MA 16020 Lesson 8: Separation of variables III

Exercise 1. Find the general solution to the differential equations.

(a) $6x^3y' = 2y' + x^2e^{-4y}$ <u>First task</u>: separating the variables

$6x^3y' - 2y' = x^2e^{-4y}$

$(6x^3-2)y' = x^2e^{-4y}$ ← now we can separate.

$$\boxed{e^{4y}\frac{dy}{dx} = \frac{x^2}{6x^3-2}}$$

$$\underbrace{\int e^{4y}\,dy}_{=\frac{1}{4}e^{4y}} = \int \frac{x^2\,dx}{6x^3-2} = \left| \begin{matrix} u = 6x^3-2 \\ du = 18x^2\,dx \end{matrix} \right| = \frac{1}{18}\int \frac{1}{u}\,du = \frac{1}{18}\ln|u| = \frac{1}{18}\ln|6x^3-2|$$

$\rightsquigarrow \frac{1}{4}e^{4y} = \frac{1}{18}\ln|6x^3-2| + C$

$e^{4y} = \frac{2}{9}\ln|6x^3-2| + C$ („new" C)

$4y = \ln\left(\frac{2}{9}\ln|6x^3-2| + C\right)$

$$\underline{\underline{y = \frac{1}{4}\ln\left(\frac{2}{9}\ln|6x^3-2| + C\right)}}$$

(b) $t^2y' = 5t^3 + 12t^3y$

$y' = 5t + 12ty = t(5+12y)$

$$\boxed{\frac{y'}{5+12y} = t}$$

$$\underbrace{\int \frac{dy}{5+12y}}_{} = \int t\,dt = \frac{t^2}{2} + C$$

$= \left| \begin{matrix} u = 5+12y \\ du = 12\,dy \end{matrix} \right| = \frac{1}{12}\int \frac{1}{u}\,du = \frac{1}{12}\ln|5+12y|$

$\rightsquigarrow \frac{1}{12}\ln|5+12y| = \frac{t^2}{2} + C$

$\ln|5+12y| = 6t^2 + C$ („new" C)

$5+12y = \underbrace{\pm e^C}_{\text{get again „new" C}} \cdot e^{6t^2}$

$12y = C\cdot e^{6t^2} - 5$

$$\underline{\underline{y = C\cdot e^{6t^2} - \frac{5}{12}}}$$

and, again, „new" C

Exercise 2. Find the general solution to the differential equation

$$y' = \sin(3x)\sqrt{3y}.$$

$$\frac{dy}{dx} = \sin(3x)\cdot\sqrt{3y}$$

$$\frac{1}{\sqrt{3y}}\frac{dy}{dx} = \sin(3x)$$

$$\int \frac{dy}{\sqrt{3y}} = \int \sin(3x)\,dx$$

1) $\int \frac{dy}{\sqrt{3y}} = \int \frac{1}{\sqrt{3}} y^{-\frac{1}{2}} dy = \frac{1}{\sqrt{3}}\cdot 2y^{\frac{1}{2}} = \frac{2}{\sqrt{3}}\sqrt{y}$

2) $\int \sin(3x)\,dx = \left| \begin{matrix} u = 3x \\ du = 3\,dx \end{matrix} \right| = \frac{1}{3}\int \sin(u)\,du = \frac{1}{3}(-\cos(u)) + C$

$= C - \frac{1}{3}\cos(3x)$

$\rightsquigarrow \frac{2}{\sqrt{3}}\sqrt{y} = C - \frac{1}{3}\cos(3x)$

$\sqrt{y} = C - \frac{\sqrt{3}}{6}\cos(3x)$ („new" C)

$$\underline{\underline{y = \left(C - \frac{\sqrt{3}}{6}\cos(3x)\right)^2}}$$

! Warning:

$$y = \left(C - \frac{\sqrt{3}}{6}\cos(3x)\right)^2 = \underbrace{C^2}_{=:D} - \underbrace{\cancel{\tfrac{\sqrt{3}}{3}\cdot C}}_{=:C \text{ („new" c)}}\cos(3x) + \frac{3}{36}\cos^2(3x) =$$

