MATH 16020 Lesson 4: Integration by Parts I

Spring 2021

Integration by Substitution: Stems from Chain Rule

Integration by Parts: Stems from Product Rule.

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \Rightarrow \int [f(x)g(x)]'\,dx = \int f'(x)g(x) + f(x)g'(x)\,dx$$

$$\Rightarrow f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

If $u = f(x)$, $v = g(x)$, then $du = f'(x)\,dx$, $dv = g'(x)\,dx$

$$\Rightarrow \boxed{\int u\,dv = uv - \int v\,du} \quad (*)$$

Integration by parts formula. MEMORIZE!!

Why do we care? $(*)$ evaluates SOME integrals that substitution can't.

Example 1. Use integration by parts to evaluate $\int x \ln(x)\,dx$.

Idea: Choose appropriate u and dv (to get du and v) and apply $(*)$.

1st try

$u = x$ $\quad dv = \ln(x)\,dx$

$\Downarrow \quad \Downarrow$

$du = dx$ $\quad v = ?$

Don't know $\int \ln(x)\,dx$, so instead

2nd try

$u = \ln(x)$ $\quad dv = x\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{2}x^2$

Will add $+C$ after all integration, so don't add $+C$ here!

$$\Rightarrow \int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\,dx$$

$$= \boxed{\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C}$$

How to choose u in general? Choose in this order.

L = Logarithms ← $\ln(x)$, $\ln(3x)$, $\ln(x^5)$, etc.

I = Inverse Trig ← Not for 16020

A = Algebraic ← Polynomials x^2, x^3+2x+7, etc. (NO ROOTS!)

T = Trig ← $\cos(x)$, $\sin(2x)$, etc.

E = Exponential ← e^x, e^{2x+1}, e^{x^2}, etc.

In Ex. 1, L before A, so choose $u=\ln(x)$ (so $dv = x\,dx$)

"everything else"

Example 2. Evaluate the following using integration by parts:

A. $\int x\cos(x)\,dx$

By LIATE, choose $u = x$ $\quad dv = \cos(x)dx$

$du = dx$ $\quad v = \sin(x)$

$$\Rightarrow \int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = \boxed{x\sin(x)+\cos(x)+C}$$

B. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

CAREFUL!! LIATE suggests $u=x^3$, $dv=\frac{dx}{\sqrt{1+x^2}}$, but we can't integrate $\int (1+x^2)^{-1/2}dx$; instead try:

$$\boxed{\begin{array}{ll} u=x^2 & dv=x(1+x^2)^{-1/2}dx \\ du=2x\,dx & v=(1+x^2)^{1/2} \end{array}}$$

$$= x^2\sqrt{x^2+1} - \int 2x\sqrt{1+x^2}\,dx = x^2\sqrt{x^2+1} - \int \sqrt{w}\,dw = x^2\sqrt{x^2+1} - \frac{2}{3}w^{3/2}+C$$

$$= \boxed{x^2\sqrt{x^2+1} - \frac{2}{3}(x^2+1)^{3/2}+C}$$

$w = x^2+1$
$dw = 2x\,dx$

w is sub. variable; u already used here

C. $\int \frac{(\ln(2x^5))^2}{x^2}\,dx$

$$\boxed{\begin{array}{ll} u=(\ln(2x^5))^2, & dv=x^{-2}dx \\ du=\frac{10\ln(2x^5)}{x}dx & v=-x^{-1} \end{array}}$$

$$= -\frac{(\ln(2x^5))^2}{x} + \int \frac{10\ln(2x^5)}{x^2}dx = \frac{-(\ln(2x^5))^2}{x} - \frac{10\ln(2x^5)}{x} + \int \frac{50}{x^2}dx$$

$$= \boxed{\frac{-(\ln(2x^5))^2}{x} - \frac{10\ln(2x^5)}{x} + \frac{50}{x} + C}$$

By Parts again!

$u=10\ln(2x^5)$, $dv=x^{-2}dx$
$du=\frac{50}{x}dx$ $\quad v=-x^{-1}$

D. $\int_3^4 x(x-3)^7\,dx$ ← Still apply LIATE, even w/ a definite integral!

$$\boxed{\begin{array}{ll} u = x & dv = (x-3)^7\,dx \\ du = dx & v = \frac{1}{8}(x-3)^8 \end{array}}$$

$$= \left[\frac{x}{8}(x-3)^8\right]_3^4 - \int_3^4 \frac{1}{8}(x-3)^8\,dx$$

$$= \left[\frac{x}{8}(x-3)^8 - \frac{1}{72}(x-3)^9\right]_3^4 = \left(\frac{4}{8} - \frac{1}{72}\right) - 0 = \frac{36}{72} - \frac{1}{72} = \boxed{\frac{35}{72}}$$

E. $\int (2x+1)e^{-x}\,dx$ (TIME PERMITTING)

$$\boxed{\begin{array}{ll} u = 2x+1 & dv = e^{-x}\,dx \\ du = 2\,dx & v = -e^{-x} \end{array}}$$

$$= (2x+1)(-e^{-x}) + \int +e^{-x}\,2dx$$

$$= -(2x+1)e^{-x} - 2e^{-x} + C$$

$$= \boxed{-(2x+3)e^{-x} + C}$$