MA 16020 Lesson 35: Eigenvalues and eigenvectors I

The **eigenvector** of a square ($n \times n$) matrix A is a 1-column matrix (i.e. *vector*) $v \neq 0$ such that:

$$A \cdot v = \lambda \cdot v \quad \text{, for some constant } \lambda$$

In such case, we call λ the *eigenvalue* of A.

Example: Let us check whether the vector $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for the matrix $\begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -4 \end{bmatrix}$, and if yes, find the eigenvalue.

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$$

.. yes, eigenvector with eigenvalue $\lambda = 2$

How to find an eigenvalue. To find eigenvectors of a matrix, it is practical to find the possible eigenvalues λ first.

Example: Find all the eigenvalues of the matrix $\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$.

Suppose that the eigenvalue is λ and the eigenvector is $\begin{bmatrix} x \\ y \end{bmatrix}$. Then we have:

$$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} - \lambda \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad / \cdot (-1)$$

$$\lambda \cdot \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{\begin{bmatrix} \lambda - 2 & 3 \\ 1 & \lambda - 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

We are looking for λ's such that the equation above has **nonzero** solution $\begin{bmatrix} x \\ y \end{bmatrix}$. This means that: $\begin{bmatrix} \lambda-2 & 3 \\ 1 & \lambda-4 \end{bmatrix}$ has to be singular

To check this, we employ the determinant:

$$\begin{vmatrix} \lambda-2 & 3 \\ 1 & \lambda-4 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda-4)-3\cdot 1=0$$

$$\lambda^2-6\lambda+8-3=0$$

$$\lambda^2-6\lambda+5=0$$

Solve: $\lambda_{1,2} = \frac{6 \pm \sqrt{36-4\cdot 5}}{2}$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = 3 \pm 2$$

$\underline{\underline{\lambda_1 = 5}}$, $\underline{\underline{\lambda_2 = 1}}$

Exercise. Find all the eigenvalues of the matrix $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.

$\begin{bmatrix} \lambda-3 & -1 \\ -2 & \lambda-4 \end{bmatrix}$ has to be singular

$$\to \begin{vmatrix} \lambda-3 & -1 \\ -2 & \lambda-4 \end{vmatrix} = 0$$

$$(\lambda-3)(\lambda-4)-(-1)\cdot(-2)=0$$

$$\lambda^2-7\lambda+12-2=0$$

$$\lambda^2-7\lambda+10=0$$

solve: $\lambda_{1,2} = \frac{7 \pm \sqrt{49-4\cdot 10}}{2}$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$\underline{\lambda_1 = 5}$, $\underline{\lambda_2 = 2}$

How to find the eigenvectors. Finally, let us continue the original example (matrix $\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$) to find eigenvectors to the found eigenvalues:

(recall: we found the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 1$)

$\lambda = 5$:

$$\begin{bmatrix} \lambda-2 & 3 \\ 1 & \lambda-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve:

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ reduced row echelon form

$x + y = 0$

$x = -y$

Choose some nonzero y, e.g. $y=1$ $\rightsquigarrow$ $x = -1$ } eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 1$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ -1 & 3 & 0 \end{array}\right] \longrightarrow$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ reduced row echelon form

$x - 3y = 0$

$x = 3y$

Choose nonzero y, $y = 2$

$\rightarrow x = 3 \cdot 2 = 6$

eigenvector $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Finding eigenvalues and eigenvectors - summary.

1. Set up the *characteristic equation*: $\det(\lambda \cdot I_n - A) = 0$

2. Eigenvalues of A are then obtained as: solutions λ to the characteristic equation.

3. For each eigenvalue λ, the corresponding eigenvectors are obtained as:

nonzero solutions v to the matrix equation

$$(\lambda \cdot I_n - A) \cdot v = 0$$

Example: Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix}$.

1. Characteristic equation:

$$\begin{vmatrix} \lambda - 1 & -5 \\ -1 & \lambda - 3 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-3) - (-5)\cdot(-1) = 0$$

$$\lambda^2 - 4\lambda + 3 - 5 = 0$$

$$\lambda^2 - 4\lambda - 2 = 0$$

→ solve

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4\cdot(-2)}}{2} = \frac{4 \pm \sqrt{24}}{2}$$

$$\lambda_{1,2} = \frac{4 \pm 2\cdot\sqrt{6}}{2} = 2 \pm \sqrt{6}$$

2. eigenvectors

$\lambda = 2 + \sqrt{6}$

$$\begin{pmatrix} 2+\sqrt{6}-1 & -5 \\ -1 & 2+\sqrt{6}-3 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1+\sqrt{6} & -5 & 0 \\ -1 & -1+\sqrt{6} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & -1+\sqrt{6} & 0 \\ 1+\sqrt{6} & -5 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1-\sqrt{6} & 0 \\ 1+\sqrt{6} & -5 & 0 \end{array}\right] \xrightarrow{-(1+\sqrt{6})R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1-\sqrt{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left(-(1+\sqrt{6})(1-\sqrt{6}) = -(1+\sqrt{6}-\sqrt{6}-(\sqrt{6})^2) = -1+6 = 5 \right)$$

$$x + (1-\sqrt{6})y = 0$$

$$x = (-1+\sqrt{6})y$$

$y := 1 \leadsto x = (-1+\sqrt{6})\cdot 1 = -1+\sqrt{6}$

$\leadsto$ eigenvector $\begin{bmatrix} -1+\sqrt{6} \\ 1 \end{bmatrix}$ //

$\lambda = 2 - \sqrt{6}$

$$\left[\begin{array}{cc|c} 1-\sqrt{6} & -5 & 0 \\ -1 & -1-\sqrt{6} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & -1-\sqrt{6} & 0 \\ 1-\sqrt{6} & -5 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1+\sqrt{6} & 0 \\ 1-\sqrt{6} & -5 & 0 \end{array}\right] \xrightarrow{-(1-\sqrt{6})R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1+\sqrt{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + (1+\sqrt{6})y = 0$$

$$x = (-1-\sqrt{6})y$$

$y := 1 \to x = -1-\sqrt{6}$

eigenvector $\begin{bmatrix} -1-\sqrt{6} \\ 1 \end{bmatrix}$

Example: Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 7 & -4 \\ 4 & -1 \end{bmatrix}$.

1. char. equation:

$$\begin{vmatrix} \lambda - 7 & 4 \\ -4 & \lambda + 1 \end{vmatrix} = 0$$

$$(\lambda - 7)(\lambda + 1) - 4(-4) = 0$$
$$\lambda^2 - 6\lambda - 7 + 16 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36 - 4 \cdot 9}}{2} = \frac{6 \pm \sqrt{0}}{2} = 3 \pm 0 = 3$$

$\rightsquigarrow$ only one eigenvalue $\lambda = 3$

2. eigenvectors:

$\lambda = 3$:

$$\begin{bmatrix} 3-7 & 4 \\ -4 & 3+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ -4 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -4 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x - y = 0$$
$$x = y$$

Set $y = 1$: $\rightsquigarrow$ $x = 1$

The "only" eigenvector is $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(up to scalar multiple)