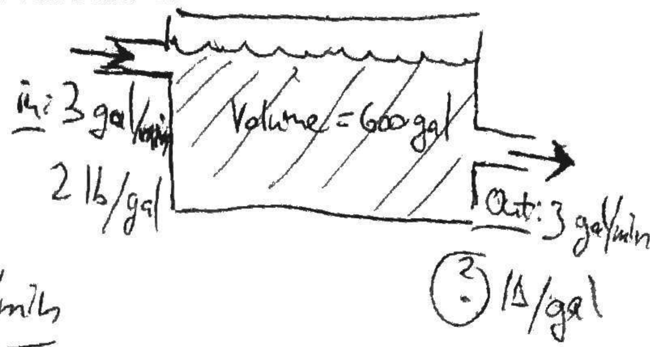
$$\cancel{= D - \ldots}$$

$$\cancel{=} \underbrace{D - C\cdot\cos(3x) + \frac{1}{12}\cos^2(3x), \quad C, D \text{ arbitrary constants}}$$

is this correct? $\boxed{\text{No!}}$: 1) C^2 is not arbitrary constant ... it is always positive

2) the constants C, D are not independent!

Exercise 3. A 800-gallon tank initially contains 600 gallons of pure water. Brine containing 2 pounds of salt per gallon flows into the tank at the rate of 3 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 3 gallons per minute. What is the amount of salt in the tank after 10 minutes?



$A(t)$ = amount of salt after t minutes

$A(0) = 0$

$\boxed{\text{Rate of change} = \text{Rate in} - \text{Rate out}}$

Rate in: $(3\ \text{gal/min}) \times (2\ \text{lb/gal}) = 6\ \text{lb/min}$

Rate out: $(3\ \text{gal/min}) \times \underbrace{\left(\frac{A(t)}{600}\ \text{lb/gal}\right)}_{\text{concentration of salt in mixture}} = \frac{A}{200}\ \text{lb/min}$

$$\rightsquigarrow \boxed{\frac{dA}{dt} = 6 - \frac{A}{200}}$$

$$\frac{1}{6 - \frac{A}{200}} \frac{dA}{dt} = 1$$

$$\int \frac{dA}{6 - \frac{A}{200}} = \int 1 \cdot dt$$

$$-200 \ln\left|6 - \frac{A}{200}\right| = t + C$$

(use sub. $u = 6 - \frac{A}{200}$)

$$\ln\left|6 - \frac{A}{200}\right| = -\frac{t}{200} + C$$

$$6 - \frac{A}{200} = \underbrace{\pm e^{C}}_{=: C \text{ ("new")}} \cdot e^{-\frac{t}{200}}$$

$$6 - \frac{A}{200} = C \cdot e^{-\frac{t}{200}}$$

solve for A, get

$$A = 200\left(6 - C \cdot e^{-\frac{t}{200}}\right) = 1200 - \underbrace{C}_{\text{"new" } C} \cdot e^{-\frac{t}{200}}$$

$A(0) = 0$:

$$0 = 1200 - C \cdot e^{0} = 1200 - C$$

$$\Rightarrow C = 1200$$

$$\Rightarrow \boxed{A(t) = 1200 - 1200 e^{-\frac{t}{200}}}$$

Finally, amount after 10 min

$$= A(10) = 1200 - 1200 e^{-\frac{10}{200}}$$

≈ 58.525 pounds

Exercise 4. In the previous problem, assume that the mixture flows out only at the rate of 2 gallons per minute. Set up a differential equation describing the amount of salt in the tank after t minutes. Is the equation separable?

Rate in: The same as above, ie $3 \cdot 2 = 6$ lb/min

Rate out: $(2\ \text{gal/min}) \times$ concentration, but now, concentration $= \frac{A(t)}{\text{volume}} = \frac{A(t)}{600+t}$

(3 gal/min go in, 2 gal/min go out $\Rightarrow$ volume increases by 1 gal/min)

$$\rightsquigarrow \text{Rate out} = 2 \cdot \frac{A}{600+t}$$

$$\rightsquigarrow \boxed{\frac{dA}{dt} = 6 - 2 \cdot \frac{A}{600+t}}$$

Not separable!