ideal meal consists of 15 g of protein and 145 g of carbohydrates, how many cans of each brand should be used?

Use x cans of brand A,

y cans of brand B

$\leadsto$ $15x + 20y = 15$ („protein")

$210x + 150y = 145$ („carbs")

$\xrightarrow{\text{aug. matrix}}$ $\left[\begin{array}{cc|c} 15 & 20 & 15 \\ 210 & 150 & 145 \end{array}\right] \xrightarrow[\frac{1}{5}R_2 \to R_2]{\frac{1}{5}R_1 \to R_1,} \left[\begin{array}{cc|c} 3 & 4 & 3 \\ 42 & 30 & 29 \end{array}\right]$

$\xrightarrow{-14R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 3 & 4 & 3 \\ 0 & -26 & -13 \end{array}\right] \xrightarrow{-\frac{1}{26}R_2 \to R_2} \left[\begin{array}{cc|c} 3 & 4 & 3 \\ 0 & 1 & 1/2 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1}$

$\to \left[\begin{array}{cc|c} 1 & 4/3 & 1 \\ 0 & 1 & \frac{1}{2} \end{array}\right] \leadsto$ $x + \frac{4}{3}y = 1$, $\leadsto$ $x + \frac{4}{3} \cdot \frac{1}{2} = 1$

$0 + 1y = \frac{1}{2}$ $\leadsto$ $\boxed{y = 1/2}$ $\quad x + \frac{2}{3} = 1$

$\boxed{x = 1 - \frac{2}{3} = \frac{1}{3}}$

$\to$ Need to use $\frac{1}{3}$ of can of brand A,

$\frac{1}{2}$ of can of brand B