Existence and uniqueness of solution of linear diff. equation

If $a_1(t), a_2(t), \dots a_n(t), g(t)$ are continuous (real-valued) functions on some interval (a, b) containing t_0, then an initial value problem of the form

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t), \quad y(t_0) = y_0, y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}$$

has a unique solution on (a, b).

Example 1 Find the largest interval on which the initial value problem

$$y''' + \sqrt{t-3}y' + y = \cos(t), \quad y(5) = 1, y'(5) = 1, y''(5) = 0$$

has a unique solution guaranteed by the existence and uniqueness theorem.

Solution: The coefficient functions are 0 (by y''), $\sqrt{t-3}$, 1 and $\cos(t)$. All of them are continusous wherever they are defined, and all of them are defined on the full real line except for $\sqrt{t-3}$, whose domain is $[3, \infty)$. Therefore the biggest interval for which the theorem guarantees existence and uniqueness is $(3, \infty)$ (note that we are taking the open interval, i.e. excluding the endpoint 3, since the theorem considers open intervals only).

Example 2 (more or less the one from class): Find the largest interval on which the initial value problem

$$(t^2 - 3t + 2)y'' + 2ty' + 3y = \frac{1}{t+4}, \quad y(0) = 3, y'(0) = 1$$

has a unique solution guaranteed by the existence and uniqueness theorem.

Solution: To apply the above theorem, we need to get rid of the coefficient by y''. Thus, we divide the equation by $(t^2 - 3t + 2)$ and obtain

$$y'' + \frac{2t}{t^2 - 3t + 2}y' + \frac{3}{t^2 - 3t + 2}y = \frac{1}{(t+4)(t^2 + 3t + 2)}.$$

Now the points where one or more of the coefficient functions is discontinuous are $t = -4, 1, 2$ (since $1, 2$ are roots of $t^2 - 3t + 2$). The biggest interval that contains 0 and on which all the coefficient functions are continuous is thus $(-4, 1)$.

Example 3 Find the largest interval on which the initial value problem

$$ty'' + \sqrt{t-3}y' + (t+2)y = 4t, \quad y(0) = 1, y'(0) = 1$$

has a unique solution guaranteed by the existence and uniqueness theorem.

Solution: Again, first we have to divide by t to make the coefficient at y'' to be 1. We obtain the equation

$$y'' + \frac{\sqrt{t-3}}{t}y' + \frac{t+2}{t}y = 4,$$

and so we see that the coefficient functions $\frac{\sqrt{t-3}}{t}$, $\frac{t+2}{t}$ are discontinuous at 0. Thus, the coefficient functions are not continuous in any interval containing 0, so in this case, the uniqueness and existence theorem does not tell us anything (we cannot make any conclusion about the existence and uniqueness).